Minimum-Degree Obstructions in the Riemann Theta Product

A certified classification and analytic limits

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October 18, 2026

Abstract

We classify the least degree of a polynomial whose square has negative signed-moment value for the normalized fixed-center Riemann theta product. On $3 \le x \le 8$ the least degree is finite, has eleven transitions, and takes the successive values $5, 6, 7, 9, 10, 9, 10, 11, 13, 14, 13, 12$. We isolate every transition and determine the degree at equality. Jacobi symmetry extends the classification to $\pi^2/8 \le x \le 8$. For each fixed degree, the scaled moment matrix tends to a Gaussian moment matrix as $x \to \infty$; hence every fixed-degree test becomes strictly positive at both center extremes. The single-summand theta model has a positive degree-five matrix for all $x \ge 3$, whereas two summands already produce the quintic obstruction at $x = 3.64$. The proof includes the complete infinite tails, a directed Taylor cover, independent exact-rational root checks, and a self-contained reproduction supplement.

All assertions concern the fixed-center product (2). **Keywords:** Riemann theta density; moment matrices; polynomial-square obstructions; minimum degree; computer-assisted proof.

1 The fixed-center moment problem

Let

$$\Phi(s) = \sum_{n=1}^{\infty} (2\pi^2 n^4 e^{9s} - 3\pi n^2 e^{5s}) e^{-\pi n^2 e^{4s}}. \tag{1}$$

For $x > 0$, write $s(x) = \frac{1}{4}\log(x/\pi)$ and set

$$p_x(t) = \frac{\Phi(s(x) + t/2)\Phi(s(x) - t/2)}{\Phi(s(x))^2}. \tag{2}$$

The density is even and strictly positive on $\mathbb{R}$ (Lemma 5), so the normalization is defined everywhere. Normal convergence of every fixed derivative on compact real intervals justifies differentiation. Define

$$m_{2j}(x) = (-1)^j p_x^{(2j)}(0), \quad m_{2j+1}(x) = 0, \qquad M_d(x) = (m_{j+k}(x))_{j,k=0}^{d}. \tag{3}$$

Here $m_0 = 1$. Let $\mathcal{L}_x$ be the real polynomial functional with $\mathcal{L}_x(X^j) = m_j(x)$. A polynomial P is an obstruction when $\mathcal{L}_x(P^2) < 0$, and

$$\delta(x) = \min\{\deg P : \mathcal{L}_x(P^2) < 0\}, \tag{4}$$

with value ∞ when the set is empty. For $P = \sum_{j=0}^{d} c_j X^j$, $\mathcal{L}_x(P^2) = c^{\mathsf{T}} M_d c$.

The initial explicit quintic witness appears in Schubr [6]. The local quintic transition a_1 on $[3.64, 3.66]$ was established in [7] (October 17; corrected October 18, 2026). Here we extend that result to the complete least-degree function on $[3, 8]$, adding ten transitions, nonmonotonicity and higher-degree equality cases, and derive analytic constraints outside that domain.

Moment optimization and Schur complements are established tools [2, 3]; polynomial-square negativity tests occur in Kot et al. [5]. Csordas [1] studies associated theta kernels obtained by integrating over the center. Certified theta-derivative inequalities and derivative-sensitive tail estimates appear in Sra [8] and Planat–Solé [9]. Michałowski [10] studies shifted Toeplitz minors of the same density in a Pólya-frequency problem; that determinant functional differs from (3).

