Mean ergodic convergence from near-diagonal subadditivity

Scott Schubr

October 24, 2026

Abstract

We prove a mean ergodic theorem whose subadditivity hypothesis is required only for nearly equal block lengths. A positive unital integral-preserving operator on a probability space need not be ergodic or invertible. Our main mechanism is conditional: convergence of the invariant projections of uniformly integrable normalized inputs, together with both ordered inequalities at length differences at most two, forces convergence in L^1. Combining this mechanism with an existing scalar shrinking-corridor theorem gives invariant L^p convergence, for every finite $p \geq 1$, when the relative corridor width has divergent square integral and eventually admits differences two. Nonnegative inputs require only individual L^p membership. For signed inputs, uniform integrability of the normalized negative-part pth powers is necessary and sufficient. The scalar width threshold and its converse are inherited; the contribution is their Markov mean-ergodic lift. Under the same scalar-admissible width and norm criterion, signed integrable inputs also satisfy $\limsup g_n/n = F$ almost surely, with a possibly nonconstant invariant limit. The upper control is uniform over macroscopic operator powers and lengths. An all-legal adjacent-halving tree has a vanishing occupation gradient; a classical maximal-inequality argument admits integrable rewards directly. No uniform boundedness assumption is needed.

1 Results and context

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and let U be a positive linear operator on real $L^1(\mu)$ satisfying

$$U1 = 1, \qquad \mathbb{E}Uh = \mathbb{E}h \quad (h \in L^1). \tag{1}$$

We call this a Markov operator, without assuming a transition-kernel representation. Positivity gives $|Uh| \leq U|h|$, so U is an L^1 contraction. It is also an L^∞ contraction on bounded functions and extends consistently as a contraction on every finite L^p. Write

$$A_J = \frac{1}{J} \sum_{j=0}^{J-1} U^j.$$

These averages converge strongly in L^1 to a positive unital integral-preserving projection P onto $\mathrm{Fix}(U)$; a proof is included below.

All inequalities between functions hold almost surely. Countably many of them may always be placed on one common full-measure set. Every pair condition below is *ordered*: both (n, m) and (m, n) are required. The operator expressions in those two conditions generally differ.

Theorem 1.1 (Conditional balanced mean convergence)**.** *Let $g_n \in L^1$ be real and put $X_n = g_n/n$. Assume that (X_n) is uniformly integrable and that, for some integer K_0, every ordered pair $n, m \geq K_0$ with $|n - m| \leq 2$ satisfies*

$$g_{n+m} \leq g_n + U^n g_m. \tag{2}$$

If $PX_n \to F$ in L^1 for some $F \in \mathrm{Fix}(U)$, then $X_n \to F$ in L^1.

For a measurable nonincreasing function $w : [T, \infty) \to [0, 1]$, the width-$w$ corridor consists of ordered integer pairs satisfying

$$|m - n| \le \min(m, n) w(\log \min(m, n)). \tag{3}$$

An eventual corridor inequality means that one fixed integer threshold applies to all such pairs.

Theorem 1.2 (The Markov lift of the scalar width criterion)**.** *Suppose*

$$\int_T^\infty w(t)^2 \, dt = \infty, \qquad nw(\log n) \ge 2 \quad \textit{eventually on integers.} \tag{4}$$

Fix $1 \le p < \infty$. Let $g_n \in L^p$ be real and assume $g_{n+m} \le g_n + U^n g_m$ eventually on the ordered corridor (3)*. Then g_n/n converges in L^p if and only if*

$$\{((g_n/n)_-)^p : n \ge 1\} \quad \textit{is uniformly integrable.} \tag{5}$$

Its limit belongs to $\mathrm{Fix}(U)$*. In particular, if all g_n are nonnegative, individual membership in L^p is sufficient.*

The width $w(t) = t^{-1/2}$ on a sufficiently late tail satisfies (4). No mixing rate, power ergodicity, reversibility, invertibility or self-adjointness is required. The lower bound two is explicit.

The square-integral criterion is inherited from the scalar theorem in [7]. If its integral is finite, that theorem supplies bounded normalized scalar counterexamples on the one-point probability space. Consequently divergence is necessary for a universal Markov norm-convergence conclusion in the width class considered here. We give the precise interface and converse in Section 4.

