MEAN DISTANCE AND THE FIRST TWO DIRICHLET EIGENVALUES OF COMPACT METRIC GRAPHS

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Abstract. We prove Mugnolo's conjecture that a universal constant $C < 1$ satisfies $\lambda_1 \rho^2 \le C$ on compact metric graphs with Dirichlet conditions on a nonempty marked set, where ρ is the mean distance to that set. A distance-based trial function gives $C = 25/36$ by a pointwise identity. A piecewise polynomial trial function with an exact rational certificate improves the estimate to $C = 331/500 < 2/3$. We also determine the sharp lower envelope of the second Dirichlet–Kirchhoff eigenvalue at prescribed total length L and Dirichlet inradius R. For $R < L < 2R$ a path of length R with one pendant edge attains the envelope. For $L > 2R$ it is approached by two equal brooms joined at their Dirichlet root and is unattained by any finite graph. Two explicit scalar equations give the envelope and show that its first derivative changes at $L/R = 2$.

1. Setting and results

Let $\mathcal{G}$ be a finite connected compact metric graph with positive edge lengths and with a nonempty set D of marked vertices. Loops and multiple edges are allowed. We use arclength measure, and write

$$L = |\mathcal{G}|, \qquad d(x) = \operatorname{dist}_{\mathcal{G}}(x, D), \qquad R = \max_{\mathcal{G}} d, \qquad \rho = \frac{1}{L}\int_{\mathcal{G}} d(x)\,dx.$$

The form domain $H^1_D(\mathcal{G})$ consists of continuous functions that are edgewise H^1 and vanish on D. The energy is $\mathcal{E}(u) = \int_{\mathcal{G}} |u'|^2$. The associated operator is the Dirichlet–Kirchhoff Laplacian. Vertices in D impose Dirichlet conditions, all other vertices impose continuity and Kirchhoff flux, and unmarked leaves impose Neumann conditions. Write its eigenvalues, with multiplicity, as $0 < \lambda_1 \le \lambda_2 \le \cdots$. A marked point in an edge is represented by subdividing the edge at that point.

Mugnolo [4, Conjecture 4.7] conjectured that $\lambda_1\rho^2$ admits a universal upper bound strictly below one. The distinction between the mean distance and the inradius is essential: the same preprint disproves any universal upper bound for $\lambda_1 R^2$. Our first result answers the mean-distance conjecture.

Theorem 1.1. *For every $(\mathcal{G}, D)$ as above,*

$$\lambda_1(\mathcal{G}, D)\rho^2 \le \frac{331}{500}.$$

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If C_* denotes the least universal constant in this inequality, then

$$\left(\frac{5}{4}\arctan\frac{\sqrt{5}}{3}\right)^2 \le C_* \le \frac{331}{500}.$$

The lower bound is approximately 0.641045.

Sharp bounds for eigenvalues of compact metric graphs have been studied under length, diameter, and connectivity constraints [2, 3, 1]. For the Dirichlet problem with joint constraints on length and inradius, the first-eigenvalue model in [5] places the surplus length at the distal end of a shortest path. We use a different allocation for the second eigenvalue. The relevant distinction is whether two disjoint regions of inradius R can fit within the total length.

Define $F(L,R) = \inf \lambda_2(\mathcal{G}, D)$, where the infimum is over the above class with the prescribed L and R. Necessarily $0 < R \le L$. Put $g(A) = \arctan(2\tan A)$ for $0 \le A < \pi/2$ and extend continuously by $g(\pi/2) = \pi/2$.

Theorem 1.2. *The sharp envelope is $F(L,R) = k^2$, where k is given by*

$$\begin{aligned}
&k = 3\pi/(2R), && L = R,\\
&k(L-R) = g\big((3\pi/2 - kR)/2\big),\ \pi/(2R) < k < 3\pi/(2R), && R < L < 2R,\\
&\cot(kR) = k(L/2 - R),\ 0 < k \le \pi/(2R), && L \ge 2R.
\end{aligned}$$

In the middle regime the minimum is attained by a path of length R with one Dirichlet endpoint and a Neumann pendant edge of length $L - R$ at distance $\big(R - \pi/(2k)\big)/2$ from that endpoint. At $L = 2R$ two intervals of length R meeting at a marked endpoint attain the value. When $L > 2R$ no finite graph attains the infimum.

