The sharp mean-distance constant and the second Dirichlet eigenvalue of compact metric graphs

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Abstract

Mugnolo (arXiv:2607.14312v2, Conjecture 4.7) conjectured that on every compact metric graph with Dirichlet conditions on a proper nonempty set of vertices, the first eigenvalue λ_1 of the Dirichlet–Kirchhoff Laplacian is at most C/ρ^2, where ρ is the arclength mean of the distance to the Dirichlet set and $C < 1$ is a universal constant. We prove the conjecture with the optimal constant $C_* = z_*^4/4 \approx 0.66167$, for every nonempty Dirichlet set. Exact and interval-arithmetic certificates give $0.6614 < C_* \le 0.66168$, and z_* is the unique initial value for which the solution of the Abel equation $z^2 z' = \sigma z - 1$ stays positive with $\sigma z \le 1$ on the half-line. No finite graph attains C_*; chains of bundles of parallel edges approach it. The coarea formula reduces the problem to weighted Sturm–Liouville problems on intervals, whose threshold solution gives both the optimal trial function and, through Barta's inequality, an extremal weight. We also sharpen the lower bound $\lambda_1 L\rho \ge 1$, recorded by Mugnolo after Plümer, where L is the total length, to $\lambda_1 \ge 1/(L\rho - (1/2 - 4/\pi^2)R^2)$ with R the Dirichlet inradius, with equality only for intervals with one Dirichlet endpoint; with Kennedy's stars this shows that the constant 1 is optimal and never attained. Finally we determine the sharp lower envelope of the second eigenvalue at prescribed L and R; its extremal graphs change at $L = 2R$, where its derivative jumps.

Keywords. Metric graph, quantum graph, Dirichlet eigenvalue, mean distance, inradius, Poincaré inequality, spectral optimization, Abel equation, computer-assisted proof.
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1 Introduction and main results

1.1 The question

Let $\mathcal{G}$ be a finite connected compact metric graph with vertex set V and positive edge lengths; loops and multiple edges are allowed. Fix a nonempty set $D \subseteq V$ of Dirichlet vertices. With arclength measure write

$$L = |\mathcal{G}|, \qquad d(x) = \operatorname{dist}_{\mathcal{G}}(x, D), \qquad R = \max_{\mathcal{G}} d, \qquad \rho_D = \rho(\mathcal{G}, D) = \frac{1}{L}\int_{\mathcal{G}} d(x)\,dx,$$

so that R is the inradius and ρ_D the mean distance to D. The form domain $H^1_D(\mathcal{G})$ consists of continuous functions that are edgewise H^1 and vanish on D, with energy $\mathcal{E}(u) = \int_{\mathcal{G}} |u'|^2$. The associated operator is the Dirichlet–Kirchhoff Laplacian: vertices in D carry Dirichlet conditions, all other vertices carry continuity and Kirchhoff conditions, and unmarked leaves carry Neumann

conditions [4, 8]. Its eigenvalues, with multiplicity, are $0 < \lambda_1 \le \lambda_2 \le \cdots$. A marked point inside an edge is represented by subdividing the edge there.

Mugnolo [9, Section 4.2] compares λ_1 with the inradius and with the mean distance. He records the two-sided information

$$\frac{1}{L\,\rho_D} \le \lambda_1(\mathcal{G}, D) < \frac{1}{\rho_D^2}, \tag{1}$$

where Mugnolo attributes the lower bound to Plümer [10, Theorem 4.4.3], and the upper bound is the test-function estimate [9, (4.22)], which Mugnolo attributes to Lemma 4.1 of his reference [4], Baptista, Kennedy, Kravitz and Mugnolo, *The mean distance function on metric graphs* (in preparation). He shows that $\lambda_1 R^2$ has no universal upper bound [9, Theorem 4.8], and he states the following conjecture [9, Conjecture 4.7], written here with D for his Dirichlet vertex set V^D and with his arclength mean distance [9, (4.19)]:

> *Let $\emptyset \ne D \subsetneq V$. Then there exists a universal constant $C \in (0,1)$ such that $\lambda_1(\mathcal{G}, D) \le C/\rho_D^2$ for all metric graphs $\mathcal{G}$.*

Edition 1 of this article [12] proved the conjecture with $C = 331/500$. The theorem below identifies the optimal constant.

Theorem 1.1. *Let $C_* = \sup_{(\mathcal{G},D)} \lambda_1(\mathcal{G}, D)\rho_D(\mathcal{G}, D)^2$, the supremum being over all finite connected compact metric graphs and all nonempty $D \subseteq V$. Then:*

1. *$C_* = z_*^4/4$, where z_* is the unique $z_0 > 0$ such that the solution of*

$$z^2 \frac{dz}{d\sigma} = \sigma z - 1, \qquad z(0) = z_0, \tag{2}$$

   exists and is positive on $[0, \infty)$ with $\sigma z(\sigma) \le 1$ throughout;

2. *$\lambda_1(\mathcal{G}, D)\rho_D(\mathcal{G}, D)^2 < C_*$ for every finite $(\mathcal{G}, D)$;*

3. *$0.6614 < C_* \le 0.66168$.*

Remark (Numerical value). Two independent high-precision integrations of the threshold problem (Section 4) give $z_* = 1.275484393011$ and $C_* = 0.661668720218$, agreeing to about 10^{-13}; the certified statement is item (3).

Theorem 1.1 proves Mugnolo's Conjecture 4.7 with the optimal constant, for every nonempty D, a class that contains the conjecture's range $D \subsetneq V$. Items (1)–(3) are proved in Sections 2–4, and item (3) is a certified enclosure obtained from two finite exact computations. Together with (1), the theorem gives the universal window

$$\frac{1}{L\rho_D} \le \lambda_1(\mathcal{G}, D) \le \frac{C_*}{\rho_D^2},$$

whose upper end is independent of length, inradius and topology; Theorem 1.3 below sharpens its lower end and shows that it is strict on every graph. The classical candidates are far from extremal: the interval of length ℓ with one Dirichlet endpoint has $\lambda_1 = \pi^2/(4\ell^2)$ and $\rho_D = \ell/2$, so $\lambda_1\rho_D^2 = \pi^2/16 = 0.6169\ldots$, and every equilateral star with Dirichlet center has the same value. By item (2) the extremal configurations exist only as limits; they are the pumpkin chains of Section 2, whose bundle sizes follow an explicit weight with total mass 2 and mean 1.

Edition 2 [16] incorporated the determination of the optimal constant, first published as the separate record [15], in place of the bracket $0.6410 \le C_* \le 331/500$ of edition 1, and kept the second-eigenvalue envelope of edition 1. *This edition* adds Theorem 1.3, the sharp form of the lower bound in (1), and keeps the results of edition 2 with their theorem, section and equation numbers, the route of the argument moving from Section 1.3 to Section 1.4.

1.2 The second eigenvalue at prescribed length and inradius

For the first eigenvalue, the sharp envelope under joint constraints on total length and Dirichlet inradius places the surplus length at the distal end of a shortest path [13]. Sharp bounds under length, diameter and connectivity constraints are studied in [6, 7, 3]. For the second eigenvalue the extremal allocation depends on whether two disjoint regions of inradius R fit within the total length, and the following theorem determines the envelope.

Define $F(L,R) = \inf \lambda_2(\mathcal{G}, D)$, the infimum being over the class above with prescribed L and R; necessarily $0 < R \le L$. Put $g(A) = \arctan(2\tan A)$ for $0 \le A < \pi/2$, extended continuously by $g(\pi/2) = \pi/2$.

Theorem 1.2. *The sharp envelope is $F(L,R) = k^2$, where k is given by*

$$\begin{aligned} &k = 3\pi/(2R), && L = R,\\ &k(L-R) = g\big((3\pi/2 - kR)/2\big),\ \pi/(2R) < k < 3\pi/(2R), && R < L < 2R,\\ &\cot(kR) = k(L/2 - R),\ 0 < k \le \pi/(2R), && L \ge 2R. \end{aligned}$$

In the middle regime the minimum is attained by a path of length R with one Dirichlet endpoint and a Neumann pendant edge of length $L - R$ at distance $\big(R - \pi/(2k)\big)/2$ from that endpoint. At $L = 2R$ two intervals of length R meeting at a marked endpoint attain the value. When $L > 2R$ the infimum is approached by brooms and attained by no finite graph.

Corollary 8.1 computes the one-sided derivatives of the envelope at $L = 2R$, where the extremal geometry changes.

1.3 The sharp lower constant

The lower bound in (1) reads $\lambda_1 L\rho_D \ge 1$. Theorem 1.1 identifies the optimal constant at the upper end of the window, and the comparison raises the same question at the lower end: *what is the best constant in $\lambda_1 L\rho_D \ge 1$, that is, the value of $\inf \lambda_1(\mathcal{G}, D)\, L\, \rho_D(\mathcal{G}, D)$ over all finite connected compact metric graphs $\mathcal{G}$ and all nonempty $D \subseteq V$, and is it attained?* The interval of length ℓ with one Dirichlet endpoint has $L = \ell$, $\rho_D = \ell/2$ and $\lambda_1 L\rho_D = \pi^2/8 = 1.2337\ldots$, so a priori the infimum lies in $[1, \pi^2/8]$. Its value, 1, follows from Kennedy's length–inradius envelope [7], as explained after Theorem 1.3. The next theorem sharpens the bound itself. It adds a correction in R^2 whose coefficient $\frac{1}{2} - \frac{4}{\pi^2} = 0.0947\ldots$ is optimal, it shows that $\lambda_1 L\rho_D > 1$ on every graph, so that the infimum is never attained, and it gives the exact rate along an explicit family.

For $N \in \mathbb{N}$ the *broom* $\mathcal{B}_N$ consists of a stem of length 1, whose root is the Dirichlet set, and N Neumann arms of length $(3N)^{-1/2}$ attached at the other end of the stem; for $N = 1$ it is an interval. Write $|E(\mathcal{G})|$ for the number of edges of $\mathcal{G}$.

