Logarithmic eigenvalue bounds on metric graphs: sharp constants for diameter and inradius, and the order for mean distance

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Abstract

Kennedy, Kurasov, Malenová and Mugnolo proved that the spectral gap μ_2 of the Kirchhoff Laplacian on a compact metric graph with diameter D and E edges satisfies $\mu_2 D^2 \le 4\pi^2 E^2$, and asked for the optimal constants for each E. The heat-kernel method of Chung, Grigor'yan and Yau gives the order $\log^2 E$ with factor 4. We prove that the optimal constant is $(1+o(1))\log^2 E$: every compact metric graph satisfies $\sqrt{\mu_2}\, D \le \log(8e^2(E+1))$, and chains of bundles of parallel edges whose sizes decay exponentially from the middle show that the coefficient 1 is optimal. The same holds with the first Betti number, or the ratio L/D of total length to diameter, in place of E. For the mean distance ρ, the product $\mu_2\rho^2$ is at most of order $\log^2 E$, and Ramanujan graphs attain this order. For the first eigenvalue λ_1 with Dirichlet conditions on a nonempty vertex set, $\lambda_1 R^2 \le \frac{1}{4}\log^2(8e^2E)$ for the inradius R, with sharp coefficient $\frac{1}{4}$, and similarly in L/R; a Betti-number bound holds once the number of Dirichlet vertices is added, which trees with Dirichlet leaves require. The optimal lower constant is of order $1/E$. These results quantify Mugnolo's examples showing that diameter, mean-distance and inradius bounds fail. The proofs combine the coarea formula for distance functions, the fact that each edge meets each level set at most twice, and a Riccati inequality for weighted Sturm–Liouville problems.

Keywords. Metric graph, quantum graph, spectral gap, Dirichlet eigenvalue, diameter, mean distance, inradius, pumpkin chain, Ramanujan graph, Hardy inequality.
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1 Introduction

1.1 Setting

Throughout, $\mathcal{G}$ is a compact connected metric graph: finitely many edges of finite positive length glued at their endpoints, loops and parallel edges allowed. We fix a model of $\mathcal{G}$ with E edges and V vertices; its first Betti number is $\beta = E - V + 1$, which does not depend on the model. Subdividing an edge by a vertex of degree two changes the model but not the metric graph. We write L for the total length, dist for the path metric, $D = \max_{x,y} \operatorname{dist}(x,y)$ for the diameter, and

$$\rho = \frac{1}{L^2}\int_{\mathcal{G}}\int_{\mathcal{G}} \operatorname{dist}(x,y)\,dx\,dy$$

for the mean distance, all integrals being with respect to arclength. The *spectral gap* is

$$\mu_2(\mathcal{G}) = \inf\Big\{ \frac{\int_{\mathcal{G}} |u'|^2}{\int_{\mathcal{G}} u^2} :\ u \in H^1(\mathcal{G}) \setminus \{0\},\ \int_{\mathcal{G}} u = 0 \Big\},$$

the first nonzero eigenvalue of the Laplacian with continuity and Kirchhoff conditions at the vertices [4]. (It is written λ_1 in [11] and λ_2 in [3, 15].) For a nonempty set V_{D} of vertices we write

$$\lambda_1(\mathcal{G}; V_{\mathrm{D}}) = \inf\Big\{ \frac{\int_{\mathcal{G}} |u'|^2}{\int_{\mathcal{G}} u^2} : \ u \in H^1(\mathcal{G}) \setminus \{0\}, \ u = 0 \text{ on } V_{\mathrm{D}} \Big\}, \qquad R = \max_{x \in \mathcal{G}} \operatorname{dist}(x, V_{\mathrm{D}}),$$

for the first eigenvalue of the Laplacian with Dirichlet conditions on V_{D} and Kirchhoff conditions elsewhere, and for the inradius.

1.2 The question

No upper bound of the form $\mu_2 \le C/D^2$ holds on metric graphs: Kennedy, Kurasov, Malenová and Mugnolo [11, Theorem 5.10] constructed chains of bundles of parallel edges with fixed diameter and $\mu_2 \to \infty$. Fixing the number of edges restores a bound. In [11, Remark 6.3] they observe that an edge of length at least D/E gives the first bound below, and that the length bound [11, (4.1)] of Nicaise gives the second:

$$\mu_2 \le \frac{4\pi^2 E^2}{D^2}, \qquad \text{and also} \qquad \mu_2 \ge \frac{\pi^2}{D^2 E^2}. \tag{1}$$

They expect both bounds to be sharp only for $E = 1$, and they pose the problem of determining the optimal constants and optimisers for each $E \ge 2$. The constants part of their problem is the following.

Question 1.1 ([11, Remark 6.3])**.** *For $E \ge 2$, determine*

$$C_E = \sup_{E(\mathcal{G})=E} \mu_2(\mathcal{G}) D(\mathcal{G})^2 \qquad \textit{and} \qquad c_E = \inf_{E(\mathcal{G})=E} \mu_2(\mathcal{G}) D(\mathcal{G})^2,$$

where $E(\mathcal{G}) = E$ means that $\mathcal{G}$ has a model with E edges.

Remark 6.3 of [11] also concerns the existence of maximisers and minimisers at fixed D and E; the results below concern the constants C_E and c_E.

In terms of total length, [11, Theorem 7.1] gives $\mu_2 D^2 \le \pi^2(4L/D - 3)$, and the authors note at the start of [11, Section 7.1] that this estimate is probably far from optimal. Berkolaiko, Kennedy, Kurasov and Mugnolo proved $\mu_2 D^2 < 24L/D$ [3, (1.14)]. Düfel, Kennedy, Mugnolo, Plümer and Täufer [8, Theorem 5.2] proved $\mu_k \le (k + \beta - 1)^2 \pi^2 / D^2$ for all $k \ge 2$, which generalises Rohleder's tree bound [16]; for $k = 2$ it reads $\mu_2 D^2 \le \pi^2(1 + \beta)^2$, recorded as [3, (1.16)], and since $\beta \le E$ it improves the upper bound of [11, Remark 6.3] to $\pi^2(E + 1)^2 < 4\pi^2 E^2$ for $E \ge 2$, as they note. These bounds are linear in L/D and quadratic in β. For the mean distance, Baptista, Kennedy and Mugnolo [2, Corollary 3.3] proved $\mu_2 \rho^2 \le \min\{\pi^2 E^2/9, \pi^2(1 + \beta)^2\}$ and remark that these bounds do not appear to be optimal. Mugnolo [15] showed with Ramanujan graphs and regular trees that none of $\mu_2 D^2$ [15, Proposition 3.5], $\mu_2 \rho^2$ [15, Proposition 3.9] and $\lambda_1(\mathcal{G}; V_{\mathrm{D}}) R^2$ [15, Theorem 4.8] is bounded on all metric graphs. Comparing these results leaves a gap: the upper bounds stated for metric graphs grow polynomially in E, β and L/D, while the known unbounded families grow only logarithmically in these parameters. The heat-kernel method of Chung, Grigor'yan and Yau gives the logarithmic order in E and in L/D with leading factor 4 (Section 1.4). This paper determines the sharp growth, with leading constants 1 for the diameter and $\frac{1}{4}$ for the inradius, and the order of growth for the mean distance.

