Mean distance and the spectral gap of unicyclic metric graphs

Scott Schubr
Luminary Oddities

November 26, 2026

Abstract

Baptista, Kennedy and Mugnolo (J. Geom. Anal., 2024) bounded the product of the spectral gap μ_2 of the standard Kirchhoff Laplacian and the squared mean distance ρ^2 by $\pi^2(1+\beta)^2$ on compact metric graphs of first Betti number β, and the sharp value on trees is $\pi^2/4$ (Schubr, 2026). For graphs with exactly one cycle their bound gives $4\pi^2$, while the circle gives $\pi^2/4$. We show that every finite connected unicyclic metric graph satisfies $\mu_2\rho^2 \le 25\pi^2/64$, and that the circle value $\pi^2/4$ is the optimal constant in four classes: all trees attached at one point of the cycle; an isometry rotating the cycle nontrivially; total length off the cycle at most $3/14$ of the cycle length; and trees at each attachment point of combined length at most $1/7$ of the cycle length, with arbitrary total length and positions. In the last two classes the circle is the only equality case. The proofs combine a rooted phase inequality for trees, the Fourier series of circular distance and covariance bounds for centered trigonometric test functions. Below the Dirichlet frequencies of the branches, the spectral problem reduces exactly to a quadratic form on the cycle with point interactions given by the branch admittances, and a sharp lower bound for each admittance in terms of branch length and mean height yields the rotational case.

Keywords. Metric graph, quantum graph, unicyclic graph, spectral gap, mean distance, Kirchhoff Laplacian.
MSC 2020. Primary 34B45. Secondary 35P15, 05C12.

1 Introduction

Let $\mathcal{G}$ be a finite connected compact metric graph of total length L, with shortest-path distance d. Its mean distance is

$$\rho = \rho(\mathcal{G}) = \frac{1}{L^2}\int_{\mathcal{G}}\int_{\mathcal{G}} d(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

as in [1, Definition 2.1], and $\mu_2 = \mu_2(\mathcal{G})$ denotes the first positive eigenvalue of the Laplacian with continuity and standard Kirchhoff conditions at the vertices, Neumann conditions at the leaves. It is given by

$$\mu_2 = \min\Big\{\frac{\int_{\mathcal{G}}|f'|^2}{\int_{\mathcal{G}} f^2} :\ 0\neq f\in H^1(\mathcal{G}),\ \int_{\mathcal{G}} f = 0\Big\}, \tag{1.1}$$

where $H^1(\mathcal{G})$ consists of the continuous functions that are H^1 on every edge [2]. The product $\mu_2\rho^2$ is invariant under dilation. The graph is *unicyclic* if its first Betti number is one, that is, if it contains exactly one cycle.

The comparison gap. For a compact metric graph with E edges and first Betti number β, Baptista, Kennedy and Mugnolo proved

$$\mu_2\rho^2 \le \min\Big\{\frac{\pi^2E^2}{9},\ \pi^2(1+\beta)^2\Big\} \tag{1.2}$$

together with a lower bound [1, Corollary 3.3]; the edge term uses [3, Theorem 4.2], which assumes $E \ge 2$. They remark in their introduction that these bounds "in general do not appear to be optimal". Some dependence on the topology is necessary, since Mugnolo's unilateral expander graphs, all of whose edges have length one, show that no universal constant bounds $\mu_2\rho^2$ from above [5, Proposition 3.9]. For trees ($\beta = 0$) the optimal constant is $\pi^2/4$, the limit of equilateral stars observed in [1, Section 3], approached but not attained [6, Theorem 2]. A circle of length C has $\mu_2 = 4\pi^2/C^2$ and $\rho = C/4$, so it also has $\mu_2\rho^2 = \pi^2/4$. For one cycle, however, the topological term of (1.2) gives $4\pi^2$. Comparing the constants $\pi^2/4$ at $\beta = 0$ and $4\pi^2$ at $\beta = 1$ identifies the question studied here:

> *What is the smallest constant that bounds $\mu_2\rho^2$ on all unicyclic metric graphs, and in which classes is it the circle value $\pi^2/4$?*

This article determines the optimal constant as $\pi^2/4$ in four classes of unicyclic graphs and improves the constant for the whole class from $4\pi^2$ to $25\pi^2/64$.

Results. Write $\mathcal{C}$ for the cycle of a unicyclic graph $\mathcal{G}$, C for its length, $M = L - C$ for the length off the cycle, and $M_{\max}$ for the largest total length of the trees attached at a single point of $\mathcal{C}$ ($M_{\max} = 0$ for a circle).

Theorem 1.1. *Every finite connected unicyclic metric graph $\mathcal{G}$ satisfies*

$$\mu_2\rho^2 \le \frac{25\pi^2}{64}.$$

Moreover $\mu_2\rho^2 \le \pi^2/4$ in each of the following cases:

(1) all trees of $\mathcal{G}$ are attached at one point of $\mathcal{C}$;

(2) $M \le 3C/14$;

(3) $M_{\max} \le C/7$.

In cases (2) and (3) equality holds only for the circle.

Theorem 1.2. *If an isometry of a unicyclic metric graph $\mathcal{G}$ induces a nonidentity rotation of its cycle, then $\mu_2\rho^2 \le \pi^2/4$.*

The circle lies in each of the four classes and attains $\pi^2/4$, so $\pi^2/4$ is the optimal constant in each class. In case (3) the number and the positions of the attachment points are arbitrary, so M/C can be arbitrarily large. Theorem 1.1 improves the case $\beta = 1$ of [1, Corollary 3.3] from $4\pi^2$ to $25\pi^2/64$; since $25/64 < 4/9 \le E^2/9$ for $E \ge 2$, the new constant is also below the edge term of (1.2) for every unicyclic graph with at least two edges. In the four classes the theorems extend the tree theorem [6, Theorem 2] to one cycle with the same constant, which for one cycle is attained. Combined with the universal lower bound $\mu_2 L\rho > 2$ of [6, Theorem 1], Theorem 1.1 gives a two-sided estimate of the spectral gap by the mean distance and the total length.

Corollary 1.3. *Every finite connected unicyclic metric graph of total length L satisfies*

$$\frac{2}{L\rho} < \mu_2 \le \frac{25\pi^2}{64\,\rho^2},$$

and in each of the four classes of Theorems 1.1 and 1.2 the right side improves to $\pi^2/(4\rho^2)$.

Route. Section 2 fixes notation and collects the two inherited tools: the phase inequalities for positive solutions on rooted trees (Lemma 2.1, from [6, Lemma 1 and (5)]) and the rooted Dirichlet bounds (Lemma 2.2, from [6, (3)] and [7, Theorem 1.1, Remark 5.1]). Section 3 proves the bound $25\pi^2/64$. A real gap eigenfunction either has a zero that separates its signs, and then a rooted Dirichlet bound applies, or it has none, and then every attached tree has rooted Dirichlet eigenvalue at least μ_2 (Lemma 3.2); in the second case the Fourier series of circular distance and centered trigonometric test functions finish the proof. Sections 4 and 5 use the first two circular moments of the attached length and the within-tree distance bound [1, Theorem 3.2] to prove cases (2) and (3). Section 6 proves case (1): folding the cycle at the point opposite the attachment point reduces a separating zero to the tree bound of [6, (3)], and otherwise an explicit secular equation on the cycle controls the phase. Section 7 eliminates the attached trees exactly through their admittances (Proposition 7.2), proves a sharp lower bound for each admittance (Lemma 7.4) and deduces Theorem 1.2. Section 8 assembles the proofs of Theorem 1.1 and Corollary 1.3.

2 Notation and inherited tools

This section records the structure of a unicyclic graph, the distance decomposition used throughout, and two results of earlier articles in the form needed here.