Theorem 1.3. *For every finite connected compact metric graph $\mathcal{G}$ and every nonempty $D \subseteq V$:*

1. $\lambda_1(\mathcal{G}, D) \ge \dfrac{1}{L\rho_D - (\frac{1}{2} - \frac{4}{\pi^2})R^2}$, *with equality if and only if $\mathcal{G}$ is a path of length R, possibly subdivided by vertices of degree two, and D consists of one of its endpoints;*
2. $\lambda_1(\mathcal{G}, D)\, L\rho_D \ge 1 + \Big(\dfrac{1}{2} - \dfrac{4}{\pi^2}\Big)\dfrac{R^2}{L\rho_D} > 1$, *and* $\lambda_1(\mathcal{G}, D)\, L\rho_D \ge 1 + \Big(\dfrac{1}{4} - \dfrac{2}{\pi^2}\Big)\dfrac{1}{|E(\mathcal{G})|}$;
3. $\inf_{(\mathcal{G},D)} \lambda_1(\mathcal{G}, D)\, L\rho_D = 1$, *a value that also follows from [7] (Proposition 9.5). The brooms satisfy $\lambda_1(\mathcal{B}_N) L\rho_D \le 1 + 1/(1 + \sqrt{3N})$ for every N, and*

$$1 + \frac{1}{1 + \sqrt{3N}} - \frac{1}{N} < \lambda_1(\mathcal{B}_N)\, L\rho_D \qquad (N \ge 3);$$

in particular $\lambda_1(\mathcal{B}_N)L\rho_D = 1 + (3N)^{-1/2} + O(N^{-1})$.

Item (1) is attained only by the interval with one Dirichlet endpoint, so its coefficient of R^2 is optimal, and by item (2) no graph attains the infimum in item (3). Together with Theorem 1.1, every $(\mathcal{G}, D)$ satisfies

$$\frac{1}{L\rho_D - (\frac{1}{2} - \frac{4}{\pi^2})R^2} \le \lambda_1(\mathcal{G}, D) < \frac{C_*}{\rho_D^2},$$

and each constant in this window is optimal. By item (2), a sequence of graphs with $\lambda_1 L\rho_D \to 1$ has $R^2/(L\rho_D) \to 0$ and an unbounded number of edges. On the brooms the Cauchy–Schwarz estimate behind (1) is nearly an equality: for the trial function $\min(d, 1)$ its loss is $\frac{1}{6}$ on the stem and $\frac{1}{6}$ on the arms (Remark 9.4), while $L\rho_D$ grows like $(N/3)^{1/2}$.

Relation to Kennedy's envelope. For $N \ge 2$ the broom $\mathcal{B}_N$ is Kennedy's star $S(L, R, N)$ [7, Definition 1.3] with $L = 1 + (N/3)^{1/2}$ and $R = 1 + (3N)^{-1/2}$. By [7, Lemma 4.1 and Proposition 1.4(1)], at fixed L and R with $L > R$ the infimum of λ_1 is Ω^2, where $\Omega = \Omega(L, R)$ is the smallest positive root of $\cot(\Omega R) = \Omega(L - R)$, and the stars $S(L, R, n)$ approach it from above as $n \to \infty$. Since $\Omega^2 LR \to 1$ as $L/R \to \infty$ (Proposition 9.5) and $\rho_D \le R$, this together with (1) already gives $\inf \lambda_1 L\rho_D = 1$. Theorem 1.3 adds the correction in R^2 with its optimal coefficient and its equality case, the strict inequality (2) with its quantitative forms, and the exact rate on $\mathcal{B}_N$. Item (1) is not a consequence of the envelope: by Proposition 9.5 it exceeds $\Omega(L, R)^2$ whenever $L\rho_D \le LR - (\frac{1}{6} + \frac{4}{\pi^2})R^2$, in particular on $\mathcal{B}_N$ for every $N \ge 4$, so the mean distance carries information beyond L and R. Theorem 1.3 is proved in Section 9.

1.4 Route through the argument

Part I proves Theorem 1.1. The coarea formula for the distance function (Lemma 2.1) shows that every graph is dominated by a weighted interval problem with the same mean distance, and pumpkin chains realize every step weight, so C_* is the supremum of a weighted Sturm–Liouville functional (Corollary 2.6). The threshold solution of the differential equation (4) in Section 3 gives the upper bound as a trial function and the lower bound as the generator of an extremal weight; the Abel form (2) follows by a change of variables. Section 4 gives the finite certificates. Part II proves Theorem 1.2 from nodal regions (Lemma 6.1) and an energy budget along a longest shortest path (Lemmas 6.2 and 6.3). Part III proves Theorem 1.3. A Cauchy–Schwarz estimate along shortest paths (Lemma 9.1) gives the lower bound in (1); its loss along one longest shortest path, measured by the mixed Dirichlet–Neumann interval (16), gives the term in R^2 (estimate (17)); the explicit ground states of the brooms (Proposition 9.3) give the rate; and Proposition 9.5 compares item (1) with Kennedy's envelope.

Part I. The optimal mean-distance constant

2 Reduction to weighted intervals

This section shows that C_* equals the supremum of a scale-free functional of weights on intervals. For the upper bound only the distance function matters; for the lower bound pumpkin chains realize the weights.

Let $T > 0$ and let w be a measurable weight on $[0, T]$ that is bounded above and below by

positive constants. Define

$$\lambda_1(w) = \inf\Big\{ \frac{\int_0^T u'^2 w}{\int_0^T u^2 w} : \ u \in H^1(0,T),\ u(0) = 0,\ u \neq 0 \Big\}, \qquad \rho(w) = \frac{\int_0^T tw}{\int_0^T w},$$

and $J(w) = \lambda_1(w)\rho(w)^2$. Both ρ and J are invariant under $w \mapsto cw$, and J is invariant under dilations $w(t) \mapsto w(t/\gamma)$, $\gamma > 0$. The infimum is attained by a positive eigenfunction of $-(wu')' = \lambda wu$ with $u(0) = 0$ and $(wu')(T) = 0$, by regular Sturm–Liouville theory [18].

For a graph, let $n_{\mathcal{G}}(t) = \#\{x \in \mathcal{G} : \ d(x) = t\}$.

Lemma 2.1. *The function $n_{\mathcal{G}}$ is finite and piecewise constant away from finitely many t, satisfies $n_{\mathcal{G}} \geq 1$ on $(0, R)$, and vanishes for $t > R$. For every bounded Borel function f,*

$$\int_{\mathcal{G}} f(d(x))\, dx = \int_0^R f(t)\, n_{\mathcal{G}}(t)\, dt.$$

Proof. On an edge $e \cong [0, \ell]$ with endpoint distances a, b, one has $d(x) = \min(a + x,\, b + \ell - x)$; the same formula covers a loop. Hence d is piecewise affine with slopes ± 1. Each affine piece contributes the indicator of its image interval to $n_{\mathcal{G}}$, which gives the coarea formula with $|d'| = 1$. A shortest path from D to a point at distance R meets every level in $(0, R)$. □

Lemma 2.2 (Distance trial functions)**.** *Suppose $h : [0, \infty) \to \mathbb{R}$ is locally Lipschitz, $h(0) = 0$, and h is nonzero on every sufficiently short positive interval. If $C > 0$ and $K \in \mathbb{R}$ satisfy*

$$h'(t)^2 - Ch(t)^2 \leq K(1 - t) \quad \text{for almost every } t \geq 0, \tag{3}$$

then $\lambda_1 \rho_D^2 \leq C$ for every graph in the setting above.

Proof. Let $u = h(d/\rho_D)$, which lies in $H^1_D(\mathcal{G})$ because d is 1-Lipschitz and vanishes on D. By Lemma 2.1, $\int_{\mathcal{G}} |u'|^2 = \rho_D^{-2} \int h'(t/\rho_D)^2 n_{\mathcal{G}}\, dt$ and $\int_{\mathcal{G}} u^2 = \int h(t/\rho_D)^2 n_{\mathcal{G}}\, dt$. Let ν be the law of d/ρ_D under normalized arclength; it has mean one. Then $\rho_D^2 \int |u'|^2 - C \int u^2 = L \int (h'^2 - Ch^2)\, d\nu \leq LK \int (1 - t)\, d\nu = 0$. Here $u \neq 0$ because ν charges every interval $[0, \varepsilon)$, and the Rayleigh principle gives the claim. □

Proposition 2.3. *For every $(\mathcal{G}, D)$, $\lambda_1(\mathcal{G}, D) \leq \lambda_1(n_{\mathcal{G}})$ and $\rho_D(\mathcal{G}, D) = \rho(n_{\mathcal{G}})$, the weight being taken on $[0, R]$.*

Proof. The second identity is Lemma 2.1 with $f(t) = t$ and $f \equiv 1$. For the first, every admissible u for the weighted problem gives the admissible $u \circ d$ on $\mathcal{G}$, with the same Rayleigh quotient by Lemma 2.1. □

Definition 2.4. Given levels $0 = t_0 < t_1 < \cdots < t_m$ and multiplicities $n_1, \ldots, n_m \in \mathbb{N}$, the *pumpkin chain* $\mathcal{P}$ has vertices $v_0, \ldots, v_m$, with n_i parallel edges of length $t_i - t_{i-1}$ joining v_{i-1} and v_i, and $D = \{v_0\}$. Subdividing every edge at its midpoint yields a simple graph with the same spectrum and the same d.