Kingman's theorem and its operator extensions impose subadditivity for all block lengths. Derriennic and Krengel [3] develop mean convergence and approximate invariance for positive contractions and Banach lattices. Their full subadditivity permits appending a fixed or unit block. The mean-Kingman argument in [5, Section 5.4] uses fixed blocks and a short remainder. Almost-subadditive and gapped versions allow error terms or inserted gaps while retaining arbitrary block lengths [4, 6]. These methods are classical ancestry; the hypothesis change here is that unbalanced append operations are unavailable. Theorem 1.1 replaces them by finite-leaf Cesaro control and a maximal strictly convex energy argument at consecutive balanced splits.

Theorems 1.1 and 1.2 are norm theorems. Theorem 5.1 additionally gives conditional pointwise upper control from ordered differences at most two and an explicit pointwise projected slope. Corollary 5.2 identifies the almost-sure limsup for signed inputs under the same full scalar-admissible width and norm criterion, without ergodicity. Corollary 5.3 makes the upper control uniform over macroscopic operator powers and lengths, including adaptive choices within its stated ranges. For composition operators these are physical moving starts. A signed endpoint obstruction in Section 6 has almost-sure convergence and shows only the failure of strong L^2 convergence under a mere uniform L^2 bound.

2 Markov averages and integrability

Lemma 2.1. *Under* (1)*, A_J converges strongly on L^1 to a positive unital integral-preserving contraction P, with $PU = UP = P$ and* $\mathrm{Range}(P) = \mathrm{Fix}(U)$*.*

Proof. The L^1 and L^∞ contraction properties noted above give a consistent L^2 contraction by interpolation. The Hilbert mean ergodic theorem gives convergence of $A_J h$ in L^2, hence L^1, for $h \in L^2$. For $h \in L^1$ choose $h_0 \in L^2$ close in L^1. Then

$$\|A_J h - A_I h\|_1 \le 2\|h - h_0\|_1 + \|A_J h_0 - A_I h_0\|_1.$$

Thus $(A_J h)$ is Cauchy. Its limit Ph defines a contraction. Positivity, unitality and integral preservation pass to the strong limit. The identity

$$A_J U - A_J = \frac{U^J - I}{J}$$

gives $PU = UP = P$. Hence the range is fixed, and $Uh = h$ implies $A_J h = h$ and $Ph = h$, proving the assertion. □

We use the standard de la Vallee-Poussin criterion: a nonnegative family $\mathcal{H} \subset L^1$ is uniformly integrable if and only if some nonnegative increasing convex Φ, with $\Phi(t)/t \to \infty$, satisfies $\sup_{h\in\mathcal{H}} \mathbb{E}\Phi(h) < \infty$.

Lemma 2.2. *If (X_n) is uniformly integrable, then all finite convex combinations of the functions $U^j X_n$, $j \geq 0$, form a uniformly integrable family. The same conclusion for the pth absolute powers holds if $(|X_n|^p)$ is uniformly integrable, for any finite $p \geq 1$.*

Proof. Choose Φ from the criterion and use the convex real function $\Psi(x) = \Phi(|x|)$, or $\Phi(|x|^p)$ in the second assertion. Markov Jensen and integral preservation give

$$\mathbb{E}\Psi(U^j X_n) \leq \mathbb{E}\Psi(X_n).$$

Ordinary Jensen for each convex combination preserves the same uniform bound. Jensen for unbounded inputs follows by expressing a continuous convex function as the supremum of countably many affine minorants, using positive unitality, and passing from bounded truncations by L^1 convergence and Fatou. All operators are the consistent extensions on their integrable domains; no kernel is used. □

Lemma 2.3 (Automatic positive-part integrability)**.** *Fix $1 \leq p < \infty$. Suppose $g_n \in L^p$ individually and the ordered inequality* (2) *holds eventually even just for $|n - m| \leq 1$. Then $(((g_n/n)_+)^p)$ is uniformly integrable. Hence uniform integrability of $|g_n/n|^p$ is equivalent to that of its negative-part pth powers.*

Proof. Fix one sufficiently large integer K. For $N \geq K$, split a consecutive interval of length N into floor/ceiling halves recursively to depth

$$d = \lfloor \log_2(N/K) \rfloor.$$

The leaves have integer lengths $k_i \in [K, 2K]$ and starting positions b_i. Every internal pair differs by at most one and has both children at least K. Positivity and repeated use of the inequalities give

$$X_N \leq Y_N := \sum_i \frac{k_i}{N} U^{b_i} X_{k_i}, \qquad X_n = \frac{g_n}{n}. \tag{6}$$