Unicyclic graphs. Let $\mathcal{G}$ be unicyclic. It contains a unique cycle $\mathcal{C}$, a closed curve of length $C > 0$ (possibly a single loop edge), and the closure of $\mathcal{G} \setminus \mathcal{C}$ is a finite union of trees, each meeting $\mathcal{C}$ in exactly one point. Let $v_1, \dots, v_n$ ($n \geq 0$) be the distinct points of $\mathcal{C}$ at which trees are attached, and let T_i be the union of the trees attached at v_i, a finite tree rooted at v_i of length $M_i > 0$. Put

$$M = \sum_i M_i, \qquad L = C + M, \qquad s = \frac{C}{L}, \qquad b = \frac{M}{L} = 1 - s, \qquad M_{\max} = \max_i M_i.$$

We identify $\mathcal{C}$ with $\mathbb{R}/C\mathbb{Z}$ by an arclength coordinate t, write t_i for the coordinate of v_i and $d_{\mathcal{C}}$ for arclength distance on $\mathcal{C}$. The projection $\mathrm{pr} : \mathcal{G} \to \mathcal{C}$ is the identity on $\mathcal{C}$ and maps T_i to v_i; the height is $\eta(x) = d(x, \mathrm{pr}\, x)$, and

$$h_i = \frac{1}{M_i} \int_{T_i} \eta, \qquad \bar{h} = \frac{1}{M} \int_{\mathcal{G}\setminus\mathcal{C}} \eta = \sum_i \frac{M_i}{M} h_i$$

are the mean heights of T_i and of all points off the cycle.

A path between two points of $\mathcal{C}$ that enters some T_i must leave it through v_i, and every path from T_i to the rest of $\mathcal{G}$ passes through v_i. Hence

$$d(x, y) = \eta(x) + d_{\mathcal{C}}(\mathrm{pr}\, x, \mathrm{pr}\, y) + \eta(y) \quad \text{unless } x, y \text{ lie in the same } T_i, \tag{2.1}$$

and $d(x, y) \leq \eta(x) + \eta(y) + d_{\mathcal{C}}(\mathrm{pr}\, x, \mathrm{pr}\, y)$ for all x, y.

The pushforward of normalized arclength under pr is the probability measure $\nu = L^{-1}\,\mathrm{d}t + \sum_i (M_i/L)\delta_{t_i}$ on $\mathcal{C}$. Its circular moments and the projected mean distance are

$$m_j = \int_{\mathcal{C}} e^{2\pi \mathrm{i} j t/C}\,\mathrm{d}\nu(t) = \sum_i \frac{M_i}{L} e^{2\pi \mathrm{i} j t_i/C} \quad (j \geq 1), \qquad \rho_{\mathcal{C}} = \int_{\mathcal{C}} \int_{\mathcal{C}} d_{\mathcal{C}}(t, t')\,\mathrm{d}\nu(t)\,\mathrm{d}\nu(t').$$

Only the atoms contribute to m_j, so $\sigma = |m_1|^2$ satisfies $0 \leq \sigma \leq b^2$.

Spectral notation. For a point $o \in \mathcal{G}$, subdivided into a vertex if necessary, let $\lambda_D(\mathcal{G}, o)$ be the first eigenvalue with a Dirichlet condition at o and the standard conditions elsewhere, the minimum of $\int |f'|^2 / \int f^2$ over nonzero $f \in H^1(\mathcal{G})$ with $f(o) = 0$, and let $\rho_o = L^{-1} \int_{\mathcal{G}} d(o, x)\, \mathrm{d}x$ be the mean distance to o. Functions vanishing at o restrict independently to the closures of the components of $\mathcal{G} \setminus \{o\}$, so $\lambda_D(\mathcal{G}, o)$ is the smallest of the corresponding eigenvalues of these closures; on each of them the ground state is simple and can be chosen positive away from o [2]. For a rooted tree (T, v) the same definitions give $\lambda_D(T, v)$, and the mean distance to the root of T_i is h_i. The eigenvalues of $\mathcal{G}$ are $0 = \mu_1 < \mu_2 \le \mu_3 \le \cdots$, and by the min-max principle the number of them below κ^2, counted with multiplicity, is the largest dimension of a subspace of $H^1(\mathcal{G})$ on which $\int_{\mathcal{G}} (|f'|^2 - \kappa^2 f^2)$ is negative definite [2].

Inherited tools. The first tool is the phase induction of [6, Section 4]. There it is applied to the Dirichlet ground state of a tree; its proof uses only the equation on the edges, the vertex conditions and positivity, and so it applies to every positive solution with positive value at the root.

Lemma 2.1 (Phase inequalities)**.** *Let (T, o) be a finite rooted metric tree, $\kappa > 0$, and let $u \in H^1(T)$ be positive on T, solve $-u'' = \kappa^2 u$ on every edge and satisfy the Kirchhoff condition at every vertex other than o, with Neumann conditions at the leaves. Let $u'(o)$ be the sum of the derivatives of u at o into the edges of T, and put*

$$B = \kappa |T|, \qquad J = \kappa^2 \int_T d(o, x)\, \mathrm{d}x, \qquad \theta = \arctan \frac{u'(o)}{\kappa u(o)}.$$

Then $u'(x) = \kappa^2 \int_{T_x} u > 0$ at every point $x \neq o$ that is not a vertex, where T_x is the subtree of points whose path to o passes through x and the derivative is taken away from o. Moreover $0 < \theta < \pi/2$ and

$$B \ge \theta, \qquad J \le \Big(\theta - \frac{\pi}{4}\Big) B + \frac{\pi\theta}{4} - \frac{\theta^2}{2}, \qquad J \le \frac{\theta B}{2}. \tag{2.2}$$

Proof. Rescale lengths by κ, so that $-u'' = u$, $B = |T|$ and J is the first moment of distance to o. The integration argument in the proof of [6, Lemma 1] uses only the edge equation and the Kirchhoff and Neumann conditions, and gives $u'(x) = \int_{T_x} u$, which is positive because $u > 0$. Summing over the edges at o gives $u'(o) > 0$, so $\theta \in (0, \pi/2)$. As in [6, Lemma 1], the phase decreases at unit speed along edges oriented away from o, vanishes at the leaves and satisfies the tangent sum at vertices. The proof of [6, inequalities (5)], an induction from the leaves through stem extensions and splits, uses only these properties and the bound $\theta < \pi/2$, so it applies verbatim and gives the first two inequalities in (2.2) for T rooted at o; when o has several edges, T is the split at o, whose phase is θ by the tangent sum. For the third inequality,

$$\frac{\theta B}{2} - \Big[\Big(\theta - \frac{\pi}{4}\Big) B + \frac{\pi\theta}{4} - \frac{\theta^2}{2}\Big] = \Big(\frac{\pi}{4} - \frac{\theta}{2}\Big)(B - \theta) \ge 0,$$

because $\theta < \pi/2$ and $B \ge \theta$. □

Lemma 2.2 (Rooted Dirichlet bounds)**.** *The following bounds hold.*

(1) For every finite rooted metric tree (T, o), with $\rho_o = |T|^{-1} \int_T d(o, x)\, \mathrm{d}x$,

$$\sqrt{\lambda_D(T, o)}\, \rho_o \le \frac{\pi}{4}.$$

(2) For every finite connected compact metric graph $\mathcal{G}$ and every point $o \in \mathcal{G}$,

$$\lambda_D(\mathcal{G}, o)\, \rho_o^2 \le \frac{25}{36}.$$

Proof. Part (1) is [6, inequality (3)]. Part (2) is the case of the Dirichlet set $\{o\}$, after subdividing an edge at o if necessary, of the explicit trial-function bound $C_* \le 25/36$ in [7, Remark 5.1], where C_* is the supremum of the first Dirichlet eigenvalue times the squared mean distance to the Dirichlet set. The sharp form is $\lambda_D(\mathcal{G}, o)\rho_o^2 < C_* \le 0.66168$ [7, Theorem 1.1(2),(3)]; the analytic constant $25/36$ suffices here. □

3 Separating zeros and the general bound

A real eigenfunction for μ_2 either has a zero separating its signs, which makes the graph a Dirichlet problem at that zero, or every attached tree has rooted Dirichlet eigenvalue at least μ_2. In the second case the mean heights are controlled by Lemma 2.2(1) and the remaining work is on the cycle, where Fourier analysis is exact.