Proposition 2.5. *For a pumpkin chain, $d(x)$ equals the level of x, $n_{\mathcal{P}}$ is the step function equal to n_i on (t_{i-1}, t_i), and $\lambda_1(\mathcal{P}, D) = \lambda_1(n_{\mathcal{P}})$.*

Proof. The first two assertions are immediate. The ground state of a connected graph with nonempty Dirichlet set is simple and can be chosen positive off D [4, 8]. A permutation of the edges in one bundle is an isometry fixing D, so by simplicity the ground state is invariant under it and is a function $u(t)$ of the level alone. On each level interval u solves $-u'' = \lambda_1 u$, and the Kirchhoff condition at v_i reads $n_{i+1}u'(t_i^+) - n_i u'(t_i^-) = 0$. Hence u is a positive eigenfunction of the weighted problem with weight $n_{\mathcal{P}}$, Dirichlet at 0 and Neumann at t_m. A positive eigenfunction of a regular Sturm–Liouville problem belongs to the lowest eigenvalue [18]. □

Corollary 2.6. $C_* = \sup J(w)$*, where the supremum is over continuous* $w > 0$ *on compact intervals* $[0, T]$*, or equivalently over positive step functions.*

Proof. The inequality $\le$ is Proposition 2.3, since $n_{\mathcal{G}}$ is a positive step function on $(0, R)$, and the approximation below applies equally to step weights. For $\ge$, let w be continuous and positive on $[0, T]$, and let $\eta \in (0, \frac{1}{2})$. By uniform continuity there is a step function $\tilde{w}$ with rational breakpoints and rational positive values such that $|\tilde{w}/w - 1| \le \eta$ on $[0, T]$. Each Rayleigh quotient changes by a factor in $[(1-\eta)/(1+\eta), (1+\eta)/(1-\eta)]$, so $\lambda_1(\tilde{w})$ does too, and $|\rho(\tilde{w}) - \rho(w)| \le 2T\eta/(1-\eta)$. Multiplying $\tilde{w}$ by a common denominator leaves J unchanged and produces integer multiplicities. Proposition 2.5 then gives a graph with $\lambda_1 \rho_D^2 = J(\tilde{w}) \to J(w)$ as $\eta \to 0$. □

3 The threshold solution and the exact constant

The goal of this section is a single function that is simultaneously an admissible trial function in Lemma 2.2 with equality in (3), and the eigenfunction of a weighted problem whose mean distance tends to one. The equality case of (3) with $K = 1$ is a first-order system, and the right constant is the threshold at which its solution exists for all time.

Definition 3.1. For $C > 0$ let (h_C, G_C) be the maximal solution, on the open region $\{G > 0\}$, of

$$h' = \sqrt{G}, \qquad G' = 2Ch\sqrt{G} - 1, \qquad h(0) = 0, \quad G(0) = 1. \tag{4}$$

The constant C *survives* if $G_C > 0$ on $[0, \infty)$, and *dies* otherwise. Put $C_0 = \inf\{C > 0 : \ C \text{ survives}\}$ and $q_C = 2Ch_C\sqrt{G_C}$.

Along (4), $\frac{d}{dt}(Ch^2 + 1 - t) = 2Chh' - 1 = G'$, and both sides equal 1 at $t = 0$. Hence, as long as the solution exists,

$$G = Ch^2 + 1 - t, \qquad \text{equivalently} \qquad h'^2 - Ch^2 = 1 - t, \tag{5}$$

which is the equality case of (3) with $K = 1$.

Write $\Psi_C(h, G) = (\sqrt{G},\ 2Ch\sqrt{G} - 1)$. On $\{G > 0\}$, Ψ_C is smooth, nondecreasing in C, and quasimonotone increasing: the off-diagonal partial derivatives $\partial_G \Psi_C^{(1)} = 1/(2\sqrt{G})$ and $\partial_h \Psi_C^{(2)} = 2C\sqrt{G}$ are nonnegative. The Kamke–Müller comparison theorem for quasimonotone systems [17] therefore gives: if $C_1 < C_2$, then $h_{C_1} \le h_{C_2}$ and $G_{C_1} \le G_{C_2}$ on any interval on which both solutions exist.

Lemma 3.2. *If* $q_C(t_1) > 1$ *for some* t_1*, then* C *survives, and* G_C *and* q_C *increase on* $[t_1, \infty)$*.*

Proof. At t_1, $G' = q - 1 > 0$. As long as $G' > 0$, both h and G increase, so $q = 2Ch\sqrt{G}$ increases and G' stays positive. Hence $G \ge G(t_1) > 0$ on $[t_1, \infty)$, and Ψ_C is Lipschitz there, so the solution is global. □

Lemma 3.3. *The set of dying C is open and down-closed, it contains $(0, \frac{1}{4}]$, and 4 survives. Consequently $0 < C_0 < 4$, the dying set is $(0, C_0)$, and C survives if and only if $C \geq C_0$.*

Proof. Down-closed. Let $C_1 < C_2$, and suppose C_2 dies at $t_\dagger$, so that $G_{C_2}(t) \to 0$ as $t \uparrow t_\dagger$. If C_1 were alive on $[0, t_\dagger]$, comparison on $[0, t_\dagger)$ would give $G_{C_1}(t_\dagger) \leq 0$, which is absurd.

Open. Let C die at $t_\dagger$. Then $h \leq h(t_\dagger) =: \bar{h}$ on $[0, t_\dagger)$, and $G' = q - 1 \to -1$ as $t \uparrow t_\dagger$. Choose $\varepsilon > 0$ with $4C\bar{h}\sqrt{\varepsilon} < \frac{1}{2}$, and choose $t_0 < t_\dagger$ with $G(t_0) < \varepsilon/2$. By continuous dependence on parameters on the compact interval $[0, t_0]$, every C' close to C has $G_{C'}(t_0) < \varepsilon$ and $h_{C'}(t_0) < \bar{h} + 1$. Afterwards, while $G_{C'} \leq \varepsilon$ and $h_{C'} \leq \bar{h} + 1$, we have $G'_{C'} \leq -\frac{1}{2}$ and $h'_{C'} \leq \sqrt{\varepsilon}$. So $G_{C'}$ reaches 0 within time 2ε, during which h grows by less than 1. Hence C' dies.

Small and large C. Let $C \leq \frac{1}{4}$. As long as $G \leq 1$ on $[0, t]$, we have $h \leq t$ and therefore $G = Ch^2 + 1 - t \leq Ct^2 + 1 - t$. The right side is at most 1 for $t \leq 1/C$, so the bound $G \leq 1$ persists up to $t = 2 \leq 1/C$. There $G(2) \leq 4C - 1 \leq 0$, so C dies no later than $t = 2$. For $C = 4$: $G' \geq -1$ gives $G \geq \frac{1}{2}$ on $[0, \frac{1}{2}]$, so $h(\frac{1}{2}) \geq \frac{1}{2\sqrt{2}}$ and $q(\frac{1}{2}) \geq 8 \cdot \frac{1}{2\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = 2 > 1$, and Lemma 3.2 shows that 4 survives. □

Proposition 3.4. *Let $(h, G) = (h_{C_0}, G_{C_0})$ and $q = q_{C_0}$. Then:*

1. *$G > 0$ and $h' > 0$ on $[0, \infty)$; moreover $q \leq 1$, G is nonincreasing, $G \to 0$, and $h^2 = (t - 1 + G)/C_0 \to \infty$;*
2. *with $I(t) = \int_0^t dt'/(2G(t'))$ and $w = e^{-I}/G = 2I'e^{-I}$, the weight w is smooth and positive, $w(0) = 1$, and*
$$-(wh')' = C_0\, wh \quad \text{on } [0, \infty); \tag{6}$$
3. *$\int_0^\infty w = 2$, $\int_0^\infty t\, w \leq 4$, and for every $T > 0$*
$$\int_0^T (1 - t)\, w\, dt = w(T)h(T)h'(T) = \frac{q(T)}{2C_0}\, w(T) \in \Big[0, \frac{w(T)}{2C_0}\Big]; \tag{7}$$
in particular $\int_0^\infty (1 - t)\, w\, dt = 0$.

Proof. (1) By Lemma 3.3, C_0 survives. If $q(t_1) > 1$ for some t_1, continuous dependence on $[0, t_1]$ would give $q_C(t_1) > 1$ for some $C < C_0$, and C would survive by Lemma 3.2, contradicting Lemma 3.3. Hence $q \leq 1$ and $G' = q - 1 \leq 0$, so $G \leq 1$ and $G \downarrow G_\infty \geq 0$. The formula for h^2 is (5). If $G_\infty > 0$, then $h \geq \sqrt{G_\infty}\, t$ and $q \geq 2C_0 G_\infty t \to \infty$, which is impossible.

(2) We have $h'' = G'/(2\sqrt{G}) = C_0 h - 1/(2h')$. Moreover
$$\frac{w'}{w} = -\frac{1}{2G} - \frac{G'}{G} = \frac{1 - 4C_0 h\sqrt{G}}{2G} = -\frac{2C_0 h - 1/(2h')}{h'},$$
so $(wh')' = w'h' + wh'' = -w(2C_0 h - \frac{1}{2h'}) + w(C_0 h - \frac{1}{2h'}) = -C_0 wh$.

(3) Since $G \leq 1$, $I(t) \geq t/2$ and $I(\infty) = \infty$. Therefore $\int_0^\infty w = \int_0^\infty 2I'e^{-I} = 2$, and, since $w = -2(e^{-I})'$, integration by parts gives $\int_0^\infty tw = 2\int_0^\infty e^{-I} \leq 4$. By (5) and (6), $(whh')' = (wh')'h + wh'^2 = w(h'^2 - C_0 h^2) = (1 - t)w$, which gives (7), using $h(0) = 0$ and $whh' = wh\sqrt{G} = qw/(2C_0)$. Since w is integrable, there is a sequence $T_n \to \infty$ with $w(T_n) \to 0$, and $(1 - t)w$ is integrable, so the last claim follows. □

3.1 Proof of Theorem 1.1, items (1) and (2)

We prove $C_* = C_0$, then non-attainment, then the Abel form $C_0 = z_*^4/4$.

Upper bound. By Proposition 3.4(1), $h = h_{C_0}$ is C^1 with $h(0) = 0$ and $h' > 0$, and it satisfies $h'^2 - C_0 h^2 = 1 - t$ everywhere by (5). Lemma 2.2 with $K = 1$ gives $\lambda_1 \rho_D^2 \le C_0$, hence $C_* \le C_0$.