The weights are nonnegative and sum to one. The finite family $((X_k)_+)^p$, $K \leq k \leq 2K$, admits a de la Vallee-Poussin function Φ. Now $\Psi(x) = \Phi((x_+)^p)$ is convex and increasing. Thus

$$\mathbb{E}\Phi((X_N)_+^p) \leq \mathbb{E}\Psi(Y_N) \leq \max_{K\leq k\leq 2K} \mathbb{E}\Psi(X_k).$$

The bound is uniform for $N \geq K$. Adding finitely many earlier functions preserves uniform integrability. Finally $|X_n|^p = (X_n)_+^p + (X_n)_-^p$, proving the equivalence. □

3 Proof of conditional balanced convergence

We prove Theorem 1.1. Put

$$f_n = X_n - F, \quad a_n = \mathbb{E}X_n, \quad L = \mathbb{E}F, \quad \delta_K = \sup_{n \geq K} |a_n - L|.$$

Integral preservation and $PX_n \to F$ give $\delta_K \to 0$. The family (f_n) is uniformly integrable. Since $U^j F = F$, Lemma 2.2 also controls all changing Markov translates and finite convex combinations of these centered functions.

Finite-leaf Cesaro control

Use the exact halving decomposition (6), with K in the legal tail. Integral preservation yields

$$0 \leq \mathbb{E}(Y_N - X_N) \leq 2\delta_K.$$

The slack is nonnegative, so this is exactly $\|Y_N - X_N\|_1 \leq 2\delta_K$. Because $U^j F = F$ and A_J commutes with U,

$$\|A_J f_N\|_1 \leq \max_{K \leq k \leq 2K} \|A_J(X_k - F)\|_1 + 2\delta_K \qquad (N \geq K). \tag{7}$$

At fixed K, strong mean ergodicity sends that finite maximum to

$$\max_{K \leq k \leq 2K} \|PX_k - F\|_1 \leq \epsilon_K := \sup_{k \geq K} \|PX_k - F\|_1 \longrightarrow 0.$$

Only finitely many functions are considered before $J \to \infty$. There is no uniform mean-ergodic rate for an arbitrary changing sequence.

Maximal strictly convex energy

Let

$$\phi(t) = \sqrt{1 + t^2}.$$

It is strictly convex, 1-Lipschitz, at least one, and has $\phi''(t) = (1 + t^2)^{-3/2} > 0$. Uniform L^1 boundedness gives

$$V := \limsup_{N \to \infty} \mathbb{E}\phi(f_N) < \infty, \qquad V \geq 1.$$

Choose $N_j \to \infty$ realizing this limsup. For $t = 0, 1$ put

$$r_j = \lfloor N_j/2 \rfloor + t, \quad s_j = N_j - r_j, \quad \alpha_j = r_j/N_j, \quad \beta_j = s_j/N_j.$$

Both orders are legal for large j, and both weights tend to $1/2$. The expected slack in their normalized inequality tends to zero. Thus

$$\|f_{N_j} - Z_j\|_1 \to 0, \qquad Z_j = \alpha_j f_{r_j} + \beta_j U^{r_j} f_{s_j}. \tag{8}$$

The Lipschitz property implies $\mathbb{E}\phi(Z_j) \to V$. Markov Jensen gives $\mathbb{E}\phi(U^{r_j} f_{s_j}) \leq \mathbb{E}\phi(f_{s_j})$. Both child energies have limsup at most V, so the nonnegative strict Jensen deficit

$$D_j = \alpha_j \phi(f_{r_j}) + \beta_j \phi(U^{r_j} f_{s_j}) - \phi(Z_j)$$

satisfies $\mathbb{E}D_j \to 0$.

For fixed $M, \eta > 0$, strict convexity gives a positive minimum of

$$\alpha\phi(x) + (1-\alpha)\phi(y) - \phi(\alpha x + (1-\alpha)y)$$

on $|x|, |y| \le M$, $|x-y| \ge \eta$, $\alpha \in [1/3, 2/3]$. Compactness and $\phi'' > 0$ justify positivity. Markov's inequality bounds the probability of this bounded-range separation event by $\mathbb{E}D_j$ divided by that minimum. Uniform L^1 bounds control the probabilities of the omitted tails. Letting j and then M tend to infinity gives

$$f_{r_j} - U^{r_j} f_{s_j} \longrightarrow 0 \quad \text{in probability.}$$

These child families, including their changing powers, are uniformly integrable by Lemma 2.2. Their differences are uniformly integrable, so convergence in probability upgrades to convergence in L^1.