2 Complete minimum-degree classification

Theorem 1. *There are exactly eleven transition parameters $3 < a_1 < \cdots < a_{11} < 8$ for δ on $[3, 8]$. Their strict brackets and crossing degrees d_i are given in Table 1. Put $a_0 = 3$, $a_{12} = 8$, and*

$$(v_0, \dots, v_{11}) = (5, 6, 7, 9, 10, 9, 10, 11, 13, 14, 13, 12). \tag{5}$$

For $0 \le i \le 11$ and $a_i < x < a_{i+1}$, $\delta(x) = v_i$. Also $\delta(3) = 5$, $\delta(8) = 12$, and

$$\delta(a_i) = \max(v_{i-1}, v_i) \qquad (1 \le i \le 11). \tag{6}$$

At a_i, M_{d_i} is positive semidefinite with a one-dimensional nullspace. On a neighborhood of a_i its monic Schur contraction has a simple zero. In particular, $\max_{3 \le x \le 8} \delta(x) = 14$.

i	d_i	Lower endpoint for a_i	Upper endpoint for a_i	$\delta(a_i)$
1	5	3.64888758188562	3.64888758188563	6
2	6	4.11304745500471102656	4.11304745500471102658	7
3	7	4.14157658489452707551	4.14157658489452707553	9
4	9	4.36591915914560481792	4.36591915914560481794	10
5	9	5.38557526472222776839	5.38557526472222776841	10
6	9	6.29313187387534279042	6.29313187387534279044	10
7	10	6.90841574564289048292	6.90841574564289048294	11
8	11	7.10426848885908199356	7.10426848885908199358	13
9	13	7.30607150627930169343	7.30607150627930169345	14
10	13	7.77596605119156831048	7.77596605119156831050	14
11	12	7.79865201846424367518	7.79865201846424367520	13

Table 1: Strict transition brackets. The crossing degree is $d_i = \min(v_{i-1}, v_i)$; a null direction at equality is not a negative obstruction. Printed decimals are exact rational endpoints.

Corollary 2. *For every $x > 0$, $M_d(x) = M_d(\pi^2/x)$ and $\delta(x) = \delta(\pi^2/x)$. Thus Theorem 1 determines δ on $[\pi^2/8, 8]$; reflected transitions are π^2/a_i with reversed orientation. Throughout this domain $M_4 > 0$ and an obstruction of degree at most fourteen exists.*

2.1 Schur contractions and transition windows

Parity splits M_d into even and odd monomial blocks. When $M_{d-1} = B > 0$, write

$$M_d = \begin{pmatrix} B & w \\ w^{\mathsf{T}} & m_{2d} \end{pmatrix}, \qquad S_d = m_{2d} - w^{\mathsf{T}} B^{-1} w,$$
$$P_d = X^d - (B^{-1}w)^{\mathsf{T}}(1, X, \dots, X^{d-1}).$$

For every monic $P = X^d + c^{\mathsf{T}}(1, \dots, X^{d-1})$,

$$\mathcal{L}_x(P^2) = S_d + (c + B^{-1}w)^{\mathsf{T}} B (c + B^{-1}w). \tag{7}$$

Thus $S_d < 0$ gives $\delta = d$, whereas $S_d = 0$ gives a unique monic null polynomial. The latter does not determine δ until higher degrees are checked. With the positive B_0 from (11), put

$$\mathcal{N}_x = B_0^2 \mathcal{L}_x, \qquad \widetilde{S}_d = B_0^2 S_d. \tag{8}$$

This multiplication preserves signs and nullspaces.

Lemma 3. *Let $J = [l, r]$ satisfy $M_{d-1} > 0$. Suppose $\widetilde{S}_d$ has opposite endpoint signs and a derivative of constant strict sign on J. Let $h \in \{d+1, d+2\}$. Suppose the opposite-parity monomial block through degree $h-1$ is positive definite on J, and a fixed degree-h polynomial Q satisfies $\mathcal{N}_x(Q^2) < 0$ on J. Then S_d has a unique zero $a \in J$. On its negative side $\delta = d$; on its positive side $\delta = h$; and $\delta(a) = h$.*

Proof. Strict monotonicity and endpoint signs give the unique zero. Equation (7) determines the crossing parity block through degree d. That parity contributes no additional degree below h, since $h \le d+2$. With the opposite-parity hypothesis this gives $M_{h-1} > 0$ on the positive side and $M_{h-1} \ge 0$ at a. The fixed negative degree-h witness proves the asserted minimum degrees. On the negative side, $M_{d-1} > 0$ and $S_d < 0$ give $\delta = d$. □

This accounts for the skipped minimum degrees eight and twelve at a_3 and a_8: the opposite-parity block remains positive while the next direction of the crossing parity is negative. The decreases $10 \to 9$ at a_5 and $14 \to 13$ at a_{10} show that the least degree is not monotone.