1.3 Main results

All logarithms are natural.

Theorem 1.2 (Spectral gap and diameter)**.** *Every compact connected metric graph satisfies*

$$\sqrt{\mu_2}\, D \le \log(8e^2(E+1)), \tag{2}$$

$$\sqrt{\mu_2}\, D \le \log(8e^2(3\beta+1)), \tag{3}$$

$$\sqrt{\mu_2}\, D \le \Lambda + \log\Lambda, \qquad \Lambda = \log\frac{4e^3 L}{D}. \tag{4}$$

Theorem 1.3 (Dirichlet ground state and inradius)**.** *For every compact connected metric graph and every nonempty set V_{D} of vertices,*

$$\sqrt{\lambda_1(\mathcal{G};V_{\mathrm{D}})}\, R \le \tfrac{1}{2}\log(8e^2E), \tag{5}$$

$$\sqrt{\lambda_1(\mathcal{G};V_{\mathrm{D}})}\, R \le \tfrac{1}{2}(\Lambda+\log\Lambda), \qquad \Lambda = \log\frac{2e^3L}{R}, \tag{6}$$

$$\sqrt{\lambda_1(\mathcal{G};V_{\mathrm{D}})}\, R \le \tfrac{1}{2}\log(8e^2(3\beta+2|V_{\mathrm{D}}|-1)). \tag{7}$$

The orders, and the leading constants 1 and $\frac{1}{4}$ of $\mu_2 D^2$ and $\lambda_1 R^2$, are sharp.

Theorem 1.4 (Sharpness)**.** *For every real $T \ge 2$ there are compact metric graphs $\mathcal{P}_T$ (with one Dirichlet vertex) and $\mathcal{Q}_T$ (Kirchhoff) such that*

$$\lambda_1(\mathcal{P}_T;V_{\mathrm{D}})R^2 \ge \frac{T^2}{4(e^{1/T}+1/T)}, \quad E(\mathcal{P}_T) \le 2(T^2+1)e^T, \quad \frac{L(\mathcal{P}_T)}{R(\mathcal{P}_T)} \le 2e^T, \quad |V_{\mathrm{D}}| = 1,$$

$$\mu_2(\mathcal{Q}_T)D^2 \ge \frac{T^2}{e^{1/T}+1/T}, \qquad E(\mathcal{Q}_T) \le 4(T^2+1)e^T, \quad \frac{L(\mathcal{Q}_T)}{D(\mathcal{Q}_T)} \le 2e^T, \quad \beta(\mathcal{Q}_T) < E(\mathcal{Q}_T).$$

For every integer $h \ge 1$ the complete binary tree $\mathcal{B}_h$ of depth h with unit edges, Dirichlet conditions at its 2^h leaves and a free root has $\beta = 0$ and

$$\lambda_1(\mathcal{B}_h;V_{\mathrm{D}})R^2 \ge c_0 \log^2|V_{\mathrm{D}}|, \qquad c_0 = \frac{\arccos(2\sqrt{2}/3)^2}{\log^2 2} = 0.2404\ldots.$$

Together, Theorems 1.2–1.4 determine C_E to leading order, and Proposition 1.7 below determines the order of c_E.

Corollary 1.5 (Optimal constants)**.** *The constants of Question 1.1 satisfy*

$$\lim_{E\to\infty}\frac{C_E}{\log^2 E} = 1, \qquad \frac{1}{4E} < c_E \le \frac{16}{E-2} \quad (E \ge 3).$$

Moreover, with suprema over all compact connected metric graphs (and all nonempty Dirichlet vertex sets) obeying the stated constraint,

$$\lim_{b\to\infty}\frac{\sup_{\beta\le b}\mu_2 D^2}{\log^2 b} = \lim_{x\to\infty}\frac{\sup_{L/D\le x}\mu_2 D^2}{\log^2 x} = 1,$$

$$\lim_{E\to\infty}\frac{\sup_{E(\mathcal{G})\le E}\lambda_1 R^2}{\log^2 E} = \lim_{x\to\infty}\frac{\sup_{L/R\le x}\lambda_1 R^2}{\log^2 x} = \lim_{N\to\infty}\frac{\sup_{3\beta+2|V_{\mathrm{D}}|\le N}\lambda_1 R^2}{\log^2 N} = \frac{1}{4}.$$

For the mean distance the same orders hold, now attained by expanders.

Theorem 1.6 (Mean distance)**.** *Every compact connected metric graph satisfies*

$$\mu_2\rho^2 < \min\Big\{\log^2\big(8e^2(E+1)\big),\ \log^2\big(8e^2(3\beta+1)\big),\ (\Lambda'+\log\Lambda')^2\Big\}, \qquad \Lambda' = \log\frac{4e^3L}{\rho}.$$

Conversely, there are 3*-regular graphs with unit edges,* $E\to\infty$*,* $\beta\ge E/3$ *and* $L/\rho\le E$*, on which* $\mu_2\rho^2 \ge (c_0-o(1))\log^2 E$*, with* c_0 *as in Theorem 1.4. Hence the supremum of* $\mu_2\rho^2$ *over graphs with at most* E *edges lies between* $(c_0-o(1))\log^2 E$ *and* $\log^2(8e^2(E+1))$*, and the same two-sided order holds in* β *and in* L/ρ*.*

Proposition 1.7 (Lower side)**.** *Every compact connected metric graph satisfies* $\mu_2 D^2 > 1/(4E)$*. For every* $E\ge 3$ *there is a graph with* E *edges and* $\mu_2 D^2 \le 16/(E-2)$*.*