Non-attainment. Suppose $\lambda_1 \rho_D^2 = C_0$ for a finite $(\mathcal{G}, D)$. The proof of Lemma 2.2 then shows that $u = h(d/\rho_D)$ attains the Rayleigh minimum, so it is a first eigenfunction, hence C^1 inside every edge and Kirchhoff at every unmarked vertex. Let x^* satisfy $d(x^*) = R$, and put $\beta = h'(R/\rho_D)/\rho_D > 0$. If x^* is interior to an edge, then $d = R - |x - x^*|$ near x^*, and u has one-sided derivatives $\pm\beta$ there, a corner. If x^* is a vertex, it lies outside D because $R > 0$; it is a local maximum of d, so the outward derivative of u along every incident edge equals $-\beta$, and the Kirchhoff sum is $-\beta \deg x^* \neq 0$. Either way we have a contradiction, which proves item (2).

Lower bound. The extremal weight of Proposition 3.4 is unbounded in extent, so we truncate it and certify the eigenvalue of each truncation from below by the following comparison principle.

Lemma 3.5 (Barta–Picone [2])**.** *Let $w > 0$ be continuous on $[0, T]$, and let $\varphi \in C^1[0, T]$ be piecewise C^2 with $\varphi(0) = 0$, $\varphi'(0) > 0$, $\varphi > 0$ on $(0, T]$, and $\varphi'(T) = 0$. Suppose $w\varphi'$ is piecewise C^1. Then $\lambda_1(w) \ge \operatorname{ess\,inf}_{(0,T)} \big(-(w\varphi')'/(w\varphi)\big)$.*

Proof. For $u \in H^1$ with $u(0) = 0$, Picone's identity reads $wu'^2 - w\varphi'(u^2/\varphi)' = w(u' - u\varphi'/\varphi)^2 \ge 0$ on $(0, T]$. Integrate over $[x_0, T]$ and integrate by parts. The boundary term at T vanishes because $\varphi'(T) = 0$. The one at x_0 is $O(u(x_0)^2/x_0)$, which tends to 0 as $x_0 \to 0$ because $u(x_0)^2 \le x_0 \int_0^{x_0} u'^2$. This yields $\int wu'^2 \ge \int (-(w\varphi')'/\varphi) u^2$. □

Let h, w be as in Proposition 3.4. Fix $T > 1$ and $\delta \in (0, 1)$, and set $\varphi = h$ on $[0, T - \delta]$ and

$$\varphi(t) = h(T-\delta) + h'(T-\delta)\Big((t - T + \delta) - \frac{(t - T + \delta)^2}{2\delta}\Big) \quad \text{on } [T-\delta, T],$$

so that $\varphi'(t) = h'(T-\delta)(T-t)/\delta$. Thus $\varphi \in C^1$, $\varphi'(T) = 0$, and $0 \le \varphi' \le h'(T-\delta)$. On $[0, T-\delta]$, (6) gives $-(w\varphi')'/(w\varphi) = C_0$. On $[T-\delta, T]$, with $\Lambda_T = \max_{[T-1,T]} |w'/w|$,

$$\frac{-(w\varphi')'}{w\varphi} = \frac{-\frac{w'}{w}\varphi' - \varphi''}{\varphi} \ge \frac{h'(T-\delta)\,(\delta^{-1} - \Lambda_T)}{h(T) + h'(T-\delta)},$$

using $\varphi \le h(T-\delta) + \delta h'(T-\delta) \le h(T) + h'(T-\delta)$. Since $h' > 0$ is continuous, the right side exceeds C_0 once δ is small enough, with T fixed. Lemma 3.5 gives $\lambda_1(w|_{[0,T]}) \ge C_0$ for every $T > 1$. Take $T = T_n$ from Proposition 3.4(3). By (7),

$$\rho(w|_{[0,T_n]}) = 1 - \frac{w(T_n)h(T_n)h'(T_n)}{\int_0^{T_n} w} \longrightarrow 1,$$

so $J(w|_{[0,T_n]}) \ge C_0 \rho(w|_{[0,T_n]})^2 \to C_0$. Corollary 2.6 gives $C_* \ge C_0$. Hence $C_* = C_0$.

The Abel form. Let $\alpha = (2\sqrt{C})^{-1/2}$, and substitute $\sigma = \sqrt{C}\, h/\alpha$ and $z = \sqrt{G}/\alpha$. Since $h' > 0$ while the solution exists, h is a valid independent variable. From (4), $\frac{d\sqrt{G}}{dh} = \frac{G'}{2\sqrt{G}\,h'} = \frac{Ch}{\sqrt{G}} - \frac{1}{2G}$. In the new variables this becomes (2) with $z(0) = 1/\alpha = (2\sqrt{C})^{1/2}$, that is, $C = z(0)^4/4$. Moreover $z > 0 \iff G > 0$, and $\sigma z = \sqrt{C}\, h\sqrt{G}/\alpha^2 = 2Ch\sqrt{G} = q$. Thus "$z$ positive on $[0, \infty)$ with $\sigma z \le 1$" is exactly "C survives and $q \le 1$ for all time"; for a global solution $\sigma \to \infty$ as $t \to \infty$, since $h \to \infty$. By Proposition 3.4(1), C_0 has this property. For uniqueness, let $C > C_0$. By comparison, $h_C \ge h_{C_0}$, so (5) gives $G_C - G_{C_0} = Ch_C^2 - C_0 h_{C_0}^2 \ge (C - C_0) h_{C_0}^2 \to \infty$. If $q_C \le 1$ for all time, G_C would be nonincreasing, which is a contradiction; so $q_C > 1$ somewhere. Every $C < C_0$ dies. Since $C \mapsto z(0)$ is increasing, z_* is unique and $C_* = C_0 = z_*^4/4$, which is item (1).

4 Finite certificates

Both certificates are independent of the numerical value of C_0; the scripts listed in the Statements and Declarations rerun them.

Upper bound $C_* \le 66168/10^5$

Let $C = 66168/10^5$ and $T = 3936/1000$. The file `upper_certificate.json` lists 48 consecutive intervals $[t_{i-1}, t_i]$ with $t_i = iT/48$. On each it gives a polynomial $h(t) = \sum_{j=0}^{12} c_{i,j}\zeta^j$, $\zeta = (t - t_{i-1})/(t_i - t_{i-1})$, with rational coefficients in $10^{-12}\mathbb{Z}$, except for the constant terms, which are fixed by continuity. The script `certify_upper.py` checks in exact rational arithmetic that:

- the pieces are continuous and $h(0) = 0$;
- on each piece, every degree-22 Bernstein coefficient of $\Phi = 1 - t + Ch^2 - h'^2$ is positive (the minimum exceeds $2.9 \cdot 10^{-7}$), so $\Phi > 0$ on $[0, T]$;
- with $H = h(T)$ and $m_T = (2CH)^{-1}$, the tail $h = H + m_T(t - T)$ gives $\Phi(t) = (1 - T + CH^2 - m_T^2) + Cm_T^2(t - T)^2$ for $t \ge T$, whose constant term exceeds 0.0127.

Lemma 2.2 with $K = 1$ then gives $C_* \le C$. The polynomial data come from a floating-point fit of the threshold solution; the verification uses only the exact rational data.

Lower bound $C_* > 0.6614$

Let $\mathcal{P}$ be the pumpkin chain with 46 levels of length $1/10$ and multiplicities

$$2049, 2136, 2207, 2260, 2294, 2305, 2294, 2261, 2205, 2128, 2032, 1919,$$
$$1793, 1656, 1513, 1366, 1219, 1076, 938, 808, 688, 579, 481, 395, 320,$$
$$256, 203, 159, 122, 93, 70, 52, 38, 28, 20, 14, 10, 7, 5, 3, 2, 1, 1, 1, 1, 1.$$

These are $\max(1, \mathrm{round}(2000\, w))$ evaluated at the level midpoints, with w the extremal weight of Proposition 3.4. The chain has 40,009 edges and $\rho_D = 800377/800180$ exactly. Let $\mu = 507932624/768344051$, so that $\mu\rho_D^2 \ge 0.6614$ exactly. The script `certify_lower.py` integrates the radial problem $-(n_{\mathcal{P}}u')' = \mu n_{\mathcal{P}} u$, $u(0) = 0$, $u'(0) = 1$, level by level from the closed-form sine–cosine solution, in 50-digit interval arithmetic with all step endpoints enclosing exact rationals. It verifies that $u' > 0$ on each of 20 subintervals of every level, and that $u'(4.6) > 0$. The flux $n_{\mathcal{P}}u'$ is continuous at the levels, so $u' > 0$ on $(0, 4.6]$. Hence $u > 0$, and the Prüfer angle $\theta = \arctan(u/(n_{\mathcal{P}}u'))$ lies in $(0, \pi/2)$ at the free end. Since $\theta(4.6; \lambda)$ is strictly increasing in λ and equals $\pi/2$ at λ_1 by Prüfer's comparison theory [5], we get $\mu < \lambda_1(n_{\mathcal{P}})$. By Proposition 2.5, $\lambda_1(\mathcal{P}, D)\rho_D^2 > \mu\rho_D^2 \ge 0.6614$. This proves item (3).

Numerical value

The script `compute_C0.py` locates the threshold in two ways: 30-digit RK4 on (4), where $C = 0.661668720$ dies and $C = 0.661668721$ survives (decided by $t \approx 5.3$), and bisection on (2) with an adaptive Runge–Kutta method, giving $z_* = 1.275484393011$ and $z_*^4/4 = 0.661668720218$. These are the digits of the numerical value after Theorem 1.1; the certified statement is the enclosure of item (3).

5 Explicit trial functions and the extremal weight

Lemma 2.2 already gives strict universal constants from short explicit functions, and these show how the optimal profile is approached.