Equation (8) now makes the parent close to both f_{r_j} and $U^{r_j} f_{s_j}$. Apply the same argument to the reversed ordered split to make it close to f_{s_j} as well. Contraction yields

$$\|U^{r_j} f_{N_j} - f_{N_j}\|_1 \to 0.$$

The two choices $t = 0, 1$ provide consecutive exponents r and $r+1$. The inequality

$$\|Uf - f\|_1 \le \|f - U^r f\|_1 + \|U^{r+1} f - f\|_1$$

follows from contraction and the triangle inequality. Therefore $\|Uf_{N_j} - f_{N_j}\|_1 \to 0$. The difference-two pair is used when N_j is even.

Remove the convex-energy excess

For fixed J, telescoping gives

$$\|A_J f_{N_j} - f_{N_j}\|_1 \le \frac{J-1}{2} \|Uf_{N_j} - f_{N_j}\|_1 \to 0.$$

Take $j \to \infty$ in (7), then $J \to \infty$ with K fixed:

$$\limsup_j \|f_{N_j}\|_1 \le \epsilon_K + 2\delta_K.$$

Let $K \to \infty$. The parent norms tend to zero, so their energies tend to one by the Lipschitz property. Hence $V = 1$ and $\mathbb{E}\phi(f_N) \to 1$ on the full sequence. For $\eta > 0$ put

$$c_\eta = \frac{\eta}{\sqrt{1+\eta^2} + 1} > 0.$$

Then $\phi(t) - 1 \ge c_\eta |t|$ on $|t| \ge \eta$, and

$$\mathbb{E}|f_N| \le \eta + \frac{\mathbb{E}\phi(f_N) - 1}{c_\eta}.$$

Let $N \to \infty$ and then $\eta \downarrow 0$. This proves $X_N \to F$ in L^1 and Theorem 1.1.

4 The scalar interface and endpoint norms

We record exactly the scalar input used from [7, Theorem on removing monotonicity].

Lemma 4.1 (Existing scalar corridor theorem)**.** *Let $w : [T, \infty) \to [0, 1]$ be measurable and nonincreasing, with divergent square integral and eventual $nw(\log n) \geq 1$. If a real sequence (b_n) satisfies*

$$b_{n+m} \leq b_n + b_m \quad \text{eventually for } n \leq m \leq n + nw(\log n),$$

then b_n/n converges in $\mathbb{R} \cup \{-\infty\}$. No monotonicity, sign or growth hypothesis on b_n is imposed.

This is a cited theorem, not an additional result of this paper. Its proof uses a corridor-preserving prefix envelope, a distributional convexity argument and dense record indices. The published scalar reference also constructs increasing exponentially bounded integer counterexamples when the square integral is finite.

Proof of Theorem 1.2. Assume (5). Lemma 2.3 makes $(|X_n|^p)$ uniformly integrable. In particular (X_n) is uniformly integrable. Apply P to the corridor inequalities. Positivity and $PU^n = P$ give

$$Pg_{n+m} \leq Pg_n + Pg_m.$$

On a common full-measure set, Lemma 4.1 gives $PX_n \to F$ pointwise in $\mathbb{R} \cup \{-\infty\}$. Lemma 2.2, applied to the Markov projection P, makes (PX_n) uniformly integrable, with uniformly bounded L^1 norms. Fatou excludes an infinite limit on any positive-measure set. Vitali gives $PX_n \to F$ in L^1. The range of P is closed, hence $UF = F$.

The width condition eventually includes all ordered differences at most two. Theorem 1.1 therefore gives $X_n \to F$ in L^1, hence in probability. Uniform integrability of $|X_n|^p$, Fatou and Vitali now give $F \in L^p$ and $X_n \to F$ in L^p. The invariant equation holds in the consistent L^p realization. Conversely, any L^p-norm-convergent family has uniformly integrable pth absolute powers and hence negative-part pth powers. This proves the equivalence. For nonnegative inputs the negative parts vanish. □

If $\int w^2 < \infty$, take the scalar integer counterexample u_n from [7] and put $g_n = \log u_n$ on the one-point probability space, with $U = I$. Its normalized values are bounded and nonnegative, so every finite-p integrability condition holds. Scalar commutativity gives both ordered corridor inequalities, but two distinct subsequential slopes preclude norm convergence. This proves the inherited converse for the universal width criterion.