Corollary 1.5 determines C_E to leading order, $C_E = (1+o(1))\log^2 E$, and c_E up to absolute factors, $c_E \asymp 1/E$; it replaces the quadratic growth $4\pi^2E^2$ of (1), and $\pi^2(E+1)^2$ of [8, Theorem 5.2], and the decay π^2/E^2 of (1), by the true orders. In particular the bounds in (1) are strict: since $\log(8e^2(E+1)) < 2\pi E$ for $E\ge 1$, Theorem 1.2 gives $\mu_2 D^2 < 4\pi^2E^2$ for every $E\ge 1$ (for $E\ge 2$ this also follows from [8, Theorem 5.2]), and Proposition 1.7 gives $\mu_2 D^2 > 1/(4E) > \pi^2/E^2$ for every $E\ge 40$. In the length–diameter setting of [11, Section 7.1] the optimal growth is $\log^2(L/D)$, again with leading constant 1, which replaces the linear growth of [11, Theorem 7.1] and [3, (1.14)]; in the Betti number the optimal growth is $\log^2\beta$, which replaces the quadratic growth of [8, Theorem 5.2] and of [2, Corollary 3.3]: the bound (3), $\sqrt{\mu_2}\,D \le \log(8e^2(3\beta+1))$, is smaller than the bound $\pi(1+\beta)$ of [8, Theorem 5.2] for every $\beta\ge 1$. Theorem 1.6 shows that the largest value of $\mu_2\rho^2$ over graphs with E edges is of order $\log^2 E$, with leading constant between c_0 and 1. This determines the optimal order of the dependence on E and on β in [2, Corollary 3.3], whose authors remark that these bounds do not appear to be optimal, and it quantifies [15, Proposition 3.9]. Theorem 1.3 and Theorem 1.4 quantify [15, Theorem 4.8]: along every family with $\lambda_1R^2\to\infty$ the number of edges grows at least like $e^{2\sqrt{\lambda_1}R}/(8e^2)$, and λ_1R^2 is bounded in terms of β and $|V_{\mathrm D}|$ together. As in the bipartite case of [15, Theorem 4.8], trees with Dirichlet leaves show that β alone does not suffice, and the trees $\mathcal{B}_h$ show that the order $\log^2|V_{\mathrm D}|$ in (7) is attained, with constant at least c_0.

Two of our earlier results give context. On trees, $\mu_2\rho^2 < \pi^2/4$ and the constant is sharp [17], so the growth in Theorem 1.6 is driven by the first Betti number. For the mean distance $\rho_{\mathrm D}$ to $V_{\mathrm D}$, every graph satisfies $\lambda_1\rho_{\mathrm D}^2 \le 0.6617$ [18]; the chains $\mathcal{P}_T$ of Theorem 1.4 have $R=T$ and $\rho_{\mathrm D} < 3$ (Proposition 5.4), so the growth in Theorem 1.3 reflects the gap between the inradius and the mean distance to the Dirichlet set.

A concrete consequence: if a metric graph has spectral gap $\mu_2 = K\pi^2/D^2$ with $K\ge 1$, that is, K times the value for an interval of the same diameter, then by Theorem 1.2

$$E+1 \ge \frac{e^{\pi\sqrt{K}}}{8e^2}, \qquad 3\beta+1 \ge \frac{e^{\pi\sqrt{K}}}{8e^2}, \qquad \frac{L}{D} \ge \frac{e^{\pi\sqrt{K}}}{4e^3\pi\sqrt{K}},$$

the last because $\Lambda+\log\Lambda \ge \pi\sqrt{K}$ forces $\Lambda \ge \pi\sqrt{K} - \log(\pi\sqrt{K})$. By Theorem 1.4 these exponential thresholds are attained up to polynomial factors.

1.4 Relation to earlier work and route of the proof

For combinatorial graphs, upper bounds on the diameter in terms of the spectral gap and $\log n$ go back to Alon and Milman [1] and Chung [6], and Mohar [13] treated the mean distance. Chung,

Grigor'yan and Yau [7, (0.1)] proved, on compact manifolds, that

$$\lambda_1 \le \frac{4}{\operatorname{dist}(X,Y)^2}\log^2\frac{2\mu(M)}{\sqrt{\mu(X)\mu(Y)}}$$

for disjoint sets X, Y, using the heat kernel. Their proof uses only the spectral expansion of the heat kernel and the integrated Gaussian estimate [7, (1.4)], which they derive from Grigor'yan's integral maximum principle [9]: for a solution u of the heat equation and $\xi = \operatorname{dist}(\cdot,Y)^2/(2(t+\varepsilon))$, the integral $\int u^2 e^{\xi}$ is nonincreasing in t. The computation

$$\frac{d}{dt}\int u^2e^{\xi} = -2\int u'\,(ue^{\xi})' + \int u^2\xi_t e^{\xi} = -2\int \left(u' + \tfrac{1}{2}u\xi'\right)^2 e^{\xi} + \int u^2e^{\xi}\left(\xi_t + \tfrac{1}{2}\xi'^2\right) \le 0$$

uses only integration by parts against the form $\int u'v'$ and $|\operatorname{dist}(\cdot,Y)'| \le 1$, so it holds verbatim for the Kirchhoff Laplacian on every compact metric graph, and so does [7, (0.1)]. Small balls of radius $s = D/(2\log(L/D))$ around the ends of a diameter then give $\mu_2 D^2 \le (4+o(1))\log^2(L/D)$, and, since $L \le 2DE$ (Section 6), $C_E \le (4+o(1))\log^2 E$. The present paper identifies the sharp leading constant 1 in these two bounds, proves the Betti-number bounds (which the length–diameter route does not give, since pendant trees make L/D unbounded at fixed β), the Dirichlet and mean-distance bounds, and the matching extremal families. Borthwick, Corsi and Jones [5] found the sharp bound $\mu_2 \le \pi^2(V+2)^2/(4D^2)$ in terms of the number of vertices; it concerns the regime of many parallel edges, complementary to the present one. Düfel, Kennedy, Mugnolo, Plümer and Täufer [8, Theorem 5.2] bounded every eigenvalue by $\mu_k \le (k+\beta-1)^2\pi^2/D^2$, also on infinite graphs; for the spectral gap, (3) improves the dependence on β from $\pi(1+\beta)$ to $\log(8e^2(3\beta+1))$ for $\sqrt{\mu_2}\,D$, and Theorem 1.4 shows that this logarithmic growth, with leading constant 1, is optimal.

The proofs have three ingredients.

1. A Riccati inequality (Lemma 2.1) for the ground state of a weighted Sturm–Liouville problem: the ratio of flux to amplitude decays at least like $e^{-2\sqrt{\lambda}t}$. It yields the sharp logarithmic bounds of Corollary 2.2 for weights bounded between 1 and M, or of mass at most m.

2. The coarea formula for distance functions, as in the reduction of pumpkin chains to weighted intervals in [11, Section 5], together with the observation that the distance to a vertex set is a tent function on each edge, so that each edge meets each level set at most twice, as in [15, proof of Lemma 4.11] (Lemma 3.1). Radial test functions on two disjoint balls of radius $D/2$ then give Theorem 1.2; the Betti-number bounds use the skeleton obtained by pruning leaves (Lemma 3.4).

3. For sharpness, quantitative versions of the exponential pumpkin chains of [11, Theorem 5.10] (Section 5), a ground-state identity for exponential weights (Lemma 5.1), a Barta certificate for binary trees (Lemma 5.5), and for the mean distance, Ramanujan graphs [12].

Sections 2–6 contain the proofs, and Section 7 lists the computational checks.