Definition 3.1. Let U be a real eigenfunction for μ_2. A zero o of U is *separating* if two distinct components of $\mathcal{G} \setminus \{o\}$ contain, respectively, a point where $U > 0$ and a point where $U < 0$.

Lemma 3.2 (Dichotomy)**.** *Let $\mathcal{G}$ be unicyclic, U a real eigenfunction for μ_2 and $k = \sqrt{\mu_2}$.*

(1) If o is a separating zero of U, then $\lambda_D(\mathcal{G}, o) = k^2$ and $k\rho \le 2k\rho_o \le 5/3$.

(2) If U has no separating zero, then $\lambda_D(T_i, v_i) \ge k^2$ and $kh_i \le \pi/4$ for every i.

(3) If U has no separating zero and $U(v_i) > 0$, then $U \ge U(v_i)$ on T_i, and $U'(x) = k^2 \int_{T_x} U > 0$ at every point $x \ne v_i$ of T_i that is not a vertex, with T_x and the derivative as in Lemma 2.1.

Proof. (1) Let $\Gamma_1 \ne \Gamma_2$ be components of $\mathcal{G} \setminus \{o\}$ with $U > 0$ somewhere on Γ_1 and $U < 0$ somewhere on Γ_2, and put $f_1 = U_+ \mathbf{1}_{\Gamma_1}$ and $f_2 = U_- \mathbf{1}_{\Gamma_2}$, where $U_\pm = \max\{\pm U, 0\}$. Since $U(o) = 0$ these functions are continuous, lie in $H^1(\mathcal{G})$, vanish at o and have positive integrals. Testing the weak eigenvalue equation $\int_{\mathcal{G}} U' f' = k^2 \int_{\mathcal{G}} U f$ with $f = f_j$ gives $\int |f_j'|^2 = k^2 \int f_j^2$, so $\lambda_D(\mathcal{G}, o) \le k^2$. Suppose $\lambda_D(\mathcal{G}, o) < k^2$. Then some component Γ^* carries a function $\psi \ne 0$ vanishing at o and off Γ^* with $\int |\psi'|^2 < k^2 \int \psi^2$. Choose j with $\Gamma_j \ne \Gamma^*$ and $c \in \mathbb{R}$ with $\int(\psi + cf_j) = 0$. The supports of ψ and f_j meet at most in o, so

$$\int |(\psi + cf_j)'|^2 = \int |\psi'|^2 + c^2 k^2 \int f_j^2 < k^2 \int (\psi + cf_j)^2,$$

which contradicts (1.1). Hence $\lambda_D(\mathcal{G}, o) = k^2$. The triangle inequality $d(x, y) \le d(x, o) + d(o, y)$ gives $\rho \le 2\rho_o$, and Lemma 2.2(2) gives $k\rho_o \le 5/6$.

(2) Suppose $\lambda_D(T_i, v_i) < k^2$. Then a component Γ of $T_i \setminus \{v_i\}$ has a Dirichlet ground state ψ, positive on Γ, vanishing at v_i and extended by zero, with eigenvalue $\lambda < k^2$; the closure of Γ is a tree with one edge at v_i, attached to the rest of $\mathcal{G}$ only at v_i. The integration argument of [6, Lemma 1] gives $\psi'(v_i) = \lambda \int_\Gamma \psi > 0$ for the derivative into Γ. Test the equation of U with ψ, and integrate by parts on the edges of Γ using the equation and the vertex conditions of ψ; the interior vertex and leaf terms cancel, and

$$(k^2 - \lambda) \int_\Gamma U\psi = -\psi'(v_i)\, U(v_i).$$

If $U(v_i) \neq 0$, say $U(v_i) > 0$ after replacing U by $-U$, then $U < 0$ at some $x \in \Gamma$, and a zero of U on the arc from v_i to x separates x from v_i. If $U(v_i) = 0$ and U is not identically zero on Γ, then $\int_\Gamma U\psi = 0$, so U takes both signs on Γ; every path in $\mathcal{G}$ between two such points either stays in the tree $\overline{\Gamma}$ or passes through v_i, and in both cases contains the tree arc between them, so a zero on that arc is separating. If U vanishes identically on Γ, then ψ and U_+ have disjoint supports, $\int |U_+'|^2 = k^2 \int U_+^2$ and $\int U_+ > 0$, and a mean-zero combination of ψ and U_+ has quotient below k^2, contradicting (1.1). Each case contradicts the hypothesis, so $\lambda_D(T_i, v_i) \geq k^2$, and Lemma 2.2(1) gives $kh_i \leq \sqrt{\lambda_D(T_i, v_i)}\, h_i \leq \pi/4$.

(3) If $U(x) < 0$ at some $x \in T_i$, a zero on the arc from v_i to x is separating. Hence $U \geq 0$ on T_i. For a non-vertex point $x \neq v_i$ of T_i the subtree T_x avoids v_i, and the integration argument of [6, Lemma 1] gives $U'(x) = k^2 \int_{T_x} U \geq 0$. So U is nondecreasing along every arc leaving v_i in T_i, whence $U \geq U(v_i) > 0$ on T_i and $U'(x) > 0$. □

Lemma 3.3 (Circular distance)**.** *For $t \in \mathbb{R}/C\mathbb{Z}$,*

$$d_{\mathcal{C}}(0,t) = \frac{C}{4} - \frac{2C}{\pi^2} \sum_{j \text{ odd}} \frac{\cos(2\pi jt/C)}{j^2},$$

with absolutely and uniformly convergent series, the sum running over the odd positive integers. Consequently

$$\rho_{\mathcal{C}} = \frac{C}{4} - \frac{2C}{\pi^2} \sum_{j \text{ odd}} \frac{|m_j|^2}{j^2} \leq \frac{C}{4}\Big(1 - \frac{8\sigma}{\pi^2}\Big).$$

Proof. On $[-C/2, C/2]$ the function is $|t|$. Its mean is $C/4$, and $(2/C)\int_{-C/2}^{C/2} |t| \cos(2\pi jt/C)\,\mathrm{d}t$ equals $-2C/(\pi^2 j^2)$ for odd j and 0 for even $j \geq 2$. The coefficients are absolutely summable, so the series converges uniformly to the continuous function. Integrating it against $\nu \otimes \nu$ termwise, with $\iint \cos(2\pi j(t-t')/C)\,\mathrm{d}\nu(t)\,\mathrm{d}\nu(t') = |m_j|^2$, gives the formula for $\rho_{\mathcal{C}}$, and dropping the terms with $j \geq 3$ gives the inequality. □

Lemma 3.4 (Trigonometric test functions)**.** *Let $e_1(x) = e^{2\pi \mathrm{i} t/C}$ with t the coordinate of $\operatorname{pr} x$, so that e_1 is constant on each T_i. For a complex number ω with $|\omega| = 1$, the real function $f_\omega = \operatorname{Re}(\bar{\omega}(e_1 - m_1))$ has mean zero, energy $\int_{\mathcal{G}} |f_\omega'|^2 = 2\pi^2/C$ and*

$$\int_{\mathcal{G}} f_\omega^2 = \frac{L}{2}\Big(1 - \sigma + \operatorname{Re}(\bar{\omega}^2(m_2 - m_1^2))\Big).$$

Consequently

$$\mu_2 \leq \frac{4\pi^2}{CL\,(1 - \sigma + |m_2 - m_1^2|)} \qquad \text{and} \qquad kC \leq 2\pi\sqrt{\frac{s}{1-\sigma}}, \tag{3.1}$$

where $k = \sqrt{\mu_2}$.