5 A pointwise upper law from adjacent halving

Let U and P be the Markov operator and its mean projection from (1). No ergodicity, invertibility or transition kernel is assumed.

Theorem 5.1 (Conditional pointwise upper control)**.** *Let $g_n \in L^1$ be real. Suppose $g_{n+m} \leq g_n + U^n g_m$ for every sufficiently late ordered pair with $|n - m| \leq 2$. Assume explicitly that*

$$Pg_n/n \longrightarrow F \quad \text{almost surely}, \qquad F \in \mathrm{Fix}(U), \quad |F| < \infty \text{ almost surely.}$$

Then $\limsup_n g_n/n \leq F$ almost surely. On one full-measure set, for every fixed finite $B \geq 0$,

$$\max_{0 \leq s \leq Bn} \left(\frac{U^s g_n(x)}{n} - F(x) \right)_+ \longrightarrow 0, \tag{9}$$

where maxima are over integers. If also $g_n/n \to F$ in L^1, then $\limsup_n g_n/n = F$ almost surely.

The pointwise projection hypothesis is separate from norm convergence; it will follow from the existing scalar theorem under the full width hypotheses below. Signed integrable rewards are allowed directly.

Corollary 5.2 (All scalar-admissible widths)**.** *Under the hypotheses of Theorem 1.2, including its negative-part integrability criterion, let F be its invariant norm limit. Then $\limsup_n g_n/n = F$ almost surely, and (9) holds. In particular this applies to nonnegative individual L^p inputs for every finite $p \geq 1$. The width need not be the critical example $t^{-1/2}$.*

Proof. The scalar-interface proof of Theorem 1.2 already gives $Pg_n/n \to F$ pointwise, as well as norm convergence. Its eventual width condition includes every ordered difference at most two. Apply Theorem 5.1. □

Corollary 5.3 (Macroscopic rectangles)**.** *In the norm-convergent setting of Theorem 5.1, on the same full-measure set, for every fixed $0 < c \leq B < \infty$,*

$$\max_{\substack{cN \leq m \leq BN \\ 0 \leq s \leq BN}} \left(\frac{U^s g_m(x)}{m} - F(x) \right)_+ \longrightarrow 0.$$

Empty maxima are taken as zero. Lengths and powers may be selected adaptively inside these ranges. For $Uh = h \circ T$, the powers are physical moving starts, with no assumption that T is ergodic. For a general operator they are not unaveraged stochastic sample paths.

Proof. Choose an integer $D \geq B/c$. In the rectangle, $s \leq Dm$ and $m \geq cN \to \infty$. The tail supremum of the convergent sequence of maxima in (9) tends to zero. Integer bands dominate all finite real bands, so no uncountable intersection is needed. □

The proof uses a random tree with *every* merger legal, replacing arbitrary block concatenation in classical interval arguments such as [1]. Independent unit-lattice noise makes the occupation kernel's spatial gradient vanish. Summation by parts, bounded coboundary density and the classical Dunford–Schwartz maximal inequality [2] then admit all integrable rewards. Flat occupation asymptotics or a local-limit theorem are unnecessary.

5.1 An all-legal random tree and its occupation gradient

Lemma 5.4. *Fix an integer $K \geq 8$. For $N \geq K$ put*

$$d = 2 \left\lfloor \tfrac{1}{2} \log_2(N/K) \right\rfloor, \qquad \mu = N/2^d \in [K, 4K).$$

Split each even node $2m$ into $m-1, m+1$, and each odd node $2m+1$ into $m, m+1$, with the two orientations equally likely and independent at different nodes. Continue to depth d. Let $W_{N,l}(s)$ be the expected number of leaves of length l starting at site s, extended by zero off $[0, N)$. All leaf lengths lie in $[K-2, 4K+2]$, and

$$0 \leq W_{N,l}(s) \leq 1, \qquad \sum_l l \sum_s W_{N,l}(s) = N.$$

At fixed K, along all integer $N \to \infty$,

$$\gamma_K(N) := \frac{1}{N} \sum_l \sum_{s \in \mathbb{Z}} |W_{N,l}(s) - W_{N,l}(s-1)| \longrightarrow 0. \tag{10}$$

Proof. Along a path, the difference from the ideal halving length satisfies $|e_{h+1}| \le |e_h|/2+1$, starting at zero, so $|e_h| < 2$. All node children are therefore positive and in a fixed legal tail when K is large enough. Each realized tree partitions its interval, proving the start-count and physical-mass statements.