2 A Riccati inequality for weighted intervals

Let $r > 0$ and let w be a measurable function on $[0,r]$ with $0 < \operatorname{ess\,inf} w \le \operatorname{ess\,sup} w < \infty$. Put

$$\lambda(w) = \inf\Big\{\frac{\int_0^r w\,u'^2}{\int_0^r w\,u^2} :\ u \in H^1(0,r)\setminus\{0\},\ u(0)=0\Big\},$$

the first eigenvalue of $-(wu')' = \lambda wu$ with a Dirichlet condition at 0 and the natural condition $(wu')(r) = 0$ at r. For $0 \le s \le r$ let

$$\widetilde{R}(s) = \int_0^s \frac{dt}{w(t)}, \qquad W(s) = \int_s^r w(t)\,dt.$$

Lemma 2.1 (Riccati inequality)**.** *For all $0 < s_1 < s_2 < r$,*

$$\lambda(w)\,\widetilde{R}(s_1)\,W(s_2)\,e^{2\sqrt{\lambda(w)}\,(s_2-s_1)} \le 1.$$

Proof. Write $\lambda = \lambda(w)$. The quotient is bounded below by $\operatorname{ess\,inf} w / \operatorname{ess\,sup} w$ times the unweighted quotient, and the embedding $H^1(0,r) \subset L^2(0,r)$ is compact, so the infimum is attained by some u. Since $|u|$ has the same quotient, we may take $u \ge 0$. The first variation gives $\int_0^r wu'\varphi' = \lambda \int_0^r wu\varphi$ for every $\varphi \in H^1(0,r)$ with $\varphi(0) = 0$. Put $q(t) = \lambda \int_t^r wu$. Integration by parts gives $\int_0^r q\varphi' = \lambda \int_0^r wu\varphi$ for the same φ, hence $\int_0^r (wu' - q)\psi = 0$ for every $\psi \in L^2$, taking $\varphi(t) = \int_0^t \psi$. Thus $p := wu'$ agrees almost everywhere with the absolutely continuous function q, and we use this representative: $p' = -\lambda wu \le 0$ and $p(r) = 0$.

Consequently $p \ge 0$, so u is nondecreasing. If $u(t_0) = 0$ for some $t_0 > 0$, then $u = 0$ on $[0, t_0]$, so $p = 0$ there and $\int_{t_0}^r wu = 0$, which forces $u \equiv 0$. Hence $u > 0$ on $(0, r]$ and $p(t) = \lambda \int_t^r wu > 0$ on $[0, r)$.

On $[s_1, s_2]$ the function $y = p/u$ is positive and absolutely continuous, and

$$y' = \frac{p'}{u} - \frac{p\,u'}{u^2} = -\lambda w - \frac{y^2}{w} \le -2\sqrt{\lambda}\,y$$

by the inequality of arithmetic and geometric means. Hence $y(s_2) \le y(s_1) e^{-2\sqrt{\lambda}(s_2 - s_1)}$. Since p is nonincreasing, $u(s_1) = \int_0^{s_1} p/w \ge p(s_1)\widetilde{R}(s_1)$, that is, $y(s_1) \le 1/\widetilde{R}(s_1)$. Since u is nondecreasing, $p(s_2) = \lambda \int_{s_2}^r wu \ge \lambda u(s_2) W(s_2)$, that is, $y(s_2) \ge \lambda W(s_2)$. Combining the three inequalities gives the claim. □

Corollary 2.2. *Let $w \ge 1$ almost everywhere on $[0, r]$.*

1. *If $w \le M$ almost everywhere, then $\sqrt{\lambda(w)}\,r \le \frac{1}{2}\log(4e^2 M)$.*
2. *If $\int_0^r w \le m$, then $\sqrt{\lambda(w)}\,r \le \frac{1}{2}(\Lambda + \log\Lambda)$ with $\Lambda = \log(2e^3 m/r)$.*

The same bounds hold for the first eigenvalue with the Dirichlet condition at r and the natural condition at 0.

Proof. Put $x = \sqrt{\lambda(w)}\,r$, so that $\lambda(w) = x^2/r^2$.

(1) If $x \le 1$ there is nothing to prove, since $\log(4e^2 M) \ge \log(4e^2) > 2$. If $x > 1$, take $s_1 = r/(2x)$ and $s_2 = r - r/(2x)$, so that $s_1 < s_2$. From $w \le M$ and $w \ge 1$ we get $\widetilde{R}(s_1) \ge s_1/M$ and $W(s_2) \ge r - s_2$, and Lemma 2.1 gives

$$\frac{x^2}{r^2} \cdot \frac{r}{2xM} \cdot \frac{r}{2x} \cdot e^{2x-2} \le 1, \qquad \text{that is,} \qquad 2x \le \log(4e^2 M).$$

(2) Since $w \ge 1$ we have $m \ge r$, so $\Lambda \ge \log(2e^3)$ and $\frac{1}{2}(\Lambda + \log\Lambda) > \frac{3}{2}$. If $x \le \frac{3}{2}$ we are done. If $x > \frac{3}{2}$, take $s_1 = r/x$ and $s_2 = r - r/(2x)$, so that $s_1 < s_2$. By the Cauchy–Schwarz inequality, $s_1^2 \le \int_0^{s_1} w \cdot \widetilde{R}(s_1) \le m\,\widetilde{R}(s_1)$, and $W(s_2) \ge r - s_2$. Lemma 2.1 gives

$$\frac{x^2}{r^2} \cdot \frac{r^2}{x^2 m} \cdot \frac{r}{2x} \cdot e^{2x-3} \le 1, \qquad \text{that is,} \qquad 2x - \log x \le \Lambda.$$

As $\log x < x$, this gives $x < \Lambda$, and then $x \le \frac{1}{2}(\Lambda + \log x) \le \frac{1}{2}(\Lambda + \log \Lambda)$.

The last statement follows from the substitution $t \mapsto r - t$, which preserves the hypotheses on w. □

Remark 2.3. The coefficient $\frac{1}{2}$ in Corollary 2.2 is optimal. For $w(t) = e^{c(r-t)}$ one has $M = e^{cr}$, and Lemma 5.1 below gives $\lambda(w) \ge c^2/4$, that is, $\sqrt{\lambda(w)}\, r \ge \frac{1}{2} \log M$. The quantity $\sup_s \widetilde{R}(s)W(s)$ is the Muckenhoupt constant of the weighted Hardy inequality on $[0, r]$ [14]; the truncated test function $\min\{\widetilde{R}, \widetilde{R}(s)\}$ gives $\lambda(w) \le 1/(\widetilde{R}(s)W(s))$. Lemma 2.1 strengthens this by the factor $e^{2\sqrt{\lambda}(s_2 - s_1)}$, which is what produces the sharp coefficient.