2. The mean-distance inequality

Lemma 2.1 (Distance trial functions)**.** *Suppose $h : [0,\infty) \to \mathbb{R}$ is locally Lipschitz, $h(0) = 0$, and h is nonzero on every sufficiently short positive interval. If $C > 0$ and $K \in \mathbb{R}$ satisfy*

$$h'(t)^2 - Ch(t)^2 \le K(1-t) \quad \text{for almost every } t \ge 0, \tag{1}$$

then $\lambda_1\rho^2 \le C$ for every graph in the setting above.

Proof. Normalize arclength to a probability measure and set $t = d/\rho$. Then $\mathbb{E}t = 1$, and $|d'| \le 1$ almost everywhere. The function $h(d/\rho)$ is a nonzero admissible test. Indeed d traverses an interval from 0 to R along a shortest path to a point of maximal distance, and $h(0) = 0$ gives the Dirichlet trace. Rayleigh's principle gives

$$\lambda_1\rho^2 \le \frac{\mathbb{E}[h'(t)^2]}{\mathbb{E}[h(t)^2]}.$$

The chain rule holds edgewise. On each edge, d is the minimum of the two affine distance extensions from its endpoints, so the exceptional levels of the locally Lipschitz h have arclength-null preimages. Integrating (1) proves the assertion. □

A short identity already gives a strict universal improvement. For $h(t) = t/\sqrt{t+2}$ and $C = 25/36$,

$$\frac{85}{162}(1-t) - h'(t)^2 + \frac{25}{36}h(t)^2 = \frac{(t-1)^2(55t^2 + 160t + 64)}{324(t+2)^3} \ge 0.$$

To prove the stronger constant, set $C = 331/500$, $T = 3$, and write $s = t/T$ on $[0, T]$. Define

$$h(t) = \frac{29996s - 22452s^2 + 18185s^3 - 11443s^4 + 4544s^5 - 700s^6}{10000}. \tag{2}$$

For $t \geq T$, extend it by $h(t) = H + m(t - T)$, where $H = 1813/1000$ and $m = (2CH)^{-1}$. At $t = T$ the two definitions agree. We verify (1) with $K = 1$.

On $[0, T]$ put $P(t) = 1 - t + Ch(t)^2 - h'(t)^2$. For each $j = 0, 1, 2, 3$, write the polynomial $P(T(j + x)/4) = \sum_{i=0}^{12} a_{j,i} x^i$. Its r-th degree-12 Bernstein coefficient is

$$b_{j,r} = \sum_{i=0}^{r} a_{j,i} \frac{\binom{r}{i}}{\binom{12}{i}} \qquad (0 \leq r \leq 12). \tag{3}$$

The following are rational lower bounds for all thirteen coefficients on each subinterval:

j	0	1	2	3
$10^6 \min_{0 \leq r \leq 12} b_{j,r}$ at least	37	49	20	30

For completeness, these bounds can be reproduced solely by rational operations as follows. Let $c = (0, 29996, -22452, 18185, -11443, 4544, -700)/10000$, and write $v_\ell = \sum_{i+i'=\ell} c_i c_{i'}$ and $w_\ell = \sum_{i+i'=\ell+2} ii' c_i c_{i'}$, with all products outside $0 \leq i, i' \leq 6$ set to zero. Then the coefficient p_ℓ of s^ℓ in $P(Ts)$ is

$$p_\ell = \mathbf{1}_{\ell=0} - T\mathbf{1}_{\ell=1} + Cv_\ell - T^{-2} w_\ell.$$