Remark 5.1 (The constants of edition 1). For $h(t) = t/\sqrt{t+2}$ and $C = 25/36$,

$$\frac{85}{162}(1-t) - h'(t)^2 + \frac{25}{36}h(t)^2 = \frac{(t-1)^2(55t^2+160t+64)}{324(t+2)^3} \geq 0,$$

so $C_* \leq 25/36$. Edition 1 used, with $T_1 = 3$ and $\zeta = t/T_1$, the sextic

$$h(t) = \frac{29996\zeta - 22452\zeta^2 + 18185\zeta^3 - 11443\zeta^4 + 4544\zeta^5 - 700\zeta^6}{10000} \tag{8}$$

on $[0, T_1]$, extended by the line through $H_1 = 1813/1000$ of slope $(2CH_1)^{-1}$, with $C = 331/500$. Writing $\Phi_1(T_1(j+\chi)/4) = \sum_{i=0}^{12} a_{j,i}\chi^i$ for $j = 0, 1, 2, 3$, where $\Phi_1 = 1 - t + Ch^2 - h'^2$, the degree-12 Bernstein coefficients

$$b_{j,p} = \sum_{i=0}^{p} a_{j,i}\frac{\binom{p}{i}}{\binom{12}{i}} \qquad (0 \leq p \leq 12) \tag{9}$$

are at least $37, 49, 20, 30$ (times 10^{-6}) on the four subintervals, and the tail constant exceeds $2421 \cdot 10^{-6}$; hence $C_* \leq 331/500$. Both functions approximate the threshold profile h_{C_0}, which is optimal by Theorem 1.1, so the pointwise method of Lemma 2.2 is sharp. Formally, the first variation of J in w on its support is proportional to $u'^2 - \lambda u^2 + 2\lambda\frac{\int u^2 w}{\rho \int w}(t-\rho)$, which vanishes exactly when $u'^2 - \lambda u^2 = K(\rho - t)$.

Remark 5.2 (A five-edge example). Five edges of length $2/3$ from a marked root to one junction, followed by two Neumann edges of length $1/3$, have $L = 4$, $\rho_D = 5/12$, and ground-state matching equation $5\cot(2k/3) = 2\tan(k/3)$, so $\lambda_1\rho_D^2 = (\frac{5}{4}\arctan\frac{\sqrt{5}}{3})^2 = 0.641045\ldots$. This was the lower bound of edition 1; the pumpkin chain of Section 4, built from the extremal weight, gives 0.6614.

Remark 5.3 (The extremal weight). The weight w of Proposition 3.4 rises from $w(0) = 1$ to a maximum $w \approx 1.153$ at $t \approx 0.55$ and then decays rapidly; it has total mass 2 and mean 1. The constant z_* is characterized by the Abel threshold of (2), an Abel equation of the second kind [11].

Part II. The second eigenvalue at prescribed length and inradius

6 Preparatory graph estimates

The lower bound in Theorem 1.2 comes from an energy budget: two nodal regions must each pay a positive energy along a longest shortest path, while the length off that path can lower the energy by at most an explicit amount. This section supplies the three estimates.

For a subgraph X and wavenumber $k > 0$, set

$$\mathcal{Q}_k(u; X) = \int_X (|u'|^2 - k^2|u|^2)\,dx.$$

We use the following one-dimensional rearrangement fact [6, 3]. A nonnegative continuous edgewise H^1 function on a connected compact metric graph of length ℓ that vanishes at an anchored point has

a nondecreasing equimeasurable rearrangement on $[0, \ell]$ vanishing at zero, with no greater Dirichlet energy. To recall the argument, almost every level between the minimum and maximum has at least one crossing on a path from the anchor to a maximum. At a regular level, the coarea formulas give the derivative of sublevel volume as the sum of $1/|u'|$ over crossings and the energy density as the sum of $|u'|$. The latter sum is at least the reciprocal of the former by Cauchy–Schwarz, and the rearranged interval has exactly that reciprocal as its energy density. Approximation by continuous edgewise piecewise-linear functions and lower semicontinuity give the H^1 assertion, with plateaus handled by the generalized inverse of sublevel volume. In particular, the Dirichlet ground eigenvalue on such a graph is at least $\pi^2/(4\ell^2)$.

Lemma 6.1 (Two regions)**.** *Let $k^2 = \lambda_2(\mathcal{G}, D)$. There exist two connected regions with disjoint edge interiors and nonzero nonnegative functions u_1, u_2 supported respectively on their closures. Each function has a nonempty Dirichlet boundary, vanishes there, and satisfies $\mathcal{Q}_k(u_i; U_i) \le 0$. The inradius of each closed region measured from its Dirichlet boundary is at most R.*

Proof. Cutting at the marked vertices decomposes the operator into the direct sum of the component operators, and each first eigenfunction is positive in its Dirichlet-cut component. If the first two eigenvalues are realized by first eigenfunctions on two components, use those components. Otherwise λ_2 is a second eigenvalue of a component that realizes λ_1. Choose a real eigenfunction orthogonal there to the positive ground state and take one of its positive nodal domains and one of its negative nodal domains. Truncating to one domain gives an admissible form function extended by zero, because the trace vanishes at each cut. Testing the weak eigenfunction equation with that truncation gives $\mathcal{Q}_k(u_i; U_i) = 0$. A sign may be changed after restriction.

In either case, a shortest path from an interior point to D meets the regional Dirichlet boundary at its first exit, or ends at an inherited Dirichlet point. Its initial segment gives the asserted regional distance bound. □

Lemma 6.2 (Off-path energy)**.** *Let Γ be a simple path in $\mathcal{G}$, and let U be a connected region as in Lemma 6.1 that meets Γ with a nonzero value. Let $\hat{u} = \max_{U \cap \Gamma} |u|$, computed on the closure, and let $d_U = |U \setminus \Gamma|$. If $kd_U < \pi/2$, then*

$$\mathcal{Q}_k(u; U \setminus \Gamma) \ge -k \tan(kd_U)\hat{u}^2.$$

Proof. Split $U \setminus \Gamma$ into components, keeping boundary endpoints at any split points. Each component has at least one boundary point at which $|u| \le \hat{u}$, since U is connected to its trace on Γ. On a component of length e, the function $v = (|u| - \hat{u})_+$ vanishes at such an anchor; also $\int |u'|^2 \ge \int |v'|^2$ and $|u| \le \hat{u} + v$. After rearranging v on $[0, e]$, the component contributes at least the minimum of $\mathcal{Q}_k(\omega; [0, e])$ over $\omega(0) = \hat{u}$. This minimum is attained by

$$\omega(x) = \hat{u}\,\frac{\cos(k(e - x))}{\cos(ke)}$$

and equals $-k \tan(ke)\hat{u}^2$. To see minimality, the cross term against variations vanishing at zero is zero, and the second variation is positive since $ke < \pi/2$. Finally, tan is superadditive on nonnegative arguments whose sum is less than $\pi/2$, and summation gives the stated bound. □

Lemma 6.3 (Axial interval energy)**.** *Let $\hat{u}$ be the absolute maximum of $|u|$ on an interval of length a. If both endpoint values vanish and $ka < \pi$, then*

$$\mathcal{Q}_k(u; [0, a]) \ge 2k \cot(ka/2)\hat{u}^2.$$

If only the left endpoint value is fixed to zero and $ka < \pi/2$, then

$$\mathcal{Q}_k(u;[0,a]) \geq k\cot(ka)\hat{u}^2.$$

Proof. Split at a point y of absolute maximum and minimize the energy with the fixed split value. In the two-sided case the resulting coefficient of $\hat{u}^2$ is $k[\cot(ky)+\cot(k(a-y))]$, which is at least $2k\cot(ka/2)$ because $\sin(ky)\sin(k(a-y)) \leq \sin(ka/2)^2$. In the one-sided case the free right endpoint gives instead $k[\cot(ky)-\tan(k(a-y))]$. The bracket is nonincreasing in y because $\sin(ky) \leq \cos(k(a-y))$, and its minimum at $y=a$ is $k\cot(ka)$. The comparisons follow by the same zero-cross-term argument as in Lemma 6.2; the homogeneous intervals used for the variations lie below their first Dirichlet or mixed eigenvalue. □

7 Proof of Theorem 1.2

7.1 The range $L \geq 2R$

Write $k^2 = \lambda_2$ and let k_0 solve the third equation in Theorem 1.2. If $kR \geq \pi/2$ the lower bound is immediate. Otherwise select the smaller of the two regions given by Lemma 6.1, of length $\ell \leq L/2$. Normalize its function to have maximum one, and take a regional shortest path of length $r \leq R$ from a Dirichlet boundary point to that maximum. On the path the minimum of $\mathcal{Q}_k$ at the fixed values $0, 1$ is $k\cot(kr)$. Off the path we discard the kinetic energy and use $|u| \leq 1$. Thus

$$0 \geq \mathcal{Q}_k(u;U) \geq k\cot(kr) - k^2(\ell - r) \geq k\cot(kR) - k^2(L/2 - R).$$

The last step uses monotonicity in ℓ and the derivative $-k^2\cot^2(kr)$ with respect to r. This proves $k \geq k_0$.

For $L = 2R$ the two mixed intervals in the statement supply the upper bound. If $L > 2R$, take two identical rooted brooms. In each broom the stem has length $R - b_N$ and there are N Neumann arms of length $b_N = (L/2 - R)/(N-1)$, with N large enough that $b_N < R$. The root is Dirichlet and the two brooms meet there. A positive radial ground state satisfies

$$\cot\big(k_N(R - b_N)\big) = N\tan(k_N b_N).$$

On the positive branch of this equation the left side decreases and the right side increases, and at its first positive root the corresponding profiles are positive. As $N \to \infty$, the first roots tend to the unique k_0, since the equation converges locally to $\cot(kR) = k(L/2 - R)$ on $0 < k < \pi/(2R)$. The coupled graph has the first eigenvalue of the broom twice, because Dirichlet conditions at the common root give a direct sum. The bound is therefore sharp.

If $L > 2R$ and equality holds in the lower bound, then $k = k_0$ has $kR < \pi/2$, and the chain of inequalities forces $\ell = L/2$ and $r = R$. The off-path portion then has positive length, and equality in the off-path estimate requires u to have zero kinetic energy and to equal its nonzero maximum there, which the equation $-u'' = k^2 u$ on its edges rules out. This proves nonattainment.