3 Level sets of distance functions and skeletons

Lemma 3.1 (Level counts)**.** *Let $\mathcal{G}$ have a model with E edges, let S be a nonempty set of vertices of that model, let $d = \operatorname{dist}(\cdot, S)$ and $R_S = \max_{\mathcal{G}} d$, and let $n(t) = \#\{x \in \mathcal{G} : \ d(x) = t\}$. Then, for all $t \in (0, R_S)$ outside a finite set,*

$$1 \le n(t) \le 2E.$$

Moreover, for every $\varphi \in H^1(0, R_S)$ the function $\varphi \circ d$ lies in $H^1(\mathcal{G})$, and

$$\int_{\mathcal{G}} \big|(\varphi \circ d)'\big|^2 = \int_0^{R_S} \varphi'(t)^2 n(t)\, dt, \qquad \int_{\mathcal{G}} (\varphi \circ d)^2 = \int_0^{R_S} \varphi(t)^2 n(t)\, dt, \qquad \int_{\mathcal{G}} 1 = \int_0^{R_S} n(t)\, dt.$$

Proof. Identify an edge e with $[0, \ell]$, and let a, b be the values of d at its endpoints (equal if e is a loop). A path from an interior point of e to S leaves e through an endpoint, because S consists of vertices, so $d(x) = \min(a + x,\, b + \ell - x)$ on e. Since $|a - b| \le \ell$, this function increases with slope 1 on $[0, x_*]$ and decreases with slope -1 on $[x_*, \ell]$, where $x_* = (b + \ell - a)/2$. Each of the two pieces meets each level set at most once, so e contributes at most 2 to $n(t)$, and $n \le 2E$. The change of variables $t = d(x)$ on each piece, where $|d'| = 1$, gives the three integral identities, and $\varphi \circ d$ is continuous and edgewise H^1. Finally, d is continuous on a path from S to a point where $d = R_S$ and takes the value 0 at its start, so that path meets every level $t \in (0, R_S)$, and $n \ge 1$. □

Distances to an arbitrary point $x_0 \in \mathcal{G}$ are covered by subdividing the edge containing x_0, which produces a model with at most $E + 1$ edges in which x_0 is a vertex; then $n \le 2(E + 1)$.

Remark 3.2. The bound $n \le 2E$ is attained: for a flower of E loops of equal length at a Dirichlet vertex, $n(t) = 2E$ for every $t \in (0, R_S)$. The coarea identities are the mechanism behind the reduction of pumpkin chains to weighted intervals in [11, Section 5]. The level-count bound $n \le 2E$ is observed in the proof of [15, Lemma 4.11]; here it is combined with Lemma 2.1.

To handle pendant trees we prune leaves. Call a set $F \subset \mathcal{G}$ *marked* if it is finite and nonempty; subdividing edges, we may assume that the points of F are vertices of the model.

Definition 3.3. Let $F \subset \mathcal{G}$ be marked. The *skeleton* S_F is obtained by repeatedly deleting a vertex of degree one that does not belong to F, together with its edge, until no such vertex is left.

Lemma 3.4 (Skeletons)**.** *Let $F \subset \mathcal{G}$ be marked with at least two points, and $S = S_F$. Then:*

1. *S is connected, contains F, and has the same first Betti number β as $\mathcal{G}$; every leaf of S lies in F;*
2. *the closure of each connected component of $\mathcal{G} \setminus S$ is a tree that meets S in exactly one point; in particular there is a map $\pi : \mathcal{G} \to S$, equal to the identity on S and constant on each such tree, and $\operatorname{dist}_S = \operatorname{dist}_{\mathcal{G}}$ on $S \times S$;*

3. S has a model, with every point of F a vertex, having at most $3\beta + 2|F| - 3$ edges.

Proof. (1) Deleting a leaf with its edge preserves connectedness and lowers the numbers of edges and vertices by one each, so β is unchanged; the procedure stops only when every leaf lies in F.

(2) The deleted edges form a forest. By induction on the number of deletions, each tree of this forest is attached to the remaining graph at a single vertex: when a leaf edge is deleted it is attached to the rest only at its other endpoint. A path between two points of S that enters one of these trees enters and leaves it through its attachment point, and removing that excursion shortens the path, so shortest paths between points of S stay in S.

(3) Take as vertices of S the points of F and the points of S of degree at least three in S; if S is a cycle, F supplies at least one vertex. Let B be the set of vertices of degree at least three that are not in F. Every vertex has degree at least one, and the degree-sum formula for this model reads $\sum_v(\deg v - 2) = 2E_S - 2V_S = 2(\beta - 1)$. The vertices in F contribute at least $-|F|$ and those in B at least $|B|$, so $|B| \le 2\beta - 2 + |F|$. Hence $V_S \le 2\beta - 2 + 2|F|$ and $E_S = V_S + \beta - 1 \le 3\beta + 2|F| - 3$. □

4 Proofs of the upper bounds

Proof of Theorem 1.2. Choose $x_1, x_2 \in \mathcal{G}$ with $\operatorname{dist}(x_1, x_2) = D$, and let $B_i = \{x : \operatorname{dist}(x, x_i) < D/2\}$. By the triangle inequality $B_1 \cap B_2 = \emptyset$.

Let $\Gamma \subset \mathcal{G}$ be a closed connected set containing x_1 and x_2 on which $\operatorname{dist}_\Gamma = \operatorname{dist}_\mathcal{G}$, together with a map $\pi : \mathcal{G} \to \Gamma$ as in Lemma 3.4(2) (for (2) and (4) take $\Gamma = \mathcal{G}$ and π the identity; for (3) take $\Gamma = S_F$ with $F = \{x_1, x_2\}$). Let $n_i(t)$ be the number of points of Γ at distance t from x_i, and let $\lambda^{(i)}$ be the first eigenvalue of the weighted problem on $[0, D/2]$ with weight n_i, the Dirichlet condition at $D/2$ and the natural condition at 0. Since Γ contains a shortest path from x_1 to x_2, Lemma 3.1 gives $n_i \ge 1$ on $(0, D/2)$.

For $\varphi_i \in H^1(0, D/2)$ with $\varphi_i(D/2) = 0$ put $g_i = \varphi_i(\operatorname{dist}(x_i, \cdot))$ on $\Gamma \cap B_i$, $g_i = 0$ on the rest of Γ, and $f_i = g_i \circ \pi$. Then $f_i \in H^1(\mathcal{G})$, $f_i' = 0$ off Γ, and $f_i = g_i$ on Γ, so by Lemma 3.1

$$\frac{\int_\mathcal{G} |f_i'|^2}{\int_\mathcal{G} f_i^2} \le \frac{\int_\Gamma |g_i'|^2}{\int_\Gamma g_i^2} = \frac{\int_0^{D/2} \varphi_i'^2 n_i}{\int_0^{D/2} \varphi_i^2 n_i}.$$

The functions f_1 and f_2 vanish outside the disjoint sets $\pi^{-1}(B_1 \cap \Gamma)$ and $\pi^{-1}(B_2 \cap \Gamma)$, so the Rayleigh quotient of every nonzero combination of them is at most the larger of their two quotients. By the min–max principle for the second eigenvalue, $\mu_2 \le \max_i \lambda^{(i)}$ after optimizing over φ_i.