Proposition 4. *Where $M_{d-1} > 0$, $\mathcal{L}_x(P_d Q) = 0$ for every $\deg Q < d$, and*

$$S_d'(x) = \mathcal{L}_x'(P_d^2). \tag{9}$$

At a simple zero a, if q is the coefficient vector of P_d, the eigenvalue of M_d vanishing there has derivative $S_d'(a)/\|q\|_2^2$.

Proof. The orthogonality equations are $Bc + w = 0$. Differentiation gives (9), since $2\mathcal{L}_x(P_d \partial_x P_d) = 0$ by $\deg \partial_x P_d < d$. At the zero the nullspace is spanned by q and the other eigenvalues are positive. Differentiating the symmetric eigenproblem gives $q^{\mathsf{T}} M_d' q/\|q\|_2^2 = S_d'(a)/\|q\|_2^2$. □

3 Certified evaluation and proof of the classification

3.1 Derivative recurrence and infinite tails

The factorization

$$A_x(t) = \sum_{n \ge 1} n^2 (2n^2 x e^{2t} - 3) e^{-n^2 x (e^{2t}-1)} e^{-(n^2-1)x}, \tag{10}$$
$$\Phi(s + t/2) = \pi e^{5s-x} e^{5t/2} A_x(t) \quad (x = \pi e^{4s})$$

gives $p_x(t) = A_x(t) A_x(-t)/A_x(0)^2$. Define

$$R_0(z) = 2z - 3, \qquad R_{k+1}(z) = 2z(R_k'(z) - R_k(z)),$$
$$B_k(x) = \sum_{n \ge 1} n^2 R_k(n^2 x) e^{-(n^2-1)x}. \tag{11}$$

Then $A_x^{(k)}(0) = B_k$, $\deg R_k = k+1$, and $B_0 > 0$ for $x \ge 3$. The unnormalized moments are

$$N_{2j} = B_0^2 m_{2j} = (-1)^j \sum_{u=0}^{2j} (-1)^u \binom{2j}{u} B_u B_{2j-u}, \qquad N_{2j+1} = 0. \tag{12}$$

In particular $N_0 = B_0^2$.

The primary computation retains $1 \le n \le 7$ and encloses every omitted term before forming products. For integers $k, r \ge 0$,

$$B_k^{(r)} = \sum_{n \ge 1} n^2 e^{-(n^2-1)x} \sum_{\ell=0}^{\min(r,k+1)} \binom{r}{\ell} n^{2\ell} (-(n^2-1))^{r-\ell} R_k^{(\ell)}(n^2 x). \tag{13}$$

Write $R_k(z) = \sum_{i=0}^{k+1} r_{ki} z^i$ and put

$$H_{k,r}(x) = \sum_{\ell=0}^{\min(r,k+1)} \binom{r}{\ell} \sum_{i=\ell}^{k+1} |r_{ki}| \frac{i!}{(i-\ell)!} x^{i-\ell} 8^{2i+2r-2\ell+2}, \tag{14}$$
$$\rho_{k,r}(x) = (9/8)^{2k+2r+4} e^{-17x}.$$

For $\rho_{k,r} < 1$, the absolute omitted tail is bounded by

$$\frac{e^{-63x}}{1-\rho_{k,r}(x)} H_{k,r}(x). \tag{15}$$

Indeed, replacing $(n^2-1)^{r-\ell}$ by $n^{2(r-\ell)}$ leaves exponents in n of at most $2k+2r+4$. For $n \geq 8$, each successive ratio is at most $\rho_{k,r}$. In all primary calculations $k \leq 28$, $r \leq 9$ and $x \geq 3$, so $\rho_{k,r} < 1/2$. Symmetric tail intervals are added to every derivative sum before (12), including all mixed tails.