Substitution gives $a_{j,i} = \sum_{\ell=i}^{12} p_\ell 4^{-\ell} \binom{\ell}{i} j^{\ell-i}$, where $0^0 = 1$. Equations (2) and (3) therefore give the displayed table without numerical root estimates. The Bernstein basis is nonnegative on $[0, 1]$, hence $P \geq 0$ on $[0, T]$. For $t > T$ the same expression reduces to

$$1 - T + CH^2 - m^2 + Cm^2(t - T)^2.$$

Here $1 - T + CH^2 - m^2 > 2421/10^6$, and the coefficient of the linear term vanishes by the choice of m. This completes the proof of the upper bound in Theorem 1.1.

For the lower bound on C_*, take five edges of length $2/3$ from a marked root to one junction, and two Neumann edges of length $1/3$ from that junction. The total length is 4 and integration of the root distance gives $\rho = 5/12$. The function that is equal on all five proximal edges and equal on both distal edges satisfies the ground-state matching equation

$$5 \cot(2k/3) = 2 \tan(k/3), \qquad k = 3 \arctan(\sqrt{5}/3).$$

Its sine profile on the proximal edges and its cosine profile on the distal edges are positive in the Dirichlet-cut interior. It therefore has eigenvalue $\lambda_1 = k^2$. Hence $\lambda_1 \rho^2 = (5 \arctan(\sqrt{5}/3)/4)^2$. Subdividing each proximal edge at its midpoint realizes the example on a simple equilateral graph.

3. Preparatory graph estimates

For a subgraph X and wavenumber $k > 0$, set

$$\mathcal{Q}_k(u; X) = \int_X (|u'|^2 - k^2 |u|^2)\, dx.$$

We use the following one-dimensional rearrangement fact. A nonnegative continuous edgewise H^1 function on a connected compact metric graph of length ℓ that vanishes at an anchored point has a nondecreasing equimeasurable rearrangement

on $[0, \ell]$ vanishing at zero, with no greater Dirichlet energy [2, 1]. To recall the argument, almost every level between the minimum and maximum has at least one crossing on a path from the anchor to a maximum. At a regular level s, the coarea formulas give the derivative of sublevel volume as the sum of $1/|u'|$ over crossings and the energy density as the sum of $|u'|$. The latter sum is at least the reciprocal of the former by Cauchy–Schwarz. The rearranged interval has exactly that reciprocal as its energy density. Approximation by continuous edgewise piecewise-linear functions and lower semicontinuity give the H^1 assertion, with plateaus handled by the generalized inverse of sublevel volume. In particular, the Dirichlet ground eigenvalue on such a graph is at least $\pi^2/(4\ell^2)$.

Lemma 3.1 (Two regions)**.** *Let $k^2 = \lambda_2(\mathcal{G}, D)$. There exist two connected regions with disjoint edge interiors and nonzero nonnegative functions u_1, u_2 supported respectively on their closures. Each function has a nonempty Dirichlet boundary, vanishes there, and satisfies $\mathcal{Q}_k(u_i; U_i) \le 0$. The inradius of each closed region measured from its Dirichlet boundary is at most R.*

Proof. Cutting at the marked vertices decomposes the operator into the direct sum of the component operators. Each first eigenfunction is positive in its Dirichlet-cut component. If the first two eigenvalues are realized by first eigenfunctions on two components, use those components. Otherwise λ_2 is a second eigenvalue of a component that realizes λ_1. Choose a real eigenfunction orthogonal there to the positive ground state and take one of its positive nodal domains and one of its negative nodal domains. Truncating to one domain gives an admissible form function extended by zero, because the trace vanishes at each cut. Testing the weak eigenfunction equation with that truncation gives $\mathcal{Q}_k(u_i; U_i) = 0$. A sign may be changed after restriction.