Proof. The arclength mean of e_1 is m_1, so f_ω has mean zero. On $\mathcal{C}$, f_ω is $\cos(2\pi t/C - \arg\omega)$ minus a constant, which gives the energy. For complex F one has $(\operatorname{Re} F)^2 = \frac{1}{2}|F|^2 + \frac{1}{2}\operatorname{Re}(F^2)$, and

$$\int_{\mathcal{G}} |e_1 - m_1|^2 = L(1-\sigma), \qquad \int_{\mathcal{G}} (e_1 - m_1)^2 = L(m_2 - 2m_1^2 + m_1^2) = L(m_2 - m_1^2),$$

which gives the norm. Choosing ω with $\bar{\omega}^2(m_2 - m_1^2) = |m_2 - m_1^2|$ in (1.1) gives the first bound in (3.1), and $|m_2 - m_1^2| \geq 0$ and $C/L = s$ give the second. □

Proposition 3.5 (General bound)**.** *Every finite connected unicyclic metric graph satisfies* $\mu_2\rho^2 \le 25\pi^2/64$.

Proof. Let U be a real eigenfunction for $\mu_2 = k^2$. If U has a separating zero, Lemma 3.2(1) gives $k\rho \le 5/3 < 5\pi/8$. Otherwise Lemma 3.2(2) gives $kh_i \le \pi/4$ for all i. By (2.1), $\rho \le \rho_{\mathcal{C}} + 2L^{-1}\int_{\mathcal{G}} \eta = \rho_{\mathcal{C}} + 2\sum_i (M_i/L)h_i$, so $k\rho \le k\rho_{\mathcal{C}} + \pi b/2$. Lemmas 3.3 and 3.4 then give

$$\frac{2k\rho}{\pi} \le 1 - s + \sqrt{\frac{s}{1-\sigma}}\Big(1 - \frac{8\sigma}{\pi^2}\Big).$$

If $\sigma \le 15/16$, then $8/\pi^2 > 4/5$, $1 - 4\sigma/5 > 0$ and $(1-4\sigma/5)^2 - (1-\sigma) = \sigma(16\sigma - 15)/25 \le 0$ give $1 - 8\sigma/\pi^2 \le 1 - 4\sigma/5 \le \sqrt{1-\sigma}$, so the right side is at most $1 - s + \sqrt{s} \le 5/4$. If $\sigma > 15/16$, then $1 - 8\sigma/\pi^2 < 1/4$, and $\sigma \le b^2 = (1-s)^2$ gives $1 - \sigma \ge s(2-s) \ge s$, so the right side is at most $1 - s + 1/4 \le 5/4$. In both cases $k\rho \le 5\pi/8$. □

4 Short total length off the cycle

When M is small compared with C, a test-function argument alone gives the sharp constant. The distance restored by the trees is bounded through the first circular moment, and the trigonometric test functions are tuned to the second.

Lemma 4.1 (Distance restored by the trees)**.** *Let* $\Phi(\ell) = \ell^2 - 2\ell^3/(3L)$. *Then*

$$\rho \le \rho_{\mathcal{C}} + \frac{1}{L}\sum_i \Phi(M_i). \tag{4.1}$$

If moreover $M \le C$, *then, with* $\xi = M/C$,

$$\frac{4\rho}{C} \le \Lambda(\xi) + \Big(2 - \frac{8}{\pi^2}\Big)\sigma \le \Lambda(\xi) + \frac{6}{5}\sigma, \qquad \Lambda(\xi) = 1 + \frac{2\xi^2}{1+\xi} - \frac{2\xi^3}{3(1+\xi)^2}. \tag{4.2}$$

Proof. By (2.1), pairs in different trees or on the cycle have distance equal to the projected distance plus both heights, and pairs in the same T_i have projected distance zero. Therefore

$$L^2(\rho - \rho_{\mathcal{C}}) = 2\sum_i (L - M_i)\int_{T_i} \eta + \sum_i \int_{T_i}\int_{T_i} d(x,y)\,\mathrm{d}x\,\mathrm{d}y.$$

For $r \ge 0$, the length of $\{x \in T_i : \eta(x) > r\}$ is at most $(M_i - r)_+$: if the set is nonempty, the arc from v_i to one of its points has an initial piece of length r on which $\eta \le r$. Integrating in r gives $\int_{T_i}\eta \le M_i^2/2$. The tree T_i is a metric graph of length M_i, so $\int_{T_i}\int_{T_i} d \le M_i^3/3$ by [1, Theorem 3.2]. Adding, $L^2(\rho - \rho_{\mathcal{C}}) \le \sum_i (LM_i^2 - 2M_i^3/3) = L\sum_i \Phi(M_i)$, which is (4.1).

Now let $M \le C$, so $M \le L/2$ and $\Phi''(\ell) = 2 - 4\ell/L \ge 0$ on $[0, M]$. Since $Lm_1 = \sum_j M_j e^{2\pi \mathrm{i} t_j/C}$, the reverse triangle inequality gives $L|m_1| \ge 2M_i - M$, so $M_i \le p = (M + L|m_1|)/2$, and $M/2 \le p \le M$ because $L|m_1| \le M$. Pad the vector (M_i) and the vector $(p, M-p)$ with zeros to a common length. The second majorizes the first: its largest entry dominates every M_i, its first two entries sum to M, and both vectors have sum M. All entries lie in $[0, M]$, where Φ is convex, and $\Phi(0) = 0$, so Karamata's inequality [4] gives $\sum_i \Phi(M_i) \le \Phi(p) + \Phi(M-p)$. Since $p - M/2 = L|m_1|/2$,

$$\Phi(p) + \Phi(M-p) = \frac{M^2}{2} - \frac{M^3}{6L} + 2\Big(p - \frac{M}{2}\Big)^2\Big(1 - \frac{M}{L}\Big) = \frac{M^2}{2} - \frac{M^3}{6L} + \frac{LC\sigma}{2}.$$

Insert this and Lemma 3.3 into (4.1), multiply by $4/C$ and use $M^2/(CL) = \xi^2/(1+\xi)$ and $M^3/(CL^2) = \xi^3/(1+\xi)^2$. This gives the first inequality in (4.2), and $2 - 8/\pi^2 = 1.1894\ldots \le 6/5$ gives the second. □

Lemma 4.2 (Covariance bound for the gap)**.** *If $M > 0$, then*

$$\mu_2 \le \frac{4\pi^2}{CL\,N(\sigma)}, \qquad N(\sigma) = 1 - \sigma + \max\Big\{0, \Big(\frac{2}{b} - 1\Big)\sigma - b\Big\}.$$

Proof. A rotation $t \mapsto t - t_0$ of the coordinate multiplies m_j by $e^{-2\pi i j t_0/C}$ and leaves σ and $|m_2 - m_1^2|$ unchanged, so we may assume $m_1 = |m_1|$. With $\varphi_i = 2\pi t_i/C$ in this coordinate, $\sum_i (M_i/L)\cos\varphi_i = |m_1|$, and the Cauchy–Schwarz inequality gives $\sum_i (M_i/L)\cos^2\varphi_i \ge \sigma/b$. Hence

$$|m_2 - m_1^2| \ge \operatorname{Re} m_2 - \sigma = 2\sum_i \frac{M_i}{L}\cos^2\varphi_i - b - \sigma \ge \Big(\frac{2}{b} - 1\Big)\sigma - b,$$

and the claim follows from (3.1). □

Proposition 4.3 (Short off-cycle length)**.** *If $0 < M \le 3C/14$, then $\mu_2\rho^2 < \pi^2/4$.*