Primary arithmetic uses 90-digit directed decimal intervals. For exponential bounds, reduce t to $u = t/2^q < 1$ and sum e^u through order 80. The positive remainder is bounded by $u^{81}/(81!(1-u/82))$. Positive reciprocal bounds followed by q outward squarings enclose e^{-t}. No floating-point sign estimate enters an acceptance decision.

3.2 Centered Taylor cover

For $J = [l, r]$, choose a rational $c \in J$ and put $h = x - c$. Taylor's theorem gives

$$B_k(c+h) \in \sum_{u=0}^{L} \frac{B_k^{(u)}(c)}{u!} h^u + \frac{B_k^{(L+1)}(J)}{(L+1)!} h^{L+1}, \tag{16}$$

with every coefficient enclosed using (15). Products are retained without truncation. The derivatives N'_{2j} are assembled separately from Taylor enclosures for B'_k and B_k, rather than by differentiating an interval remainder.

On an ordinary cell, choose fixed rational monic polynomials $q_0, \ldots, q_d$. Their coefficient matrix is triangular with diagonal one; exact orthogonality is unnecessary. For fixed positive rational scales σ_i, the inequalities

$$\frac{\mathcal{N}_x(q_i^2)}{\sigma_i^2} - \sum_{\substack{j<d \\ j\neq i}} \frac{|\mathcal{N}_x(q_i q_j)|}{\sigma_i \sigma_j} > 0 \qquad (0 \leq i < d) \tag{17}$$

prove positive definiteness through degree $d-1$ by symmetric strict diagonal dominance and congruence. A negative upper bound for $\mathcal{N}_x(q_d^2)$ proves $\delta = d$ on the whole cell. Taylor order $L = 8$ is used for these cells.

On each transition window, fixed-basis interval Schur elimination encloses entries and their derivatives with $L = 4$. It verifies all positive prefix pivots, a strict sign for $\widetilde{S}'_d$, and opposite endpoint signs. Separate checks establish the opposite-parity block and higher-degree negative witness in Lemma 3. These numerical derivative bounds concern $\widetilde{S}'_d$, not S'_d on the whole window.

Proof of Theorem 1. The supplied cover has 277 closed ordinary cells and eleven closed transition windows, meeting only at endpoints and covering exactly $[3, 8]$. On each ordinary cell, the strict interval inequalities (17) and negative witness establish the recorded minimum degree. Each transition window satisfies Lemma 3; independent exact-rational interval checks isolate its zero in the strict bracket of Table 1. The prefix and opposite-parity checks give positive semidefiniteness below the persistent witness at equality, proving (6).

The checker verifies exact adjacency, endpoint degrees, agreement of neighboring regimes, and all root and witness bindings. Thus no intervening region or transition of δ is omitted, and the sequence is (5). Schur factorization gives one null direction in M_{d_i} at each root; the derivative bound gives simplicity. The degree-fourteen region and upper bound everywhere give the maximum. The supplement regenerates the entire cover and checks Table 1 against its explicit rational inputs. □

The independent rational implementation rounds outward to the grid 10^{-120}, retains ten theta summands, and uses an exponential series through order 120 after binary range reduction. It checks every listed root, selected degree-minimal neighborhoods, and the finite-summand comparison below. This separately verifies those premises; it is not a second-arithmetic replay of all 277 ordinary cells.