7.2 The range $R < L < 2R$

Write $\Delta = L - R$. The second equation in Theorem 1.2 has a unique root k_* in $(\pi/(2R), 3\pi/(2R))$, since its left side increases strictly and its right side decreases strictly from $\pi/2$ to 0. Put $A_* = (3\pi/2 - k_* R)/2$. We record

$$k_*\Delta = g(A_*) < \pi/2, \qquad k_*(R + 2\Delta) = 3\pi/2 + 2[g(A_*) - A_*] < 2\pi. \tag{10}$$

For the last inequality, $g(A) - A$ is nonnegative and

$$\tan(g(A) - A) = \frac{\tan A}{1 + 2\tan^2 A} \leq \frac{1}{2\sqrt{2}} < 1.$$

Suppose for contradiction that $k^2 = \lambda_2 < k_*^2$. Choose an isometric shortest path $\Gamma = [0, R]$ from D to a point at distance R, and two regions U_1, U_2 from Lemma 6.1. Their lengths are at least $\pi/(2k)$ by the length-only Dirichlet estimate. Since $k\Delta < \pi/2$, both regions have a positive-length intersection with Γ. Also $k \geq \pi/L > \pi/(2R)$ by applying the same length bound to both regions.

Each region's intersection with Γ consists of intervals with zero endpoints, except for at most one interval meeting the terminal point of Γ with a nonzero value there. Such a terminal interval has zero left endpoint and a free right endpoint. The interval decomposition follows from edgewise analyticity of eigenfunctions, allowing whole zero edges as well as isolated zeros. Write $\Delta_i = |U_i \setminus \Gamma|$; then $\Delta_1 + \Delta_2 \leq \Delta$.

Consider first a region with only two-sided zero-endpoint intervals on Γ. It has one such interval of length a with

$$k\Delta_i \geq g\big((\pi - ka)_+/2\big), \qquad a \geq \pi/k - 2\Delta_i. \tag{11}$$

If an interval has $ka \geq \pi$, it can be selected immediately. Otherwise every axial interval has nonnegative $\mathcal{Q}_k$ by the Dirichlet interval eigenvalue. Select an interval containing the axial absolute maximum. Lemmas 6.2 and 6.3, together with the nonpositive total $\mathcal{Q}_k$ of the region, give $2\cot(ka/2) \leq \tan(k\Delta_i)$. This is the first inequality in (11); the length estimate follows from it and $g(A) \geq A$.

Two disjoint intervals each satisfying the length estimate in (11) cannot occur in the two regions, because (10) gives

$$a_1 + a_2 \geq 2\pi/k - 2\Delta > R.$$

Consequently one region, say U_2, has the terminal interval, of length b. Choose the interval of U_1 of length a as above, so $a + b \leq R$. We claim that

$$k\Delta_2 \geq \big(\pi/2 - kb\big)_+. \tag{12}$$

This is immediate for $kb \geq \pi/2$. If $kb < \pi/2$, the terminal interval has nonnegative $\mathcal{Q}_k$ by the mixed interval eigenvalue. No other interval of U_2 can have length at least π/k, by the disjoint interval contradiction, so all axial contributions have nonnegative $\mathcal{Q}_k$. If the axial absolute maximum of U_2 were attained on a two-sided interval, the argument of (11) would give a disjoint second interval of length at least $\pi/k - 2\Delta_2$. Therefore the maximum is attained on the terminal interval, and Lemmas 6.2 and 6.3 give $\cot(kb) \leq \tan(k\Delta_2)$, proving (12).

Set $A = (\pi - ka)_+/2$, $B = (\pi/2 - kb)_+$, and $\tau = 3\pi/2 - kR \in (0, \pi)$. The length constraint $a + b \leq R$ gives $2A + B \geq \tau$. We have $g' \leq 2$ on $[0, \pi/2)$, since

$$g'(A) = \frac{2}{1 + 3\sin^2 A}.$$

If $A \geq \tau/2$ then $g(A) + B \geq g(\tau/2)$ by monotonicity. Otherwise $B \geq \tau - 2A$ and the same inequality follows by integrating $g' \leq 2$ from A to $\tau/2$. Combining with (11) and (12) gives

$$k\Delta \geq g(A) + B \geq g((3\pi/2 - kR)/2),$$

which contradicts $k < k_*$ and proves the lower bound.

To attain it, take $k = k_*$ and set $a = R - \pi/(2k)$. Attach one Neumann edge of length Δ to the axial path at $a/2$, with its root marked at 0. The pendant endpoint is strictly below distance R, since

$$k\Delta = g(A_*) < \pi/2 < \pi - A_* = k(R - a/2).$$

Impose an auxiliary Dirichlet condition at the axial point a. The resulting proximal piece has two axial halves of length $a/2$ with Dirichlet outer endpoints and the single Neumann pendant. Its positive ground state satisfies $2\cot(ka/2) = \tan(k\Delta)$, which is precisely the defining equation for k_*. The distal Dirichlet–Neumann interval has length $\pi/(2k)$ and the same ground eigenvalue. The two ground states extended by zero span a two-dimensional space on which the Rayleigh quotient equals k^2. The auxiliary Dirichlet form domain has codimension one and first eigenvalue k^2, so min–max gives equality in the original graph as well.

If $L = R$, a longest shortest path uses the entire length and is the full graph up to degree-two subdivision, and its root is its only marked point. The Dirichlet–Neumann interval spectrum gives $\lambda_2 = 9\pi^2/(4R^2)$, completing the proof of Theorem 1.2.

8 The transition at $L = 2R$

Corollary 8.1. *For fixed R, put $\xi = L/R$ and $\mathfrak{F}(\xi) = R^2 F(\xi R, R)$. Then*

$$\mathfrak{F}(\xi) = \frac{9\pi^2}{4}\bigl(1 - 2(\xi - 1) + O((\xi - 1)^2)\bigr) \quad \text{as } \xi \downarrow 1.$$

The one-sided derivatives at the transition are $\mathfrak{F}'(2-) = -2\pi^2/5$ and $\mathfrak{F}'(2+) = -\pi^2/4$.

Proof. Let $\kappa = kR$. Near $\xi = 1$, differentiation of $\kappa(\xi - 1) = g((3\pi/2 - \kappa)/2)$ and $g'(0) = 2$ give $\kappa'(1+) = -3\pi/2$. At $\xi = 2$ the left equation has $\kappa = \pi/2$ and $g'(\pi/2) = 1/2$, giving $\kappa'(2-) = -2\pi/5$. The right equation $\cot\kappa = \kappa(\xi - 2)/2$ gives $\kappa'(2+) = -\pi/4$. Differentiate $\mathfrak{F} = \kappa^2$. □

Theorem 1.2 concerns the second eigenvalue counted with multiplicity. It is the counterpart, for the second eigenvalue, of the length–inradius envelope of the first eigenvalue in [13].

Part III. The sharp lower constant

9 Proof of Theorem 1.3

The lower bound in (1) comes from one Cauchy–Schwarz estimate along shortest paths. We first prove that estimate, then measure its loss along a single longest shortest path, then compute the brooms, for which the loss is small, and finally compare item (1) with Kennedy's length–inradius envelope.

9.1 The pointwise estimate

A *shortest path* from D to $x \in \mathcal{G}$ is a map $\Gamma_x : [0, d(x)] \to \mathcal{G}$, parametrized by arclength, with $\Gamma_x(0) \in D$ and $\Gamma_x(d(x)) = x$; since $\mathcal{G}$ is a finite metric graph, one exists for every x. For $0 \le s \le d(x)$ the initial segment of Γ_x gives $d(\Gamma_x(s)) \le s$, and $d(x) \le d(\Gamma_x(s)) + d(x) - s$ gives the reverse inequality, so

$$d(\Gamma_x(s)) = s \qquad (0 \le s \le d(x)). \tag{13}$$

In particular Γ_x is injective, so it passes through each vertex at most once and consists of finitely many segments of edges. On an edge $e \cong [0,\ell]$ whose endpoints have distances a and b to D, the triangle inequality along e gives $|a-b| \le \ell$, and, as in the proof of Lemma 2.1,

$$d(x) = \min(a + x,\, b + \ell - x) \qquad (0 \le x \le \ell); \tag{14}$$

the same formula covers a loop. Thus d has slopes ± 1 on every edge, and it takes the value $(a+b+\ell)/2 \ge \ell/2$ at the point $x = (b+\ell-a)/2$ of e.

Lemma 9.1 (Pointwise estimate)**.** *Let $u \in H^1_D(\mathcal{G})$, $x \in \mathcal{G}$, and let Γ_x be a shortest path from D to x. Then*

$$|u(x)|^2 \le d(x) \int_0^{d(x)} \bigl|(u \circ \Gamma_x)'(s)\bigr|^2 \, ds \le d(x)\, \mathcal{E}(u). \tag{15}$$

Consequently $\int_{\mathcal{G}} |u|^2 \le L\rho_D\, \mathcal{E}(u)$, and $\lambda_1(\mathcal{G}, D) \ge 1/(L\rho_D)$.

Proof. The function $\psi = u \circ \Gamma_x$ is continuous, and it is H^1 on each of the finitely many edge segments of Γ_x; hence $\psi \in H^1(0, d(x))$ with $|\psi'| = |u'| \circ \Gamma_x$ almost everywhere. Since $\psi(0) = 0$, the Cauchy–Schwarz inequality gives $|u(x)|^2 = |\int_0^{d(x)} \psi'|^2 \le d(x) \int_0^{d(x)} |\psi'|^2$. The last integral is the energy of u on the image of Γ_x, which is at most $\mathcal{E}(u)$ because Γ_x is injective. Integrating (15) over x gives $\int_{\mathcal{G}} |u|^2 \le \mathcal{E}(u) \int_{\mathcal{G}} d = L\rho_D\, \mathcal{E}(u)$, and the Rayleigh principle gives the bound on λ_1. □

The bound $\lambda_1 \ge 1/(L\rho_D)$ is the lower bound recorded by Mugnolo [9, Section 4.2], who attributes it to Plümer [10, Theorem 4.4.3]; for a single Dirichlet point the same Cauchy–Schwarz argument is the first step of the proof of [1, Theorem 3.5]. We include the proof because Theorem 1.3 comes from an analysis of its loss.