For the limiting assertion we may assume $d \ge 4$. Choose a uniform leaf. Write $n_0 = N, n_1, \dots, n_d = L$ for its path lengths and B_h for its right-child indicators. A fresh orientation and a fair branch choice make the selected child length independent of the direction bit. Recursively, the length chain has its two possible transitions each with probability $1/2$, independently of all the independent fair bits B_h. Its start is

$$S = \sum_{h=1}^{d} B_h(n_{h-1} - n_h) = B_1 N + \sum_{h=1}^{d-1} (B_{h+1} - B_h) n_h - B_d L. \tag{11}$$

Consequently $W_{N,l}(s) = 2^d \Pr(S = s, L = l)$.

Condition on every even-level length and every bit. The odd-level lengths are independent two-step Markov bridges; L is fixed. For an odd parent $2m+1$, its intermediate lengths $m, m+1$ have one common descendant: if $m = 2r$, their child sets are $\{r-1, r+1\}$ and $\{r, r+1\}$, with common value $r+1$; if $m = 2r+1$, the sets are $\{r, r+1\}$ and $\{r, r+2\}$, with common value r. The two paths to that endpoint each have probability $1/4$. Conditional on it, the intermediate length is a fair choice of consecutive integers. If its coefficient in (11) is nonzero, it supplies a signed fair unit Bernoulli summand.

Use the second two steps of each disjoint four-level block. Call the block good when n_{4r+2} is odd, n_{4r+4} is its common descendant, and $B_{4r+3} \ne B_{4r+4}$. Let q count these blocks. Goodness is determined by the conditioned even lengths and bits, so the q corresponding fair bridge variables remain independent under that conditioning, and independent of the remaining odd variables.

To bound q, now expose blocks in chronological order, rather than conditioning on future endpoints. From any current sufficiently large integer, two transitions give an odd value with probability at least $1/4$: for an even current value it is $1/2$ in either residue class modulo four; for an odd value its odd child alone contributes $1/4$. The common endpoint then has probability $1/2$, and the two fresh bits differ with probability $1/2$. Thus each block is good with conditional probability at least $1/16$ given the past. For $t > 0$,

$$\mathbb{E}e^{-tq} \le \left(1 - \frac{1 - e^{-t}}{16}\right)^{\lfloor d/4 \rfloor}.$$

This proves $q \to \infty$ in probability uniformly as $d \to \infty$. It is not a lower bound conditional on all future even endpoints.

Given the even states and bits, S is a sum of q independent signed fair Bernoulli variables plus an independent remainder. Sign reversal only translates a fair Bernoulli. The zero-extended unit-shift ℓ^1 difference of a binomial law is twice its largest atom, by unimodality, and is at most $C_0/\sqrt{q+1}$. For example its largest atom is bounded by the Fourier integral of $|\cos(t/2)|^q$, which has this bound. Convolution cannot increase that difference. Since L is fixed, convexity upon removing the conditioning gives the joint estimate

$$\sum_l \sum_s |\Pr(S = s, L = l) - \Pr(S = s-1, L = l)| \le C_0 \mathbb{E}(q+1)^{-1/2} \longrightarrow 0.$$

The expectation tends to zero by boundedness and the probability bound. Multiply by $2^d/N = 1/\mu \le 1/K$ to obtain (10). No divisibility assumption on N was used. □

5.2 Weighted pointwise convergence and the upper bound

For fixed K, l define

$$A_{N,l}h = \frac{1}{N}\sum_s W_{N,l}(s)U^s h, \qquad \theta_{N,l} = \frac{1}{N}\sum_s W_{N,l}(s).$$

Lemma 5.5. *For every $h \in L^1$, $A_{N,l}h - \theta_{N,l}Ph \to 0$ almost surely. For every fixed finite B, this convergence also holds uniformly after applying U^s, $0 \le s \le BN$, with invariant target $\theta_{N,l}Ph$.*

Proof. For a bounded coboundary $h = (I-U)v$, summation by parts and zero extension give

$$\|A_{N,l}h\|_\infty \le \|v\|_\infty \frac{1}{N}\sum_s |W_{N,l}(s) - W_{N,l}(s-1)| \longrightarrow 0.$$