In either case, a shortest path from an interior point to D encounters the regional Dirichlet boundary at its first exit, or ends at an inherited Dirichlet point. Its initial segment gives the asserted regional distance bound. □

Lemma 3.2 (Off-path energy)**.** *Let P be a simple path in $\mathcal{G}$, and let U be a connected region as in Lemma 3.1 that meets P with a nonzero value. Let $A = \max_{U \cap P} |u|$, computed on the closure, and let $d_U = |U \setminus P|$. If $kd_U < \pi/2$, then*

$$\mathcal{Q}_k(u; U \setminus P) \ge -k \tan(kd_U) A^2.$$

Proof. Split $U \setminus P$ into components, keeping boundary endpoints at any split points. Each component has at least one boundary point at which $|u| \le A$, since U is connected to its trace on P. On a component of length e, the function $v = (|u| - A)_+$ vanishes at such an anchor. Also $\int |u'|^2 \ge \int |v'|^2$ and $|u| \le A + v$. After rearranging v on $[0, e]$, the component contributes at least the minimum of $\mathcal{Q}_k(w; [0, e])$ over $w(0) = A$. This minimum is attained by

$$w(x) = A \frac{\cos(k(e - x))}{\cos(ke)}$$

and equals $-k \tan(ke) A^2$. To see minimality, the cross term against variations vanishing at zero is zero, and the second variation is positive since $ke < \pi/2$. Finally, tan is superadditive on nonnegative arguments whose sum is less than $\pi/2$. Summation gives the stated bound. □

Lemma 3.3 (Axial interval energy)**.** *Let the absolute maximum of u on an interval of length a be A. If both endpoint values vanish and $ka < \pi$, then*

$$\mathcal{Q}_k(u;[0,a]) \geq 2k\cot(ka/2)A^2.$$

If only the left endpoint is fixed to zero and $ka < \pi/2$, then

$$\mathcal{Q}_k(u;[0,a]) \geq k\cot(ka)A^2.$$

Proof. Split at a point y of absolute maximum and minimize the energy with the fixed split value. In the two-sided case the resulting coefficient of A^2 is $k[\cot(ky) + \cot(k(a-y))]$. It is at least $2k\cot(ka/2)$, since $\sin(ky)\sin(k(a-y)) \leq \sin(ka/2)^2$. In the one-sided case the free right endpoint gives instead $k[\cot(ky) - \tan(k(a-y))]$. The bracket is nonincreasing in y because $\sin(ky) \leq \cos(k(a-y))$. The minimum at $y = a$ is $k\cot(ka)$. The comparisons follow by the same zero-cross-term argument as in Lemma 3.2; the homogeneous intervals used for the variations lie below their first Dirichlet or mixed eigenvalue. □

4. The second eigenvalue

4.1. **The range $L \geq 2R$.** Write $k^2 = \lambda_2$ and let k_0 solve the third equation in Theorem 1.2. If $kR \geq \pi/2$ the lower bound is immediate. Otherwise select the smaller of the two regions given by Lemma 3.1, of length $\ell \leq L/2$. Normalize its function to have maximum one, and take a regional shortest path of length $h \leq R$ from a Dirichlet boundary point to that maximum. On the path the minimum of $\mathcal{Q}_k$ at the fixed values $0, 1$ is $k\cot(kh)$. Off the path we discard the kinetic energy and use $|u| \leq 1$. Thus

$$0 \geq \mathcal{Q}_k(u;U) \geq k\cot(kh) - k^2(\ell - h) \geq k\cot(kR) - k^2(L/2 - R).$$

The last step uses monotonicity in ℓ and the derivative $-k^2\cot^2(kh)$ with respect to h. This proves $k \geq k_0$.