9.2 The loss along a longest shortest path

Fix $R > 0$ and put $\varphi_R(s) = \sin(\pi s/(2R))$, so that $-\varphi_R'' = (\pi/(2R))^2 \varphi_R$, $\varphi_R(0) = 0$, $\varphi_R > 0$ on $(0, R]$ and $\varphi_R'(R) = 0$. For $\psi \in H^1(0,R)$ with $\psi(0) = 0$, integrate Picone's identity $|\psi'|^2 - \varphi_R'(|\psi|^2/\varphi_R)' = |\psi' - \psi\varphi_R'/\varphi_R|^2$ over $[x_0, R]$ as in the proof of Lemma 3.5 (with $w \equiv 1$). The boundary term at R vanishes, the one at x_0 tends to 0 as $x_0 \to 0$, and the right side converges by monotone convergence. This gives

$$\int_0^R |\psi'|^2 - \frac{\pi^2}{4R^2} \int_0^R |\psi|^2 = \int_0^R \Bigl|\psi' - \psi \frac{\varphi_R'}{\varphi_R}\Bigr|^2 \ge 0, \tag{16}$$

the variational form of the first mixed Dirichlet–Neumann eigenvalue $\pi^2/(4R^2)$ of $[0,R]$.

Proof of Theorem 1.3, items (1) and (2). Let $u \in H^1_D(\mathcal{G})$ be nonzero. Choose x^* with $d(x^*) = R$ and a shortest path $\Gamma = \Gamma_{x^*}$ from D to x^*. By (13), Γ is an isometry of $[0,R]$ onto its image, which we also denote by Γ, with $d(\Gamma(s)) = s$. Put $\psi = u \circ \Gamma$. The function $\Theta = d\,\mathcal{E}(u) - |u|^2$ is continuous and, by Lemma 9.1, nonnegative on $\mathcal{G}$. Hence, using $\int_{\mathcal{G}} d = L\rho_D$, then (16), and finally $\int_0^R |\psi'|^2 \le \mathcal{E}(u)$,

$$\begin{aligned}
\int_{\mathcal{G}} |u|^2 &= L\rho_D\, \mathcal{E}(u) - \int_{\mathcal{G}} \Theta \le L\rho_D\, \mathcal{E}(u) - \int_0^R \bigl(s\,\mathcal{E}(u) - |\psi(s)|^2\bigr)\, ds \\
&= \Bigl(L\rho_D - \frac{R^2}{2}\Bigr)\mathcal{E}(u) + \int_0^R |\psi|^2 \le \Bigl(L\rho_D - \frac{R^2}{2}\Bigr)\mathcal{E}(u) + \frac{4R^2}{\pi^2} \int_0^R |\psi'|^2 \\
&\le \Bigl(L\rho_D - \Bigl(\frac{1}{2} - \frac{4}{\pi^2}\Bigr)R^2\Bigr)\, \mathcal{E}(u).
\end{aligned} \tag{17}$$

Here $L\rho_D \geq \int_\Gamma d = R^2/2$, so the last bracket is positive, and the Rayleigh principle gives item (1) as an inequality.

Suppose equality holds in item (1), and let u be a real first eigenfunction, so that $\mathcal{E}(u) = \lambda_1 \int_{\mathcal{G}} u^2 > 0$ and every inequality in (17) is an equality. The first gives $\int_{\mathcal{G}\setminus\Gamma} \Theta = 0$, so $\Theta = 0$ on $\mathcal{G} \setminus \Gamma$. The last gives $\int_0^R |\psi'|^2 = \mathcal{E}(u)$, so $u' = 0$ almost everywhere on $\mathcal{G} \setminus \Gamma$. If the open set $\mathcal{G} \setminus \Gamma$ were nonempty, it would contain an open subinterval of an edge, on which u would be constant and $u^2 = \mathcal{E}(u)\, d$; then d would be constant there, contradicting (14), by which d has slopes ± 1. Hence $\mathcal{G} = \Gamma$ is a path of length R, and by (13) its only point in D is the endpoint $\Gamma(0)$. Conversely, a path of length R with one Dirichlet endpoint has $\lambda_1 = \pi^2/(4R^2)$ and $L\rho_D = R^2/2$, so $L\rho_D - (\frac{1}{2} - \frac{4}{\pi^2})R^2 = 4R^2/\pi^2 = 1/\lambda_1$. This proves item (1).

For item (2), item (1) gives

$$\lambda_1 L\rho_D - 1 \geq \frac{(\frac{1}{2} - \frac{4}{\pi^2})R^2}{L\rho_D - (\frac{1}{2} - \frac{4}{\pi^2})R^2} \geq \Big(\frac{1}{2} - \frac{4}{\pi^2}\Big)\frac{R^2}{L\rho_D} > 0,$$

which is the first bound. By (14), every edge of length ℓ contains a point with $d \geq \ell/2$. Hence every edge has length at most $2R$, and $L \leq 2|E(\mathcal{G})|R$. Since $\rho_D \leq R$, we get $R^2/(L\rho_D) \geq R/L \geq 1/(2|E(\mathcal{G})|)$, which gives the second bound. □

9.3 Brooms

Elementary inequalities for cot and tan control the broom eigenvalue and the comparison in Section 9.4.

Lemma 9.2. *For $0 < k \leq 1$,*

$$\cot k \geq \frac{1}{k} - \frac{k}{3} - \frac{k^3}{40} \qquad \textit{and} \qquad \tan k \leq k + k^3,$$

and for $0 < k < \pi/2$, $\cot k < 1/k - k/3$.

Proof. Integrating $\cos k \leq 1$ repeatedly from 0 gives, for $k \geq 0$, $\sin k \leq k - k^3/6 + k^5/120$ and $\cos k \geq 1 - k^2/2 + k^4/24 - k^6/720$. Multiplied by $40k \sin k > 0$, the first inequality becomes $40k\cos k \geq (40 - \frac{40}{3}k^2 - k^4)\sin k$. The bracket is positive for $k \leq 1$, so it suffices that

$$40k\Big(1 - \frac{k^2}{2} + \frac{k^4}{24} - \frac{k^6}{720}\Big) - \Big(40 - \frac{40}{3}k^2 - k^4\Big)\Big(k - \frac{k^3}{6} + \frac{k^5}{120}\Big) = \frac{k^5(1-k^2)}{9} + \frac{k^9}{120}$$

is nonnegative, which it is. Since $\cos k > 0$, the second inequality is $\sin k \leq (k + k^3)\cos k$, and it suffices that

$$(k + k^3)\Big(1 - \frac{k^2}{2} + \frac{k^4}{24} - \frac{k^6}{720}\Big) - \Big(k - \frac{k^3}{6} + \frac{k^5}{120}\Big) = k^3\Big(\frac{2}{3} - \frac{7k^2}{15} + \frac{29k^4}{720} - \frac{k^6}{720}\Big)$$

is nonnegative, which holds because the bracket is at least $\frac{2}{3} - \frac{7}{15} - \frac{1}{720} > 0$. Both polynomial identities follow by expanding. For the third inequality, put $h(k) = (1 - k^2/3)\sin k - k\cos k$. Then $h(0) = 0$ and $h'(k) = \frac{k}{3}(\sin k - k\cos k) > 0$ on $(0, \pi/2)$, because $\tan k > k$; hence $h > 0$, and dividing by $k \sin k > 0$ gives $\cot k < (1 - k^2/3)/k$. □

Proposition 9.3. *Let $N \geq 1$, $\ell_N = (3N)^{-1/2}$ and $M_N = (N/3)^{1/2}$, so that $N\ell_N = M_N$ and $M_N\ell_N = \frac{1}{3}$. The broom $\mathcal{B}_N$ has*

$$L = 1 + M_N, \qquad L\rho_D = M_N + \tfrac{2}{3}, \qquad R = 1 + \ell_N,$$

and $\lambda_1(\mathcal{B}_N) = \hat{k}^2$*, where* $\hat{k}$ *is the unique root in* $(0, \pi/2)$ *of*

$$\cot k = N \tan(k\ell_N). \tag{18}$$

Moreover $\lambda_1(\mathcal{B}_N) \le 1/(M_N + \frac{1}{3})$ *for every* N*, and*

$$\lambda_1(\mathcal{B}_N) > \frac{1 - \vartheta_N}{M_N + \frac{1}{3}}, \qquad \vartheta_N = \frac{1/40 + 1/(9M_N)}{(M_N + \frac{1}{3})^2} \qquad (N \ge 3). \tag{19}$$

Proof. The distance d runs from 0 to 1 along the stem and from 1 to $1 + \ell_N$ along each arm, so $L\rho_D = \frac{1}{2} + N(\ell_N + \frac{1}{2}\ell_N^2) = \frac{1}{2} + M_N + \frac{1}{6}$ and $R = 1 + \ell_N$.

Upper bound. The trial function $u = \min(d, 1)$ has $\mathcal{E}(u) = 1$ and $\int u^2 = \frac{1}{3} + N\ell_N = M_N + \frac{1}{3}$.

Ground state. Since $\ell_N \le 3^{-1/2} < 1$, every $k \in (0, \pi/2)$ has $k\ell_N < \pi/2$. The function equal to $\sin(kd)$ on the stem and to $\sin k \cos(k(1 + \ell_N - d))/\cos(k\ell_N)$ on each arm solves $-u'' = k^2 u$ on every edge, vanishes at the root, is continuous at the junction, satisfies the Neumann condition at the tips, and is positive off D. The Kirchhoff condition at the junction, $k \cos k = Nk \sin k \tan(k\ell_N)$, is (18). The function $\mathcal{S}_N(k) = \cot k - N \tan(k\ell_N)$ decreases strictly on $(0, \pi/2)$ from $+\infty$ to $-N \tan(\pi\ell_N/2) < 0$, so it has exactly one root $\hat{k}$. The resulting eigenfunction is positive off D, as is the ground state [4, 8], and eigenfunctions of different eigenvalues are orthogonal; hence $\hat{k}^2 = \lambda_1(\mathcal{B}_N)$.