Proof. Put $\xi = M/C \in (0, 3/14]$, so $L/C = 1 + \xi$ and $b = \xi/(1+\xi)$. By Lemmas 4.1 and 4.2,

$$\mu_2\rho^2 \le \frac{\pi^2}{4} \cdot \frac{(\Lambda(\xi) + 6\sigma/5)^2}{(1+\xi)N(\sigma)},$$

so it suffices to prove $D(\sigma) < 0$ for $0 \le \sigma \le b^2$, where $D(\sigma) = \Lambda(\xi) + 6\sigma/5 - \sqrt{(1+\xi)N(\sigma)}$. Put $\sigma_0 = b^2/(2-b) = \xi^2/((1+\xi)(2+\xi)) \le b^2$, the point where $(2/b - 1)\sigma - b$ changes sign. On $[0, \sigma_0]$, $N(\sigma) = 1 - \sigma$ and D is increasing. On $[\sigma_0, b^2]$, $(1+\xi)N(\sigma) = 1 + 2(1+\xi)\sigma/\xi \le 1 + 2\xi/(1+\xi) < 4$, so the derivative of $\sqrt{(1+\xi)N}$ exceeds $(1+\xi)/(2\xi) \ge 17/6 > 6/5$ and D is decreasing. Thus $D \le D(\sigma_0)$. Since $(1+\xi)(1-\sigma_0) = 1 + 2\xi/(2+\xi)$, the identity

$$1 + \frac{2\xi}{2+\xi} - \Big(\Lambda(\xi) + \frac{6\xi^2}{5(1+\xi)(2+\xi)}\Big)^2 = \frac{2\xi P(\xi)}{225(1+\xi)^4(2+\xi)^2},$$

$$P(\xi) = 450 - 315\xi - 4890\xi^2 - 11772\xi^3 - 13464\xi^4 - 7727\xi^5 - 2060\xi^6 - 200\xi^7,$$

reduces $D(\sigma_0) < 0$ to $P(\xi) > 0$. Every nonconstant coefficient of P is negative, so P decreases on $[0, 3/14]$, and $P(3/14) = 264672531/26353376 > 0$. □

5 Small attachments at each point

Without the majorization step, (4.1) keeps the individual lengths M_i, and it combines with the trigonometric test functions into a criterion for each graph. A bound on each M_i alone then gives the sharp constant for arbitrary total length.

Proposition 5.1 (Criterion)**.** *Put $K = \frac{4}{CL}\sum_i M_i^2(1 - \frac{2M_i}{3L})$ and $\gamma = 8/\pi^2$. If*

$$(1 + K - \gamma\sigma)^2 \le \frac{L}{C}(1 - \sigma + |m_2 - m_1^2|), \tag{5.1}$$

then $\mu_2\rho^2 \le \pi^2/4$, and strict inequality in (5.1) *gives $\mu_2\rho^2 < \pi^2/4$.*

Proof. Inequality (4.1) and Lemma 3.3 give $0 < 4\rho/C \le 1 + K - \gamma\sigma$, and (3.1) gives $\mu_2\rho^2 \le (\pi^2/4)(C/L)(1 + K - \gamma\sigma)^2/(1 - \sigma + |m_2 - m_1^2|)$. □

Proposition 5.2 (Small attachments)**.** *If $M > 0$ and $M_{\max} \le C/7$, then $\mu_2\rho^2 < \pi^2/4$.*

Proof. If $M \le 3C/14$, Proposition 4.3 applies, so assume $M > 3C/14$, that is $3/17 < b < 1$. Then $K \le \frac{4}{CL} M_{\max} \sum_i M_i \le \frac{4}{7} b$, and $1 + K - \gamma\sigma > 0$, so by Proposition 5.1 and $|m_2 - m_1^2| \ge 0$ it suffices to prove

$$\Big(1 + \frac{4}{7}b - \gamma\sigma\Big)^2 < \frac{1-\sigma}{1-b} \qquad (0 \le \sigma \le b^2).$$

The difference of the two sides is convex in σ, so it suffices to check the endpoints. At $\sigma = 0$, the binomial series of $(1-b)^{-1/2}$ has positive coefficients, so

$$(1-b)^{-1/2} \ge 1 + \frac{b}{2} + \frac{3b^2}{8} + \frac{5b^3}{16} > 1 + \frac{4}{7}b,$$

because $\frac{1}{2} + \frac{3}{8} \cdot \frac{3}{17} + \frac{5}{16}\big(\frac{3}{17}\big)^2 > \frac{4}{7}$ and the left side increases with b. At $\sigma = b^2$ the right side is $1 + b$, and $1 + \frac{4}{7}b - \gamma b^2 \ge 1 + \frac{4}{7} - \gamma > 0$. The third derivative of $\sqrt{1+b}$ is positive, so Taylor's formula gives

$$\sqrt{1+b} \ge 1 + \frac{b}{2} - \frac{b^2}{8} > 1 + \frac{4}{7}b - \gamma b^2,$$

where the strict inequality follows from $\gamma > 4/5$ and $\frac{1}{2} + \big(\frac{4}{5} - \frac{1}{8}\big)\frac{3}{17} > \frac{4}{7}$. Squaring the positive quantities proves both endpoint comparisons. □

6 All trees attached at one point

When every tree is attached at the same point v, the reflection of the cycle fixing v is an isometry of $\mathcal{G}$. It reduces a separating zero to the tree bound, and otherwise the eigenfunction on the cycle is an explicit cosine whose phase is determined by the attached trees.

Proposition 6.1 (One attachment point)**.** *If all trees of $\mathcal{G}$ are attached at one point v of $\mathcal{C}$, then $\mu_2\rho^2 \le \pi^2/4$.*

Proof. If $M = 0$, $\mathcal{G}$ is a circle and $\mu_2\rho^2 = \pi^2/4$. Let $M > 0$, let $T = T_1$ be the tree at v, let U be a real eigenfunction for $\mu_2 = k^2$, and let v^* be the point of $\mathcal{C}$ at distance $C/2$ from v. The reflection R of $\mathcal{C}$ fixing v and v^*, extended by the identity on T, is an isometry of $\mathcal{G}$.

A separating zero. Let o be a separating zero. For $o \in \mathcal{C} \setminus \{v\}$ the set $\mathcal{G} \setminus \{o\}$ is connected, so $o = v$ or $o \in T \setminus \{v\}$; in both cases R fixes o. Let Γ be the component of $\mathcal{G} \setminus \{o\}$ containing $\mathcal{C} \setminus \{v\}$. The map R preserves Γ, so the Dirichlet ground state of $\overline{\Gamma}$, being simple and positive, is invariant under R, and its derivative at v^* vanishes. Cut $\mathcal{C}$ at v^*, obtaining a tree $\mathcal{G}'$ in which $\mathcal{C}$ is replaced by two intervals of length $C/2$ attached at v. The restricted ground state satisfies the equation, the vertex conditions, Neumann conditions at the two new leaves and the Dirichlet condition at o, and it is positive; since eigenfunctions for distinct eigenvalues are orthogonal, it is the ground state of the cut component. The other components are unchanged, so $\lambda_D(\mathcal{G}', o) = \lambda_D(\mathcal{G}, o) = k^2$ by Lemma 3.2(1). Every point of $\mathcal{C}$ keeps its distance to v, hence to o, so ρ_o is the same for $\mathcal{G}$ and $\mathcal{G}'$. Lemma 2.2(1) for the tree $(\mathcal{G}', o)$ and $\rho \le 2\rho_o$ give $k\rho \le \pi/2$.