For $L = 2R$ the two mixed intervals in the statement supply the upper bound. If $L > 2R$, take two identical rooted brooms. In each broom the stem has length $R - b_n$ and there are n Neumann arms of length $b_n = (L/2 - R)/(n-1)$, with n large enough that $b_n < R$. The root is Dirichlet and the two brooms meet there. A positive radial ground state satisfies

$$\cot\big(k_n(R - b_n)\big) = n\tan(k_n b_n).$$

On the positive branch of this equation the left side decreases and the right side increases. At its first positive root the corresponding profiles are positive. As $n \to \infty$, the first roots tend to the unique k_0, since the equation converges locally to $\cot(kR) = k(L/2 - R)$ on $0 < k < \pi/(2R)$. The coupled graph has the first eigenvalue of the broom twice, because Dirichlet conditions at the common root give a direct sum. The bound is therefore sharp.

If $L > 2R$ and equality holds in the lower bound, then $k = k_0$ has $kR < \pi/2$, and the chain of inequalities forces $\ell = L/2$ and $h = R$. Its off-path portion has positive length. Equality in the estimate off the path requires u to have zero kinetic energy and to equal its nonzero maximum there. The equation $-u'' = k^2u$ on its edges rules this out. This proves nonattainment.

4.2. **The range $R < L < 2R$.** Write $\delta = L - R$. The second equation in Theorem 1.2 has a unique root k_* in $(\pi/(2R), 3\pi/(2R))$, since its left side increases strictly and its right side decreases strictly from $\pi/2$ to 0. Put $A_* = (3\pi/2 - k_*R)/2$. We record

$$k_*\delta = g(A_*) < \pi/2, \qquad k_*(R + 2\delta) = 3\pi/2 + 2[g(A_*) - A_*] < 2\pi. \tag{4}$$

For the last inequality, $g(A) - A$ is nonnegative and

$$\tan(g(A) - A) = \frac{\tan A}{1 + 2\tan^2 A} \le \frac{1}{2\sqrt{2}} < 1.$$

Suppose for contradiction that $k^2 = \lambda_2 < k_*^2$. Choose an isometric shortest path $P = [0, R]$ from D to a point at distance R, and two regions U_1, U_2 from Lemma 3.1. Their lengths are at least $\pi/(2k)$ by the length-only Dirichlet estimate. Since $k\delta < \pi/2$, both regions have a positive-length intersection with P. Also $k \ge \pi/L > \pi/(2R)$ by applying the same length bound to both regions.

Each region's intersection with P consists of intervals with zero endpoints, except for at most one interval meeting the terminal point of P with a nonzero value there. Such a terminal interval has zero left endpoint and a free right endpoint. The interval decomposition follows from edgewise analyticity of eigenfunctions, allowing whole zero edges as well as isolated zeros. Write $\delta_i = |U_i \setminus P|$; then $\delta_1 + \delta_2 \le \delta$.

Consider first a region with only two-sided zero-endpoint intervals on P. It has one such interval of length a with

$$k\delta_i \ge g\big((\pi - ka)_+/2\big), \qquad a \ge \pi/k - 2\delta_i. \tag{5}$$

If an interval has $ka \ge \pi$, it can be selected immediately. Otherwise every axial interval has nonnegative $\mathcal{Q}_k$ by the Dirichlet interval eigenvalue. Select an interval containing the axial absolute maximum A. Lemmas 3.2 and 3.3, together with the nonpositive total $\mathcal{Q}_k$ of the region, give $2\cot(ka/2) \le \tan(k\delta_i)$. This is the first inequality in (5). The length estimate follows also from $g(A) \ge A$.

Two disjoint intervals each satisfying the length estimate in (5) cannot occur in the two regions, by (4):

$$a_1 + a_2 \ge 2\pi/k - 2\delta > R.$$

Consequently one region, say U_2, has the terminal interval of length b. Choose the interval of U_1 of length a as above, so $a + b \le R$. We claim that

$$k\delta_2 \ge \big(\pi/2 - kb\big)_+. \tag{6}$$

This is immediate for $kb \ge \pi/2$. If $kb < \pi/2$, the terminal interval has nonnegative $\mathcal{Q}_k$ by the mixed interval eigenvalue. No other interval of U_2 can have length at least π/k, by the disjoint interval contradiction. Thus all axial contributions have nonnegative $\mathcal{Q}_k$. If the axial absolute maximum of U_2 were attained on a two-sided interval, the argument of (5) would give a disjoint second interval of length at least $\pi/k - 2\delta_2$. Therefore its maximum is attained on the terminal interval. Lemmas 3.2 and 3.3 give $\cot(kb) \le \tan(k\delta_2)$, proving (6).