We now bound $\lambda^{(i)}$ with Corollary 2.2, applied with $r = D/2$ and the Dirichlet condition at r.

- For (2), $\Gamma = \mathcal{G}$. Subdividing only the edge that contains x_i gives a model with at most $E + 1$ edges in which x_i is a vertex, so $n_i \le 2(E + 1)$ by Lemma 3.1. Part (1) with $M = 2(E + 1)$ gives $\sqrt{\lambda^{(i)}}\, D/2 \le \frac{1}{2}\log(8e^2(E + 1))$.

- For (3), subdivide at x_1 and x_2, which does not change β, and let $\Gamma = S_F$. By Lemma 3.4(3) with $|F| = 2$, Γ has a model with at most $3\beta + 1$ edges in which x_i is a vertex, so $n_i \le 2(3\beta + 1)$. Part (1) gives $\sqrt{\lambda^{(i)}}\, D/2 \le \frac{1}{2}\log(8e^2(3\beta + 1))$.

- For (4), $\int_0^{D/2} n_i = |B_i| \le L$. Part (2) with $m = L$ and $r = D/2$ gives $\sqrt{\lambda^{(i)}}\, D/2 \le \frac{1}{2}(\Lambda + \log\Lambda)$ with $\Lambda = \log(4e^3 L/D)$.

Multiplying by 2 gives the three bounds. □

Proof of Theorem 1.3. Let $z \in \mathcal{G}$ satisfy $\operatorname{dist}(z, V_{\mathrm{D}}) = R$. As above, let Γ be either $\mathcal{G}$ with its given model (for (5) and (6)) or, after subdividing at z, the skeleton S_F with $F = V_{\mathrm{D}} \cup \{z\}$ (for (7)), with its map π. Every shortest path from a point of Γ to V_{D} stays in Γ by Lemma 3.4(2), and $z \in \Gamma$, so the distance to V_{D} restricted to Γ is the distance within Γ, with maximum R. Let n be its level-count function on Γ and λ the first eigenvalue of the weighted problem on $[0, R]$ with weight n, the Dirichlet condition at 0 and the natural condition at R. For $\varphi \in H^1(0, R)$ with $\varphi(0) = 0$, the function $f = \varphi(\operatorname{dist}(\pi(\cdot), V_{\mathrm{D}}))$ vanishes on V_{D} and, by Lemma 3.1, has Rayleigh quotient at most $\int \varphi'^2 n / \int \varphi^2 n$. Hence $\lambda_1(\mathcal{G}; V_{\mathrm{D}}) \le \lambda$.

For (5), $n \le 2E$, and Corollary 2.2(1) with $M = 2E$ gives $\sqrt{\lambda}\, R \le \frac{1}{2}\log(8e^2E)$. For (6), $\int_0^R n = L$, and part (2) with $m = L$ gives the bound. For (7), the model of Lemma 3.4(3) has every Dirichlet vertex as a vertex and at most $3\beta + 2(|V_{\mathrm{D}}| + 1) - 3 = 3\beta + 2|V_{\mathrm{D}}| - 1$ edges, so $n \le 2(3\beta + 2|V_{\mathrm{D}}| - 1)$ and part (1) gives the bound. □

Remark 4.1. For a tree with a single Dirichlet vertex, the skeleton with $F = V_{\mathrm{D}} \cup \{z\}$ is one shortest path, so $n \equiv 1$ and the proof gives $\lambda_1 R^2 \le \pi^2/4$, the value of an interval with one Dirichlet end.

Proof of the upper bounds in Theorem 1.6. The function $\operatorname{dist}(x, y)$ is at most D and is smaller than $D/2$ on a neighbourhood of the diagonal, so $\rho < D$, and the first two bounds follow from (2) and (3) (compare [2, Theorem 3.1]). For the third, $t \mapsto t + \log t$ is increasing on $[1, \infty)$, and $\Lambda \le \Lambda'$ because $\rho < D$, so (4) gives $\sqrt{\mu_2}\,\rho < \sqrt{\mu_2}\, D \le \Lambda' + \log \Lambda'$. □

5 Sharpness

Lemma 5.1 (Exponential weights)**.** *Let $0 \le z < T$, $c > 0$, $K > 0$ and $\omega(t) = Ke^{-ct}$ on $[z, T]$.*

1. *If $u \in H^1(z, T)$ and $u(z) = 0$, then $\int_z^T \omega u'^2 \ge \frac{c^2}{4} \int_z^T \omega u^2$.*
2. *If $u \in H^1(0, T)$ and $\int_0^T \omega u = 0$, then $\int_0^T \omega u'^2 \ge \frac{c^2}{4} \int_0^T \omega u^2$.*

Proof. (1) Write $u = e^{c(t-z)/2} v$. Then $\omega u'^2 = Ke^{-cz}(v' + cv/2)^2 = Ke^{-cz}(v'^2 + cvv' + c^2v^2/4)$, and $\int_z^T cvv' = \frac{c}{2} v(T)^2 \ge 0$ because $v(z) = 0$. Hence $\int \omega u'^2 \ge \frac{c^2}{4} Ke^{-cz} \int v^2 = \frac{c^2}{4} \int \omega u^2$.

(2) Since ω is bounded above and below on $[0, T]$, the minimum μ of $\int \omega u'^2 / \int \omega u^2$ over nonzero $u \in H^1(0, T)$ with $\int \omega u = 0$ is attained. Testing the first variation with constants shows that a minimizer u satisfies $\int \omega u' \varphi' = \mu \int \omega u \varphi$ for all $\varphi \in H^1(0, T)$, so u is a smooth solution of $-(\omega u')' = \mu \omega u$ with $u'(0) = u'(T) = 0$. Since $\int \omega u = 0$ and $u \ne 0$, u changes sign and has a zero $z \in (0, T)$. If u vanished identically on $[z, T]$, then $u(z) = u'(z) = 0$ and uniqueness for the equation would force $u \equiv 0$. Multiplying the equation by u and integrating over $[z, T]$, using $u(z) = 0$ and $u'(T) = 0$, gives $\int_z^T \omega u'^2 = \mu \int_z^T \omega u^2$, and (1) on $[z, T]$ gives $\mu \ge c^2/4$. □

Definition 5.2. A *pumpkin chain* with levels $0 = t_0 < t_1 < \cdots < t_m$ and multiplicities $n_1, \dots, n_m \ge 1$ has vertices $v_0, \dots, v_m$ and n_i parallel edges of length $t_i - t_{i-1}$ between v_{i-1} and v_i [11, Definition 5.3]. Its *weight* is the step function $n(t) = n_i$ on (t_{i-1}, t_i). A point of an edge of the i-th bundle at distance s from v_{i-1} has *level* $t_{i-1} + s$.