No separating zero. First, $kC < 2\pi$: the function equal to $\cos(2\pi t/C)$ on $\mathcal{C}$, with t measured from v, and to 1 on T, minus its mean b, has energy $2\pi^2/C$ and squared norm $C/2 + M - M^2/L = C/2 + MC/L > C/2$. Second, $U(v) \ne 0$. Indeed, suppose $U(v) = 0$. If U is not identically zero on some component Γ of $T \setminus \{v\}$, then either U takes both signs on Γ, and a zero on a tree arc between them is separating, or U has one sign on Γ, and then U has the other sign somewhere outside $\overline{\Gamma}$ because it has mean zero, so v is separating. If U vanishes on T, then on $\mathcal{C}$, parametrized by $[0, C]$ with both ends at v, it is a nonzero multiple of a Dirichlet eigenfunction $\sin(j\pi t/C)$ with equal end

derivatives by the Kirchhoff condition, so j is even and $kC \geq 2\pi$, which is excluded. Replace U by $-U$ so that $U(v) > 0$. By Lemma 3.2(3), $U > 0$ on T with $U' > 0$ away from v, and

$$Q = \frac{U'_T(v)}{kU(v)} > 0,$$

where $U'_T(v)$ is the sum of the derivatives of U at v into the edges of T.

On $\mathcal{C}$, write $U = c_1 \cos(k(t - C/2)) + c_2 \sin(k(t - C/2))$ for $t \in [0, C]$ and put $\zeta = kC/2 \in (0, \pi)$. The two end values of U agree, so $c_2 \sin\zeta = 0$, and $\sin\zeta \neq 0$ gives $c_2 = 0$. Then $U(v) = c_1 \cos\zeta \neq 0$, and the outgoing derivatives of U along $\mathcal{C}$ at v are both $kc_1 \sin\zeta$. The Kirchhoff condition at v reads $2kc_1 \sin\zeta + kU(v)Q = 0$, that is,

$$2\tan\zeta + Q = 0.$$

Since $Q > 0$, $\zeta \in (\pi/2, \pi)$; put $\tau = \pi - \zeta \in (0, \pi/2)$, so $Q = 2\tan\tau$, and $A = \arctan Q = \arctan(2\tan\tau)$.

Lemma 2.1 applies to U on (T, v) with $\kappa = k$, and its phase is $\arctan Q = A$. The inequality $J \leq \theta B/2$ of (2.2) reads $k^2 M h_1 \leq AkM/2$, that is $kh_1 \leq A/2$. The measure ν is s times the uniform probability measure on $\mathcal{C}$ plus $b\delta_v$, so $\rho_{\mathcal{C}} = s^2 C/4 + 2sbC/4 = C(1 - b^2)/4$, and (2.1) gives $\rho \leq C(1 - b^2)/4 + 2bh_1$. Hence

$$k\rho \leq \frac{\zeta}{2}(1 - b^2) + bA \leq \frac{\zeta}{2} + \frac{A^2}{2\zeta},$$

by maximizing $-\zeta b^2/2 + Ab$ over b. The right side is at most $\pi/2$ if and only if $A^2 \leq \zeta(\pi - \zeta) = \tau(\pi - \tau)$. For $\tau \leq \pi/10$, $\tan(2\tau) \geq 2\tan\tau$ gives $A \leq 2\tau$, and $4\tau^2 \leq \tau(\pi - \tau)$. For $\tau \geq \pi/10$, put $\epsilon = (\pi/2 - \tau)/2 \in (0, \pi/4)$; then $2\tan\tau = 2\cot(2\epsilon)$, and $\tan(2\epsilon) \geq 2\tan\epsilon$ gives $\arctan(\frac{1}{2}\tan 2\epsilon) \geq \epsilon$, so $A = \pi/2 - \arctan(\frac{1}{2}\tan 2\epsilon) \leq \pi/4 + \tau/2$. Finally

$$16\Big[\Big(\frac{\pi}{4} + \frac{\tau}{2}\Big)^2 - \tau(\pi - \tau)\Big] = 20\tau^2 - 12\pi\tau + \pi^2,$$

whose roots are $\pi/10$ and $\pi/2$, so it is nonpositive on $[\pi/10, \pi/2]$. Thus $k\rho \leq \pi/2$. □

7 Exact reduction to the cycle and rotational symmetry

Below the rooted Dirichlet frequencies of the trees, each tree can be eliminated exactly through its Dirichlet-to-Neumann response at the attachment point, a Schur-complement reduction familiar for quantum graphs [2]. What remains is a quadratic form on the cycle with point interactions, whose weights are bounded from below by the length and the mean height of each tree.

Lemma 7.1 (Branch solutions)**.** *Let (T, v) be a finite rooted metric tree and $0 < \kappa^2 < \lambda_D(T, v)$. There is a unique $w \in H^1(T)$ with $w(v) = 1$ that solves $-w'' = \kappa^2 w$ on every edge and satisfies the Kirchhoff condition at every vertex other than v, with Neumann conditions at the leaves. It satisfies $w \geq 1$ on T, $w'(x) = \kappa^2 \int_{T_x} w > 0$ at every non-vertex point $x \neq v$, and*

$$\alpha = w'(v) = \kappa^2 \int_T w \geq \kappa^2 |T| > 0,$$

where $w'(v)$ is the sum of the derivatives at v into the edges of T. Moreover, with $\mathfrak{q}_T(f) = \int_T (|f'|^2 - \kappa^2 f^2)$ and its bilinear form $\mathfrak{q}_T(f, g)$, one has $\mathfrak{q}_T(w) = -\alpha$, $\mathfrak{q}_T(w, g) = 0$ and $\mathfrak{q}_T(g) > 0$ for every nonzero $g \in H^1(T)$ with $g(v) = 0$.

Proof. Let $g \in H^1(T)$ with $g(v) = 0$. The definition of $\lambda_D(T, v)$ gives $\int_T g^2 \le \lambda_D(T, v)^{-1} \int_T |g'|^2$, hence

$$\mathfrak{q}_T(g) \ge \Big(1 - \frac{\kappa^2}{\lambda_D(T, v)}\Big) \int_T |g'|^2,$$

so $\mathfrak{q}_T$ is coercive on this subspace. Minimizing $\mathfrak{q}_T$ over the closed affine subspace $\{f(v) = 1\}$ gives a unique w with $\mathfrak{q}_T(w, g) = 0$ for all g vanishing at v, which is the weak form of the stated boundary value problem; a second solution would differ from w by a function g vanishing at v with $\mathfrak{q}_T(g) = 0$, hence $g = 0$.

Nodal domains. The negative part $w_- = \max\{-w, 0\}$ lies in $H^1(T)$ and vanishes at v, because $w(v) = 1$. On $\{w < 0\}$ one has $w = -w_-$ and $w' = -w'_-$ almost everywhere, and w_-, w'_- vanish almost everywhere elsewhere, so $0 = \mathfrak{q}_T(w, w_-) = -\mathfrak{q}_T(w_-)$. Positive definiteness forces $w_- = 0$: the solution has no negative nodal domain, and $w \ge 0$.