Set $A = (\pi - ka)_+/2$, $B = (\pi/2 - kb)_+$, and $\tau = 3\pi/2 - kR \in (0, \pi)$. The length constraint gives $2A + B \ge \tau$. We have $g' \le 2$ on $[0, \pi/2)$, since

$$g'(A) = \frac{2}{1 + 3\sin^2 A}.$$

If $A \geq \tau/2$ then $g(A) + B \geq g(\tau/2)$ by monotonicity. Otherwise $B \geq \tau - 2A$ and the same inequality follows by integrating $g' \leq 2$ from A to $\tau/2$. Combining with (5) and (6) now gives

$$k\delta \geq g(A) + B \geq g((3\pi/2 - kR)/2).$$

This contradicts $k < k_*$ and proves the lower bound.

To attain it, take $k = k_*$ and set $a = R - \pi/(2k)$. Attach one Neumann edge of length δ to the axial path at $a/2$, with its root marked at 0. The pendant endpoint is strictly below distance R, since

$$k\delta = g(A_*) < \pi/2 < \pi - A_* = k\big(R - a/2\big).$$

Impose an auxiliary Dirichlet condition at the axial point a. The resulting proximal piece has two axial halves of length $a/2$ with Dirichlet outer endpoints and the single Neumann pendant. Its positive ground state satisfies $2\cot(ka/2) = \tan(k\delta)$, which is precisely the defining equation for k_*. The distal Dirichlet–Neumann interval has length $\pi/(2k)$ and the same ground eigenvalue. The two ground states extended by zero give a two-dimensional space with Rayleigh quotient k^2 everywhere away from zero. The auxiliary Dirichlet form domain has codimension one and first eigenvalue k^2, so min–max gives equality in the original graph as well.

If $L = R$, a longest shortest path uses the entire length and is the full graph up to degree-two subdivision. Its root is its only marked point. The Dirichlet–Neumann interval spectrum gives $\lambda_2 = 9\pi^2/(4R^2)$, completing the proof of Theorem 1.2.

5. Consequences and scope

Corollary 5.1. *For fixed R, put $s = L/R$ and $\mathfrak{F}(s) = R^2 F(sR, R)$. Then*

$$\mathfrak{F}(s) = \frac{9\pi^2}{4}\big(1 - 2(s-1) + O((s-1)^2)\big) \quad \text{as } s \downarrow 1.$$

The one-sided derivatives at the transition are $\mathfrak{F}'(2-) = -2\pi^2/5$ and $\mathfrak{F}'(2+) = -\pi^2/4$.

Proof. Let $\kappa = kR$ and $x = s - 1$. At $x = 0$, differentiation of $\kappa x = g((3\pi/2 - \kappa)/2)$ and $g'(0) = 2$ gives $\kappa'(0+) = -3\pi/2$. At $x = 1$ the left equation has $\kappa = \pi/2$ and $g'(\pi/2) = 1/2$, giving $\kappa'(1-) = -2\pi/5$. The right equation $\cot\kappa = \kappa(x-1)/2$ gives $\kappa'(1+) = -\pi/4$. Differentiate $\mathfrak{F} = \kappa^2$. □

Theorem 1.1 supplies a strict universal mean-distance constant and bounds its optimal value. Determining C_* remains a separate sharp-constant problem. Theorem 1.2 concerns the second eigenvalue counted with multiplicity. The transition at $L = 2R$ concerns the choice of extremizing graph geometry. The value is a squared wavenumber for the lossless wave equation on a network with the specified vertex conditions.

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