4 Symmetry and the fixed-degree Gaussian limit

Lemma 5. *The density Φ is even and strictly positive on $\mathbb{R}$, and $p_x(t) = p_{\pi^2/x}(t)$ for $x > 0$ and real t.*

Proof. Let $\vartheta(u) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 u}$. Jacobi's transformation $\vartheta(u) = u^{-1/2}\vartheta(1/u)$ makes $F(s) = e^s\vartheta(e^{4s})$ even [4]. Direct differentiation gives $F'' - F = 16\Phi$: the zero lattice term cancels, and the two equal nonzero terms give the factor 16. Hence Φ is even. For $s \geq 0$ each summand of (1) is positive because $2\pi n^2 e^{4s} - 3 > 0$; evenness covers $s < 0$. Replacing x by π^2/x sends s to $-s$ and interchanges the numerator factors of (2). □

Proof of Corollary 2. Differentiate the identity at $t = 0$. The reflection of $[3, 8]$ is $[\pi^2/8, \pi^2/3]$; these intervals overlap. Every ordinary cell and transition window establishes $M_4 > 0$, including equality cases. The witnesses and symmetry give $\delta \leq 14$ on the stated union. □

Theorem 6. *For every fixed integer $d \geq 0$, put $T_d(x) = \mathrm{diag}((8x)^{-j/2})_{j=0}^{d}$. Then*

$$T_d(x)M_d(x)T_d(x) \longrightarrow G_d \qquad (x \to \infty), \tag{18}$$

where G_d is the standard real Gaussian moment matrix. For each d there exists X_d such that $M_d(x) > 0$ for $x \geq X_d$ and for $0 < x \leq \pi^2/X_d$. Thus $\delta(x)$ escapes every fixed bound at both extremes, allowing the value ∞.

Proof. The single-summand normalized product is

$$p_x^{[1]}(t) = \left(1 - \frac{12x(\cosh(2t) - 1)}{(2x-3)^2}\right) e^{-2x(\cosh(2t)-1)} \qquad (x \geq 3). \tag{19}$$

Under $t = z/\sqrt{8x}$ its quadratic exponent is $-z^2/2$, each exponent coefficient of order $2k > 2$ is $O(x^{1-k})$, and every fixed nonconstant coefficient of the prefactor tends to zero. Coefficientwise at each fixed order this gives $p_x^{[1]}(z/\sqrt{8x}) \to e^{-z^2/2}$, hence $m_{2j}^{[1]}(x)/(8x)^j \to (2j-1)!!$.

For fixed k and $x \geq 3$, polynomial growth gives

$$B_k(x) = R_k(x) + O_k(x^{k+1}e^{-3x}). \tag{20}$$

Factor e^{-3x} from $n \geq 2$ and bound the remaining polynomial-weighted series by its convergent value at $x = 3$. Since $B_0 \sim 2x$, substitution into (12), including the denominator change, gives

$$m_{2j}(x) - m_{2j}^{[1]}(x) = O_j(x^{2j}e^{-3x}).$$

After division by $(8x)^j$ this proves (18); odd entries vanish identically. For every nonzero real polynomial P of degree at most d, the Gaussian integral of P^2 is positive, so $G_d > 0$. Entrywise convergence in fixed finite dimension preserves strict positive definiteness for sufficiently large x. Invertible congruence and Lemma 5 finish the proof. □

The threshold depends on d. The theorem asserts neither positivity of all degrees at one center nor existence of a finite obstruction at every center outside the classified domain.

5 The two-summand quintic mechanism

Write $p_x^{[N]}$ for the normalized product retaining $n \le N$ in (1), and use $[N]$ for its moments and contractions.

Theorem 7. *For all $x \ge 3$, $M_5^{[1]}(x) > 0$. At $x = 91/25$, the degree-four prefix is positive definite in the one-, two-, and three-summand models and in the full model, while*

$$\begin{aligned} 2418877279 &< S_5^{[1]} < 2418877280, \\ -43193403 &< S_5^{[2]} < -43193402, \\ -45634396 &< S_5^{[3]} < -45634395, \\ -45634396 &< S_5 < -45634395, \\ -2440994 &< S_5 - S_5^{[2]} < -2440993. \end{aligned}$$

Thus two summands already produce the quintic obstruction at this center, and the remaining infinite sum preserves it.