The endpoint terms use only nonnegative powers of U. Applying U^s preserves the same bound by its L^∞ contraction. Strong mean ergodicity gives $\ker P = \overline{\operatorname{ran}(I-U)}$ in L^1. Bounded functions are dense, so bounded coboundaries are dense in $\ker P$. For an approximation error $e \in L^1$, positivity and $W \le 1$ imply

$$|U^s A_{N,l}e| \le \frac{1}{N}\sum_{r=0}^{N-1} U^{s+r}|e| \le \frac{1}{N}\sum_{r=0}^{\lfloor (B+1)N \rfloor} U^r|e| \quad (0 \le s \le BN).$$

The last expression is bounded by $(B+2)$ times the ordinary positive Dunford–Schwartz maximal average of $|e|$. Its weak $(1,1)$ inequality [2, Theorem11.11] bounds the exceptional measure at tolerance η by $(B+2)\|e\|_1/\eta$. Approximate an element of $\ker P$ arbitrarily closely by bounded coboundaries and let that error norm tend to zero. This is the usual Banach closure argument [2, Proposition11.9]; it proves the uniform zero limit on $\ker P$. Finally $U^s A_{N,l}Ph = \theta_{N,l}Ph$ exactly. No convergence of $\theta_{N,l}$ or rate for a changing observable is assumed. □

Proof of Theorem 5.1. Put the countably many inequalities and their integer Markov-power images on one full-measure set. Choose K large enough that every node's children in Lemma 5.4 exceed the legal cutoff. Repeated ordered subadditivity along each tree and positivity give

$$g_N/N \le \sum_l A_{N,l} g_l.$$

This is valid for signed rewards: no illegal node is dropped and no extra positive reward is inserted. Apply U^s and Lemma 5.5 to the fixed finite integrable family g_l, $K-2 \le l \le 4K+2$. At fixed K and integer B the errors tend uniformly to zero over $0 \le s \le BN$. Since $\sum_l l\theta_{N,l} = 1$, the projected term is a convex combination of $Pg_l(x)/l$. Hence

$$\limsup_N \max_{0 \le s \le BN} \frac{U^s g_N(x)}{N} \le \sup_{l \ge K-2} \frac{Pg_l(x)}{l}.$$

Intersect the full-measure sets for countably many K, B and use the explicit pointwise projected limit. Letting $K \to \infty$ after the N limsup proves (9); $s = 0$ gives the upper bound. If norm convergence holds, choose a deterministic subsequence converging almost surely to F, which gives the reverse limsup inequality. This proves equality of the limsup, not the lower limit. □

6 A signed integrability obstruction

A uniform bound on $\|g_n/n\|_2$ cannot replace uniform integrability of its negative-part squares. The following explicit spike/activation construction follows the classical obstruction mechanism in [3, Example 6.1]; we provide all details on an ergodic Bernoulli probability space.

For $k \geq 1$ let

$$n_k = 2^{2^k}, \quad e_k = 2^{-k}, \quad p_k = e_k^2/n_k, \quad B_k = n_k/e_k.$$

Let $\omega_{i,k}$, $i \in \mathbb{Z}$, $k \geq 1$, be mutually independent Bernoulli variables with probabilities p_k. The row shift is an ergodic measure-preserving transformation; let U be its Koopman operator. Put

$$h_k = B_k \omega_{0,k}, \qquad H_k = \sum_{j \leq k} h_j, \qquad H_0 = 0.$$

Thus $\mathbb{E}h_k = e_k$ and $\mathrm{Var}(h_k) = n_k - e_k^2$. Define $k(n) = \max\{k : n_k \leq n\}$, taking zero before $n_1 = 4$, and

$$F_n = \sum_{i=0}^{n-1} U^i H_{k(n)}, \qquad g_n = -F_n, \qquad X_n = g_n/n.$$

Every g_n is finite and belongs to L^2. Monotonicity of H_k gives

$$F_{n+m} \geq F_n + U^n F_m,$$

so g_n is subadditive for *all* ordered pairs.