Lemma 5.3 (Spectra of pumpkin chains)**.** *Let $\mathcal{P}$ be a pumpkin chain with weight n on $[0, T]$, $T = t_m$, and $h = \max_i(t_i - t_{i-1})$.*

1. *With $V_{\mathrm{D}} = \{v_0\}$, $\lambda_1(\mathcal{P}; V_{\mathrm{D}}) \ge \lambda(n)$.*

2. *Let $\mathcal{Q}$ be two copies of $\mathcal{P}$ glued at v_0, with Kirchhoff conditions everywhere, and let $\tilde{n}(t) = n(|t|)$ on $[-T,T]$. Then $\mu_2(\mathcal{Q}) \geq \min\{\mu_2(\tilde{n}), \pi^2/h^2\}$, where $\mu_2(\tilde{n})$ is the first nonzero eigenvalue of $-(\tilde{n}u')' = \mu\tilde{n}u$ on $[-T,T]$ with natural conditions.*

Proof. Let Γ_0 be the finite group of isometries that permute the edges inside each bundle (of both copies, in case (2)) and fix all vertices. It acts transitively on the points of each level (each signed level in case (2)), so a Γ_0-invariant function is a function of the level. Each $\gamma \in \Gamma_0$ preserves the Rayleigh quotient and the form domain, and maps eigenfunctions to eigenfunctions with the same eigenvalue.

(1) Let f be a minimizer for $\lambda_1(\mathcal{P}; V_{\mathrm{D}})$. Then $|f|$ and all $|f| \circ \gamma$ are minimizers, hence eigenfunctions, and their average g over Γ_0 is a nonzero nonnegative eigenfunction that depends only on the level, $g = G(\text{level})$. The level is the distance to v_0, so by Lemma 3.1 the Rayleigh quotient of g equals the weighted quotient of G, and $G(0) = 0$. Hence $\lambda_1(\mathcal{P}; V_{\mathrm{D}}) \geq \lambda(n)$.

(2) Let $\mathcal{E}$ be the eigenspace of $\mu_2(\mathcal{Q})$, and let P be the average over Γ_0, which maps $\mathcal{E}$ to itself. If $P\mathcal{E} \neq \{0\}$, it contains a nonzero eigenfunction $g = G(\text{signed level})$. Its Rayleigh quotient is $\mu_2(\mathcal{Q})$, and it is orthogonal to the constants because $\mu_2(\mathcal{Q}) > 0$. Applying Lemma 3.1 to each copy of $\mathcal{P}$, with the distance to the glued vertex, the Rayleigh quotient of g equals $\int \tilde{n}G'^2 / \int \tilde{n}G^2$, and $\int \tilde{n}G = \int_{\mathcal{Q}} g = 0$. Hence $\mu_2(\mathcal{Q}) \geq \mu_2(\tilde{n})$. If $P\mathcal{E} = \{0\}$, take $f \in \mathcal{E} \setminus \{0\}$. Every $\gamma \in \Gamma_0$ fixes the vertices, so $f(v_i) = (Pf)(v_i) = 0$ for all i. Then f restricted to an edge on which it is nonzero is a Dirichlet eigenfunction of that edge, so $\mu_2(\mathcal{Q}) \geq \pi^2/h^2$. □

Proposition 5.4 (Exponential pumpkin chains)**.** *Let $T \geq 2$, $m = \lceil T^2 \rceil$, $h = T/m$, $t_i = ih$, and $n_i = \lceil Te^{T-(i-1)h} \rceil$ for $1 \leq i \leq m$. Let $\mathcal{P}_T$ be the pumpkin chain with these data and $V_{\mathrm{D}} = \{v_0\}$, and $\mathcal{Q}_T$ the doubled chain of Lemma 5.3(2). Then the conclusions of Theorem 1.4 for $\mathcal{P}_T$ and $\mathcal{Q}_T$ hold, and the mean distance to V_{D} on $\mathcal{P}_T$ satisfies $\rho_{\mathrm{D}} < 3$.*

Proof. Put $\omega(t) = Te^{T-t}$ on $[0,T]$, so $\omega \geq T$. On the i-th bundle, $t \in [(i-1)h, ih]$, we have $n_i \geq Te^{T-(i-1)h} \geq \omega(t)$ and $n_i \leq e^h\omega(t) + 1 \leq (e^h + 1/T)\omega(t)$. With $\kappa = e^h + 1/T \leq e^{1/T} + 1/T$ we therefore have $\omega \leq n \leq \kappa\omega$, and the same holds for $\tilde{n}$ and $\tilde{\omega}(t) = \omega(|t|)$. The Rayleigh quotients of n and of ω differ by a factor in $[1/\kappa, \kappa]$, so by the min–max principle $\lambda(n) \geq \lambda(\omega)/\kappa$ and $\mu_2(\tilde{n}) \geq \mu_2(\tilde{\omega})/\kappa$.

By Lemma 5.1(1) with $z = 0$, $\lambda(\omega) \geq \frac{1}{4}$. For $\mu_2(\tilde{\omega})$, split $u \in H^1(-T,T)$ with $\int \tilde{\omega}u = 0$ into its even and odd parts; the cross terms of both quadratic forms vanish because $\tilde{\omega}$ is even. The odd part vanishes at 0, so on $[0,T]$ Lemma 5.1(1) applies to it; the even part has $\int_0^T \omega u_{\mathrm{e}} = 0$, so Lemma 5.1(2) applies. Hence $\mu_2(\tilde{\omega}) \geq \frac{1}{4}$. Since $h \leq 1/T$, $\pi^2/h^2 \geq 4\pi^2 > 1/(4\kappa)$, and Lemma 5.3 gives

$$\lambda_1(\mathcal{P}_T; V_{\mathrm{D}}) \geq \frac{1}{4\kappa}, \qquad \mu_2(\mathcal{Q}_T) \geq \frac{1}{4\kappa}.$$

The vertex v_m is at distance T from v_0, so $R(\mathcal{P}_T) \geq T$ (in fact $= T$), and the two ends of $\mathcal{Q}_T$ are at distance $2T$, so $D(\mathcal{Q}_T) \geq 2T$. This gives the eigenvalue bounds of Theorem 1.4.