Proof. Put $D = (2x-3)^2$ and

$$Q_j = (-1)^j \sum_{u=0}^{2j} (-1)^u \binom{2j}{u} R_u R_{2j-u}, \qquad m_{2j}^{[1]} = Q_j / D.$$

For $e \in \{0,1\}$ and $k \le 3$, the leading parity minor is

$$D^{-k} \det(Q_{i+j+e})_{i,j=0}^{k-1} = D^{-k} c_{e,k} (2x-3)^{h_{e,k}} F_{e,k}(x-3).$$

Exact polynomial arithmetic gives Table 2. Every coefficient of each $F_{e,k}$ is strictly positive. Thus all six leading parity minors are positive for $x \ge 3$, and Sylvester's criterion gives $M_5^{[1]} > 0$. The supplement records all coefficients, reconstructs the determinants, and independently checks the moment numerators from (19).

At $x = 91/25$, evaluate (11)–(12) with the indicated finite sums and full tail bounds. With $u = m_2$, $v = m_4$, $w = m_6$, $y = m_8$, $z = m_{10}$,

$$S_5 = z - \frac{w^3 - 2vwy + uy^2}{uw - v^2}. \tag{21}$$

The interval checks give $u > 0$, $uw - v^2 > 0$, $v - u^2 > 0$, and $\det \left(\begin{smallmatrix} 1 & u & v \\ u & v & w \\ v & w & y \end{smallmatrix}\right) > 0$, which establish the degree-four prefix. Formula (21) gives the printed strict bounds. A separate exact-rational implementation reproduces these contraction bounds. □

e	k	$c_{e,k}$	$h_{e,k}$	$\deg F_{e,k}$	Minimum coefficient of $F_{e,k}$
0	1	1	2	0	1
0	2	32	0	6	64
0	3	393216	0	12	2048
1	1	8	0	3	4
1	2	6144	0	8	64
1	3	6039797760	0	15	1024

Table 2: Exact single-summand certificate after $x = 3 + y$. The minimum ranges over all integer coefficients, including the constant term.

The comparison concerns the entire normalized two-summand product. It does not isolate cross terms from the second-summand square. Small scalar corrections need not be small in the high derivatives and Schur cancellations determining the witness degree.

6 Consequence for positive definiteness

Corollary 8. *For each $x \in [\pi^2/8, 8]$, p_x is not positive definite. With $d = \delta(x)$, all sufficiently small $h > 0$ give a negative direction in $(p_x((r-s)h))_{r,s=0}^{d}$.*

Proof. Let $(F_h)_{j,r} = h^{-j}(-1)^{j-r}\binom{j}{r}$ for $r \le j$, and zero otherwise. Repeated finite differences give

$$F_h(p_x((r-s)h))_{r,s=0}^{d} F_h^{\mathsf{T}} \longrightarrow A = ((-1)^k p_x^{(j+k)}(0))_{j,k=0}^{d}.$$

For $U = \operatorname{diag}(1, i, \dots, i^d)$, $UAU^* = M_d$, with U^* the conjugate transpose. The classified negative direction persists for small h; invertibility of F_h and unitary congruence preserve negative inertia. A real symmetric matrix with a negative Hermitian quadratic value has a real negative direction. Positive definiteness of p_x would require every finite translation Gram matrix to be positive semidefinite. □

Reproducibility and scope

The supplement uses Python's standard library. Its README gives commands for a fresh source-only run in a new directory, including runtime and source identities, cover generation, root verification, analytic certificates, and final claim checks. Current executable sources, generated certificates, independent checks, and historical provenance are distinguished. No saved pass status substitutes for regeneration. The classification is complete on the stated compact domain.

AI assistance. AI assisted derivation, verification code, and preparation; the author takes responsibility for the manuscript.

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