For $k = k(n) \geq 1$, independence gives

$$\mathbb{E}X_n = -(1 - 2^{-k}), \qquad \mathrm{Var}(X_n) = \frac{1}{n} \sum_{j \leq k} (n_j - 4^{-j}).$$

Since $n_{k+1} = n_k^2$ and $n_k \geq 4$, induction gives $\sum_{j \leq k} n_j \leq 2n_k$. Consequently $\|X_n\|_2^2 \leq 3$. At $n = n_k$ the variance tends to one: its current term is $1 - 4^{-k}/n_k$ and all previous terms total at most $2/\sqrt{n_k}$. At $m_k = n_{k+1} - 1$ it is at most $2n_k/(n_k^2 - 1) \to 0$. The means tend to -1 along both subsequences. Hence

$$\|X_{n_k} + 1\|_2^2 \to 1, \qquad \|X_{m_k} + 1\|_2^2 \to 0.$$

There is no strong L^2 limit.

Nevertheless $X_n \to -1$ almost surely. The monotone sum $H_\infty = \sum_k h_k$ is integrable with expectation one. For fixed K and all large n,

$$\sum_{i<n} U^i H_K \leq F_n \leq \sum_{i<n} U^i H_\infty.$$

Birkhoff's theorem on the common full-measure set for these countably many functions gives lower limit at least $1 - 2^{-K}$ and upper limit at most one. Let $K \to \infty$. Thus this is an integrability obstruction, not an almost-sure counterexample to our open restricted-pair question. Its normalized negative-part squares fail uniform integrability, as the exact norm calculation requires.

7 Comparison and further questions

Theorems 1.1 and 1.2 restrict the allowed gluing pairs, rather than improving the classical mean-ergodic conclusion under full subadditivity. The distinction persists even in a bounded normalized scalar example. Set

$$g_n = n + \sqrt{n}(100 + (-1)^n).$$

If $\min(n,m) \le \max(n,m) \le 2\min(n,m)$ and $N = n+m$, the child corrections total at least $198\sqrt{N/3}$ while the parent correction is at most $101\sqrt{N}$. The strict inequality follows from $198^2 > 3 \cdot 101^2$. Thus near-half subadditivity holds and $0 \le g_n \le 102n$. For every fixed odd k and odd n,

$$g_{n+k} - g_n - g_k \ge 2\sqrt{n} - 101\sqrt{k} \longrightarrow \infty.$$

A uniform-in-n expected positive-defect bound at each fixed k, of the type in [4], therefore does not follow from our restricted hypotheses. This is a hypothesis comparison, not an exclusion of every possible reduction to other existing results.

The present proof deliberately uses both ordered splits and differences two. It leaves the adjacent-only conditional problem and widths with missing such pairs untreated. Finite-leaf control and strict convex energy extend the cited scalar results to the Markov setting. Norm convergence alone does not give a pathwise conclusion. The adjacent-halving argument in Section 5, combined with classical coboundary density and the Dunford–Schwartz maximal inequality, supplies a pointwise upper law under the same full scalar-admissible width and norm criterion, including signed integrable rewards. The conditional form requires pointwise convergence of projected slopes explicitly; it does not infer this from norm convergence alone.

Use of AI tools. OpenAI ChatGPT/Codex assisted with discovery, proof exploration, drafting and checking. Separate AI-assisted internal proof checks were performed.

References

[1] J. M. Steele, *Kingman's subadditive ergodic theorem*, Ann. Inst. H. Poincaré Probab. Statist. **25** (1989), 93–98. Numdam.

[2] T. Eisner, B. Farkas, M. Haase and R. Nagel, *Operator Theoretic Aspects of Ergodic Theory*, Graduate Texts in Mathematics **272**, Springer, 2015, Theorems 8.24 and 11.11, Proposition 11.9. Author manuscript.

[3] Y. Derriennic and U. Krengel, *Subadditive mean ergodic theorems*, Ergodic Theory Dynam. Systems **1** (1981), 33–48. doi:10.1017/S0143385700001140.

[4] Y. Derriennic, *Un théorème ergodique presque sous-additif*, Ann. Probab. **11** (1983), 669–677. doi:10.1214/aop/1176993511.

[5] S. Filip, *Notes on the multiplicative ergodic theorem*, Ergodic Theory Dynam. Systems **39** (2019), 1153–1189. doi:10.1017/etds.2017.68. Mean-Kingman proof: author notes, Section 5.4.

[6] R. Raquépas, *A gapped generalization of Kingman's subadditive ergodic theorem*, J. Math. Phys. **64** (2023), 062702. doi:10.1063/5.0142431; arXiv:2211.13134.

[7] S. Schubr, *Oscillating sequences with subadditivity on shrinking corridors*, author publication, edition 1 (2026). Public edition 1.