Slopes. The integration argument of [6, Lemma 1] uses only the equation and the vertex conditions, and gives $w'(x) = \kappa^2 \int_{T_x} w \ge 0$ at every non-vertex $x \ne v$. Hence w is nondecreasing along every arc leaving v, so $w \ge w(v) = 1$ on T, and then $w'(x) \ge \kappa^2 |T_x| > 0$. Summing over the edges at v gives $\alpha = \kappa^2 \int_T w \ge \kappa^2 |T|$. Finally, integrating by parts on each edge, the terms at the leaves and at the vertices other than v cancel by the vertex conditions, and $\mathfrak{q}_T(w) = -w'(v)w(v) = -\alpha$. □

Proposition 7.2 (Exact reduction)**.** *Let $\mathcal{G}$ be unicyclic and $\kappa > 0$ with $\kappa^2 < \lambda_D(T_i, v_i)$ for every i, and let w_i and α_i be given by Lemma 7.1 for (T_i, v_i). Define on $H^1(\mathcal{C})$*

$$\mathfrak{c}_\kappa(g) = \int_{\mathcal{C}} (|g'|^2 - \kappa^2 g^2)\, \mathrm{d}t - \sum_i \alpha_i g(t_i)^2.$$

Then the number of eigenvalues of $\mathcal{G}$ below κ^2, counted with multiplicity, equals the largest dimension of a subspace on which $\mathfrak{c}_\kappa$ is negative definite. In particular this dimension is at least one, and it equals one if and only if $\kappa^2 \le \mu_2$.

Proof. For $f \in H^1(\mathcal{G})$ write $f|_{T_i} = f(v_i)w_i + g_i$, so $g_i(v_i) = 0$. The map $f \mapsto (f|_{\mathcal{C}}, (g_i)_i)$ is a linear bijection from $H^1(\mathcal{G})$ onto $H^1(\mathcal{C}) \times \prod_i \{g \in H^1(T_i) : g(v_i) = 0\}$. By Lemma 7.1 the cross terms vanish and

$$\int_{\mathcal{G}} (|f'|^2 - \kappa^2 f^2) = \mathfrak{c}_\kappa(f|_{\mathcal{C}}) + \sum_i \mathfrak{q}_{T_i}(g_i),$$

an orthogonal sum of $\mathfrak{c}_\kappa$ and positive definite forms. Let S be a subspace on which the left side is negative definite. A nonzero element of S with $f|_{\mathcal{C}} = 0$ would have a positive value, so $f \mapsto f|_{\mathcal{C}}$ is injective on S; and for $0 \ne f \in S$, $\mathfrak{c}_\kappa(f|_{\mathcal{C}})$ is at most the value of the left side, which is negative. So the image of S is a subspace of the same dimension on which $\mathfrak{c}_\kappa$ is negative definite. Conversely, a subspace of $H^1(\mathcal{C})$ on which $\mathfrak{c}_\kappa$ is negative definite, extended by $g_i = 0$, is one for the left side. The min-max principle identifies the left side with the eigenvalue count. Since $\mu_1 = 0 < \kappa^2$, the count is at least one, and it is one exactly when $\mu_2 \ge \kappa^2$. □

Corollary 7.3 (First-harmonic inequality)**.** *Under the hypotheses of Proposition 7.2, put*

$$W = \kappa^2 C + \sum_i \alpha_i, \qquad Z = \sum_i \alpha_i e^{2\pi \mathrm{i} t_i / C}.$$

Then $W > 0$, and if $\kappa^2 \le \mu_2$,

$$W - \frac{|Z|^2}{W} \le \frac{4\pi^2}{C}.$$

Proof. By Lemma 7.1, every $\alpha_i > 0$, so $W > 0$ and $\varpi_\kappa = \kappa^2\,\mathrm{d}t + \sum_i \alpha_i \delta_{t_i}$ is a positive measure on $\mathcal{C}$ of mass W. The constant function has $\mathfrak{c}_\kappa(1) = -W < 0$, and the bilinear form satisfies $\mathfrak{c}_\kappa(1, g) = -\int g\,\mathrm{d}\varpi_\kappa$. For $g \in H^1(\mathcal{C})$ let $\tilde{g} = g - W^{-1}\int g\,\mathrm{d}\varpi_\kappa$, which is $\mathfrak{c}_\kappa$-orthogonal to 1. If $\tilde{g} \neq 0$ and $\mathfrak{c}_\kappa(\tilde{g}) < 0$, then 1 and $\tilde{g}$ would span a two-dimensional subspace on which $\mathfrak{c}_\kappa$ is negative definite, which Proposition 7.2 excludes. Hence

$$0 \le \mathfrak{c}_\kappa(\tilde{g}) = \mathfrak{c}_\kappa(g) + \frac{1}{W}\Big(\int g\,\mathrm{d}\varpi_\kappa\Big)^2.$$

Apply this to $g = \cos(2\pi t/C)$ and $g = \sin(2\pi t/C)$ and add: the energies sum to $4\pi^2/C$, the remaining terms of $\mathfrak{c}_\kappa$ to $-\kappa^2 C - \sum_i \alpha_i = -W$, and the squared integrals to $|Z|^2$. □

This inequality retains the individual attachment weights lost in (3.1). The proof of Theorem 1.2 applies it when the branch weights have vanishing first circular moment, $Z = 0$, together with the following lower bound for each weight.

Lemma 7.4 (Admittance bound)**.** *Let (T, o), κ and u be as in Lemma 2.1, with B, J and θ as there, and put $\chi = 2J/B$ and $F(\chi) = \tan(\chi)/\chi$. Then $0 < \chi \le \theta < \pi/2$ and*

$$\tan\theta \ge B\,F(\chi), \tag{7.1}$$

with equality when T is an interval rooted at an endpoint.

Proof. Rescale lengths by κ. The bound $\chi \le \theta$ is the last inequality of (2.2). The function $F(\chi) = \int_0^1 \sec^2(r\chi)\,\mathrm{d}r$ is convex on $[0, \pi/2)$, since $\sec^2$ is convex there, and $F + \chi F' = \sec^2\chi$. We argue by induction on the number of edges. For an interval of length B rooted at an endpoint, the phase runs from 0 at the leaf to $\theta = B$ at o, and $J = B^2/2$, so $\chi = B$ and $\tan\theta = BF(B)$.

Splits. If o has several edges, T is the union of pieces (S_j, o), each with one edge at o and data B_j, J_j, θ_j. Then $B = \sum_j B_j$, $J = \sum_j J_j$, the tangent sum gives $\tan\theta = \sum_j \tan\theta_j$, and χ is the B_j-weighted average of the $\chi_j = 2J_j/B_j$. By the induction hypothesis and Jensen's inequality, $\tan\theta \ge \sum_j B_j F(\chi_j) \ge BF(\chi)$.

Stems. If o has one edge, of length ℓ_0, and its other end x_0 is not a leaf, let $S^{(\ell)}$ for $0 \le \ell \le \ell_0$ be the subtree of points whose path to o passes through the point of that edge at distance ℓ from x_0, rooted at that point. Then $S^{(\ell_0)} = T$, and $S^{(0)}$, rooted at x_0, has fewer edges than T, so it satisfies (7.1) by the induction hypothesis, applied directly or through the split step. Along ℓ the data satisfy $\theta' = B' = 1$ and $J' = B$, so $\chi' = 2 - \chi/B$. The difference $\Delta = \tan\theta - BF(\chi)$ satisfies

$$\Delta' = \big(\tan\theta + BF(\chi)\big)\Delta + (B - \chi)\big[(B + \chi)F(\chi)^2 - 2F'(\chi)\big],$$

as follows from $F + \chi F' = \sec^2\chi$ and $\chi F = \tan\chi$. The last term is nonnegative: $B \ge \theta \ge \chi$ by (2.2) applied to $S^{(\ell)}$, and $(B + \chi)F^2 - 2F' \ge 2(\chi F^2 - F') = 2(\tan\chi - \chi)/\chi^2 \ge 0$. Since $\Delta(0) \ge 0$ and the coefficient of Δ is continuous, $\Delta \ge 0$ on $[0, \ell_0]$, and $\ell = \ell_0$ gives (7.1) for T. □

Proof of Theorem 1.2. If $M = 0$, $\mathcal{G}$ is a circle and $\mu_2\rho^2 = \pi^2/4$. Let $M > 0$ and let the isometry Θ induce the rotation of $\mathcal{C}$ by $t_0 \not\equiv 0 \pmod{C}$. A nonidentity rotation has no fixed point, so Θ maps each T_i isometrically onto a tree $T_{i'}$ attached at $v_{i'} \neq v_i$, disjoint from T_i, with $t_{i'} = t_i + t_0$.