Lower bound. Let $N \ge 3$, so that $M_N \ge 1$, $\vartheta_N < 1$ and $k_-^2 = (1 - \vartheta_N)/(M_N + \frac{1}{3}) < 1$. For $0 < k \le 1$, Lemma 9.2 and $N\ell_N^3 = M_N \ell_N^2 = 1/(9M_N)$ give

$$\mathcal{S}_N(k) \ge \frac{1}{k} - \frac{k}{3} - \frac{k^3}{40} - N(k\ell_N + k^3\ell_N^3) = \frac{1}{k}\Big(1 - k^2(M_N + \tfrac{1}{3}) - \vartheta_N(M_N + \tfrac{1}{3})^2 k^4\Big).$$

At $k = k_-$ the bracket equals $1 - (1 - \vartheta_N) - (1 - \vartheta_N)^2\vartheta_N = \vartheta_N^2(2 - \vartheta_N) > 0$. Hence $\mathcal{S}_N(k_-) > 0$, and since $\mathcal{S}_N$ decreases, $\hat{k} > k_-$. □

Proof of Theorem 1.3, item (3). By Proposition 9.3, $\lambda_1(\mathcal{B}_N) L\rho_D \le (M_N + \frac{2}{3})/(M_N + \frac{1}{3}) = 1 + 1/(3M_N + 1)$, and $3M_N = \sqrt{3N}$. For $N \ge 3$, (19) gives

$$\lambda_1(\mathcal{B}_N) L\rho_D > (1 - \vartheta_N)\Big(1 + \frac{1}{1 + \sqrt{3N}}\Big) \ge 1 + \frac{1}{1 + \sqrt{3N}} - \frac{5}{4}\vartheta_N,$$

because $1/(1 + \sqrt{3N}) \le \frac{1}{4}$; moreover $\vartheta_N < (\frac{1}{40} + \frac{1}{9})/M_N^2 = \frac{49}{360} \cdot \frac{3}{N}$, so that $\frac{5}{4}\vartheta_N < \frac{49}{96N} < \frac{1}{N}$. This is the two-sided estimate. Since $(3N)^{-1/2} - (3N)^{-1} \le 1/(1 + \sqrt{3N}) \le (3N)^{-1/2}$, it gives $\lambda_1(\mathcal{B}_N) L\rho_D = 1 + (3N)^{-1/2} + O(N^{-1})$. Together with item (2), the infimum over all $(\mathcal{G}, D)$ is 1, and it is not attained; the value 1 also follows from [7], by Proposition 9.5(1). □

Remark 9.4. Item (1) alone gives, for every N,

$$\lambda_1(\mathcal{B}_N) L\rho_D \ge \frac{M_N + \frac{2}{3}}{M_N + \frac{2}{3} - (\frac{1}{2} - \frac{4}{\pi^2})R^2} > 1 + \frac{\frac{3}{2} - \frac{12}{\pi^2}}{2 + \sqrt{3N}},$$

since $R > 1$; with the upper bound of item (3), the excess of the brooms is of exact order $N^{-1/2}$. For the trial function $\min(d, 1)$ the loss $\int_{\mathcal{G}} \Theta$ in (17) is $\frac{1}{6}$ on the stem and $N\ell_N^2/2$ on the arms; the arm length $(3N)^{-1/2}$ makes the two equal.

9.4 Comparison with the length–inradius envelope

For $0 < R < L$ let $\Omega = \Omega(L,R)$ be the smallest positive root of $\cot(\Omega R) = \Omega(L-R)$; it lies in $(0, \pi/(2R))$, where the difference of the two sides decreases strictly from $+\infty$ to a negative value. By Kennedy [7, Lemma 4.1 and Proposition 1.4(1)], every $(\mathcal{G}, D)$ with total length L and inradius R has $\lambda_1(\mathcal{G}, D) > \Omega^2$, and the stars $S(L,R,n)$ of [7, Definition 1.3] have $\lambda_1(S(L,R,n)) \downarrow \Omega^2$ as $n \to \infty$.

Proposition 9.5. *Let $0 < R < L$ and $\Omega = \Omega(L,R)$.*

1. *$1 \le \Omega^2 LR < 1 + \frac{2/3}{L/R - 2/3}$; in particular $\Omega^2 LR \to 1$ as $L/R \to \infty$, and the stars $S(L,R,n)$ give $\inf \lambda_1 L\rho_D = 1$.*
2. *If $(\mathcal{G}, D)$ has total length L, inradius R and $L\rho_D \le LR - (\frac{1}{6} + \frac{4}{\pi^2})R^2$, then*

$$\frac{1}{L\rho_D - (\frac{1}{2} - \frac{4}{\pi^2})R^2} > \Omega^2.$$

This hypothesis holds for $\mathcal{B}_N$ for every $N \ge 4$.

Proof. (1) Put $x = \Omega R \in (0, \pi/2)$. By Lemma 9.2, $x(L-R)/R = \cot x < 1/x - x/3$, that is, $x^2(L/R - \frac{2}{3}) < 1$. Hence $\Omega^2 R^2 < 1/(L/R - \frac{2}{3})$ and $\Omega^2 LR < 1 + \frac{2}{3}\Omega^2 R^2 < 1 + \frac{2/3}{L/R - 2/3}$. For the lower bound, $\rho_D \le R$ and Lemma 9.1 give $\lambda_1(S(L,R,n))\, LR \ge \lambda_1(S(L,R,n))\, L\rho_D \ge 1$, and the left side tends to $\Omega^2 LR$. The stars therefore have $\lambda_1 L\rho_D \le \lambda_1 LR$, with limit $\Omega^2 LR$, which tends to 1 as $L/R \to \infty$.

(2) Put $\mu = 1/(L\rho_D - (\frac{1}{2} - \frac{4}{\pi^2})R^2)$ and $x = \sqrt{\mu}\, R$. Since $\frac{1}{6} + \frac{4}{\pi^2} + \frac{1}{2} - \frac{4}{\pi^2} = \frac{2}{3}$, the hypothesis reads $\mu(LR - \frac{2}{3}R^2) \ge 1$, that is, $1 - x^2/3 \le x^2(L-R)/R$. Since $L > R$, the graph is not the path Γ of the proof of Theorem 1.3, so $L\rho_D = \int_{\mathcal{G}} d > \int_\Gamma d = R^2/2$ and $x^2 < \pi^2/4$. By Lemma 9.2, $\cot x < (1 - x^2/3)/x \le x(L-R)/R$, so x lies beyond the root ΩR, and $\mu > \Omega^2$.

For $\mathcal{B}_N$, Proposition 9.3 gives $L\rho_D = M_N + \frac{2}{3}$ and $LR = (1+M_N)(1+\ell_N) = M_N + \frac{4}{3} + \ell_N$, so the hypothesis is $q(\ell_N) \le 0$ with $q(\ell) = (\frac{1}{6} + \frac{4}{\pi^2})(1+\ell)^2 - \frac{2}{3} - \ell$. The function q is convex, $q(0) = \frac{4}{\pi^2} - \frac{1}{2} < 0$, and, since $\pi > 3.14$ and $0.2886 < 12^{-1/2} < 0.2887$, $q(12^{-1/2}) < 0.5724 \cdot 1.6608 - 0.9552 < 0$. Hence $q < 0$ on $[0, 12^{-1/2}]$, which contains $\ell_N = (3N)^{-1/2}$ for $N \ge 4$. □

Remark 9.6 (Flowers). Replace the arms of $\mathcal{B}_N$ by N loops of length $(2/(3N))^{1/2}$ at the end of the stem; these are the stowers of [7, Remark 1.5(a)], see also [13]. The loops have total length $(2N/3)^{1/2}$, and $L\rho_D = (2N/3)^{1/2} + \frac{2}{3}$. The trial function $\min(d, 1)$ has energy 1 and $\int u^2 = (2N/3)^{1/2} + \frac{1}{3}$, so $\lambda_1 L\rho_D \le 1 + 1/(1+\sqrt{6N})$, with $N+1$ edges as for $\mathcal{B}_N$. By the symmetry argument of Proposition 2.5, the ground state is symmetric on every loop, and its secular equation is $\cot k = 2N \tan\bigl(k(6N)^{-1/2}\bigr)$.

Remark 9.7 (The Kirchhoff analogue). For the pairwise mean distance ρ_{pr} and the standard Kirchhoff spectral gap μ_2, the analogue of the lower bound in (1) is the sharp inequality

$$\mu_2 L\rho_{\mathrm{pr}} > 2$$

of [14, Theorem 1], which improves the bound $\mu_2 L\rho_{\mathrm{pr}} \ge 1$ of Baptista, Kennedy and Mugnolo [1, Theorem 3.5]. In the Dirichlet problem the constant 1 produced by the Cauchy–Schwarz argument is already optimal (Theorem 1.3(3) and Proposition 9.5), and Theorem 1.3(1) gives its correction in R^2.

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Data and code availability. No external datasets were generated or analyzed. The following files accompany the article and reproduce its computations:

- `certify_upper.py` with the data file `upper_certificate.json`: the upper certificate of Section 4;
- `certify_lower.py`: the lower certificate of Section 4;
- `compute_C0.py`: the numerical value after Theorem 1.1;
- `verify-mean-bound.py`: the edition 1 certificate of Remark 5.1;
- `verify_lower_constant.py`: the polynomial identities of Lemma 9.2, the broom and flower data of Proposition 9.3 and Remark 9.6, and interval-arithmetic enclosures of their first eigenvalues and of Kennedy's envelope $\Omega(L, R)^2$, compared with Theorem 1.3 and Proposition 9.5; the proofs of Section 9 are complete without it.

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