First, $\lambda_D(T_i, v_i) \ge \mu_2$ for every i. Otherwise a Dirichlet ground state $\psi \ge 0$ on T_i, extended by zero, and $\psi \circ \Theta^{-1}$ have disjoint supports, equal positive integrals and quotient $\lambda_D(T_i, v_i) < \mu_2$, and their difference contradicts (1.1).

Fix $\kappa \in (0, \sqrt{\mu_2})$. Then Lemma 7.1 and Corollary 7.3 apply. Since Θ maps (T_i, v_i) onto $(T_{i'}, v_{i'})$, the branch solutions correspond and $\alpha_{i'} = \alpha_i$, so $Z = e^{2\pi\mathrm{i}t_0/C}Z$, and $e^{2\pi\mathrm{i}t_0/C} \neq 1$ gives $Z = 0$. Thus $W \le 4\pi^2/C$.

For each i, Lemma 7.4 applies to w_i on (T_i, v_i), with $B_i = \kappa M_i$, $J_i = \kappa^2 M_i h_i$, $\chi_i = 2\kappa h_i < \pi/2$ and $\tan\theta_i = \alpha_i/\kappa$; it gives $\alpha_i \geq \kappa^2 M_i F(2\kappa h_i)$. By Jensen's inequality, $\sum_i \alpha_i \geq \kappa^2 M F(\bar{\chi})$ with $\bar{\chi} = 2\kappa\bar{h} < \pi/2$. Hence $\kappa^2(C + MF(\bar{\chi})) \leq 4\pi^2/C$, that is,

$$\frac{\kappa C}{2\pi} \leq \sqrt{\frac{s}{s + (1-s)F(\bar{\chi})}}.$$

By (2.1) and Lemma 3.3, $\rho \leq \rho_{\mathcal{C}} + 2b\bar{h} \leq C/4 + 2b\bar{h}$. With $a = 2\bar{\chi}/\pi = 4\kappa\bar{h}/\pi \in (0,1)$,

$$\frac{2\kappa\rho}{\pi} \leq \sqrt{\frac{s}{s + (1-s)F(\pi a/2)}} + (1-s)a.$$

We show that the right side is at most 1. Put $z = 1 - (1-s)a \in (1-a, 1)$. Squaring and solving for F shows that the claim is equivalent to

$$F\Big(\frac{\pi a}{2}\Big) \geq 1 + \frac{a}{z} - \frac{1-a}{z^2}.$$

As a function of $1/z \in [1, 1/(1-a)]$ the right side is a concave quadratic with vertex at $1/z = a/(2(1-a))$. Its maximum is therefore $2a$, attained at $z = 1$, if $a \leq 2/3$, and $1 + a^2/(4(1-a))$ if $a > 2/3$. Since $\sec^2 y \geq 1 + y^2$, $F(\chi) \geq 1 + \chi^2/3$, so $F(\pi a/2) \geq 1 + \pi^2 a^2/12 \geq 2a$ for $a \leq 2/3$, the smaller root of $1 + \pi^2 a^2/12 - 2a$ being $0.7035\ldots$. For $a > 2/3$ use the partial fraction expansion, obtained by logarithmic differentiation of the product for the cosine,

$$F\Big(\frac{\pi a}{2}\Big) = \frac{8}{\pi^2} \sum_{j \geq 1} \frac{1}{(2j-1)^2 - a^2}.$$

Its value at $a = 0$ is 1; subtracting it and keeping the first of the nonnegative differences gives $F(\pi a/2) \geq 1 + (8/\pi^2)a^2/(1-a^2)$. Since $8/(\pi^2(1+a)) > 4/\pi^2 > 1/4$, this is at least $1 + a^2/(4(1-a))$. Hence $\kappa\rho \leq \pi/2$ for every $\kappa < \sqrt{\mu_2}$, and letting κ increase to $\sqrt{\mu_2}$ gives $\mu_2\rho^2 \leq \pi^2/4$. □

8 Proofs of Theorem 1.1 and Corollary 1.3

Proof of Theorem 1.1. The universal bound is Proposition 3.5. Case (1) is Proposition 6.1. In cases (2) and (3) the circle has $\mu_2\rho^2 = \pi^2/4$, and every other graph has $M > 0$ and satisfies $\mu_2\rho^2 < \pi^2/4$ by Propositions 4.3 and 5.2. □

Proof of Corollary 1.3. The lower bound is $\mu_2 L\rho > 2$ [6, Theorem 1], valid on every compact metric graph. The upper bounds are Theorems 1.1 and 1.2. □

Statements and Declarations

Funding. This research was self-funded.

Competing interests. The author owns Luminary Oddities and has no other relevant competing interests.

Data and code availability. No external datasets were generated or analyzed. All results have analytic proofs in the article. The script `verify_unicyclic_identities.py` (Python with sympy and mpmath) checks the identities and constants of Sections 3–7 exactly, and `fem_check.py` (numpy and scipy) gives illustrative finite-element checks of the theorems; both accompany the article.

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AI assistance. OpenAI ChatGPT assisted with derivations, proof development and checking. Anthropic Claude reviewed, edited and cross-referenced the article and checked computations. Scott Schubr directed the research and is responsible for the content.

References

[1] L. N. Baptista, J. B. Kennedy, and D. Mugnolo, *Mean distance on metric graphs*, J. Geom. Anal. **34** (2024), no. 5, Paper No. 137, https://doi.org/10.1007/s12220-024-01574-0; version read: https://arxiv.org/abs/2312.04952v2.

[2] G. Berkolaiko and P. Kuchment, *Introduction to Quantum Graphs*, Math. Surveys Monogr., vol. 186, Amer. Math. Soc., Providence, RI, 2013, https://doi.org/10.1090/surv/186.

[3] J. B. Kennedy, P. Kurasov, G. Malenová, and D. Mugnolo, *On the spectral gap of a quantum graph*, Ann. Henri Poincaré **17** (2016), no. 9, 2439–2473, https://doi.org/10.1007/s00023-016-0460-2; version read: https://arxiv.org/abs/1504.01962v2.

[4] A. W. Marshall, I. Olkin, and B. C. Arnold, *Inequalities: Theory of Majorization and Its Applications*, 2nd ed., Springer Ser. Statist., Springer, New York, 2011, https://doi.org/10.1007/978-0-387-68276-1.

[5] D. Mugnolo, *The role of expanders in the spectral geometry of metric graphs*, preprint, arXiv:2607.14312v2 [math.SP], 2026, https://arxiv.org/abs/2607.14312v2.

[6] S. Schubr, *Sharp mean-distance bounds for spectral gaps of compact metric graphs and trees*, edition 2, Luminary Research Archive, 2026, https://research.luminaryoddities.org/sharp-mean-distance-bounds-for-spectral-gaps-of-compact-metric-graphs-and-trees/v/2, https://doi.org/10.5281/zenodo.24550489.

[7] S. Schubr, *The sharp mean-distance constant and the second Dirichlet eigenvalue of compact metric graphs*, edition 2, Luminary Research Archive, 2026, https://research.luminaryoddities.org/mean-distance-and-the-first-two-dirichlet-eigenvalues-of-compact-metric-graphs/v/2, https://doi.org/10.5281/zenodo.24550456.