For the size, $n_i \leq Te^T e^{-(i-1)h} + 1$, so

$$E(\mathcal{P}_T) = \sum_{i=1}^{m} n_i \leq \frac{Te^T}{1 - e^{-h}} + m, \qquad L(\mathcal{P}_T) = h\sum_{i=1}^{m} n_i \leq \frac{h}{1 - e^{-h}}\, Te^T + T.$$

For $0 < h \leq \frac{1}{2}$ one has $1 - e^{-h} \geq h - h^2/2 \geq \frac{3}{4}h$ and $h/(1 - e^{-h}) \leq 1 + h$. With $1/h = m/T \leq (T^2{+}1)/T$ and $T \geq 2$ this gives $E(\mathcal{P}_T) \leq \frac{4}{3}(T^2{+}1)e^T + T^2 + 1 \leq 2(T^2{+}1)e^T$ and $L(\mathcal{P}_T) \leq \frac{3}{2}Te^T + T \leq 2Te^T$. Finally $E(\mathcal{Q}_T) = 2E(\mathcal{P}_T)$, $L(\mathcal{Q}_T) = 2L(\mathcal{P}_T)$, and $\beta(\mathcal{Q}_T) = E(\mathcal{Q}_T) - 2m < E(\mathcal{Q}_T)$.

Finally, the mean distance to V_{D} on $\mathcal{P}_T$ is $\rho_{\mathrm{D}} = \int_0^T tn / \int_0^T n \leq \kappa \int_0^T te^{-t}dt / \int_0^T e^{-t}dt \leq \kappa < 3$ by Lemma 3.1 and $\omega \leq n \leq \kappa\omega$. □

Lemma 5.5 (Barta inequality for step weights)**.** *Let n be a positive step function on $[0,r]$, and let $\kappa > 0$. Suppose $\varphi \in H^1(0,r)$ is positive on $(0,r]$, $\varphi(0) = 0$, $\varphi'(0^+) > 0$, $\varphi'' = -\kappa\varphi$ on each interval where n is constant, the flux $n\varphi'$ has equal one-sided limits at each jump of n, and $\varphi'(r) \geq 0$. Then $\lambda(n) \geq \kappa$.*

Proof. Let $u \in H^1(0,r)$ with $u(0) = 0$, and put $v = u/\varphi$ on $(0,r]$. On $[\varepsilon, r]$ we have $nu'^2 = n\varphi^2 v'^2 + n\varphi'\varphi(v^2)' + n\varphi'^2 v^2$, and integrating the middle term by parts, using $(n\varphi')' = -\kappa n\varphi$ on each step and the continuity of $n\varphi'$,

$$\int_\varepsilon^r nu'^2 = \int_\varepsilon^r n\varphi^2 v'^2 + \Big[\frac{n\varphi' u^2}{\varphi}\Big]_\varepsilon^r + \kappa \int_\varepsilon^r nu^2.$$

The boundary term at r is nonnegative because $\varphi'(r) \geq 0$. At $\varepsilon \to 0$ it tends to 0, because $u(\varepsilon)^2 \leq \varepsilon \int_0^\varepsilon u'^2 = o(\varepsilon)$ while $\varphi(\varepsilon)/\varepsilon \to \varphi'(0^+) > 0$ and φ' is bounded near 0. Hence $\int_0^r nu'^2 \geq \kappa \int_0^r nu^2$. □

Proposition 5.6 (Binary trees)**.** *Let $\mathcal{B}_h$ be the complete binary tree of depth $h \geq 1$ with unit edges, Dirichlet conditions at its 2^h leaves and the natural condition at the root, and let $k_0 = \arccos(2\sqrt{2}/3)$. Then $\lambda_1(\mathcal{B}_h; V_\mathrm{D}) \geq k_0^2$ and $R = h$, so $\lambda_1 R^2 \geq c_0 \log^2 |V_\mathrm{D}|$ with $c_0 = k_0^2/\log^2 2$.*

Proof. The root is at distance h from every leaf, so $R = h$. The automorphisms of $\mathcal{B}_h$ fixing the root act transitively on the points at each distance from the leaves, so the averaging argument of Lemma 5.3(1) gives $\lambda_1(\mathcal{B}_h; V_\mathrm{D}) \geq \lambda(n)$, where $n = 2^{h-j}$ on $(j, j+1)$, $0 \leq j < h$, is the level count of the distance to the leaves.

Put $x_j = j\,2^{j/2}$ for $0 \leq j \leq h$, and on $[j, j+1]$ let

$$\varphi(t) = \frac{x_j \sin k_0(j+1-t) + x_{j+1} \sin k_0(t-j)}{\sin k_0},$$

so that $\varphi(j) = x_j$, $\varphi'' = -k_0^2\varphi$ on each step, $\varphi = x_1 \sin(k_0 t)/\sin k_0$ on $[0,1]$, and $\varphi > 0$ on $(0,h]$ because $0 < k_0 < \pi$ and $x_j > 0$ for $j \geq 1$. At an interior integer j the weight halves, and continuity of $n\varphi'$ means $\varphi'(j^+) = 2\varphi'(j^-)$, that is, $x_{j+1} - x_j \cos k_0 = 2(x_j \cos k_0 - x_{j-1})$, or $x_{j+1} - 2\sqrt{2}\,x_j + 2x_{j-1} = 0$. This recurrence has the double characteristic root $\sqrt{2}$, and $x_j = j\,2^{j/2}$ solves it. At $t = h$, $\varphi'(h^-) = k_0(x_h \cos k_0 - x_{h-1})/\sin k_0$, which is positive because $x_{h-1} = 0$ when $h = 1$, and $x_h/x_{h-1} = \sqrt{2}\,h/(h-1) > \sqrt{2} > 1/\cos k_0$ when $h \geq 2$. Lemma 5.5 gives $\lambda(n) \geq k_0^2$, and $\log |V_\mathrm{D}| = h \log 2$ gives the last claim. □

Remark 5.7. The constant k_0^2 is the bottom of the spectrum found by Mugnolo for balls of the 3-regular tree with Dirichlet leaves [15, proof of Theorem 4.8], through Kesten's formula; the radial weights $3 \cdot 2^{h-1-j}$ of those balls are proportional to the weights of $\mathcal{B}_h$, so the two families have the same first eigenvalue, since for both, by averaging and radial test functions, λ_1 equals the weighted eigenvalue of the radial level count. By (7) with $\beta = 0$, the best constant C in $\lambda_1 R^2 \leq (C + o(1)) \log^2 |V_\mathrm{D}|$ on trees therefore lies in $[c_0, \frac{1}{4}]$.

Proof of Theorem 1.4 and Corollary 1.5. Theorem 1.4 is Propositions 5.4 and 5.6. Subdividing edges does not change μ_2, λ_1, D, R or L, so each supremum in Corollary 1.5 is nondecreasing in its parameter and equals the supremum over graphs with exactly E edges where relevant. The upper limits follow from Theorems 1.2 and 1.3, since $\log(8e^2(E+1))$, $\log(8e^2(3b+1))$, $\Lambda + \log\Lambda$ with $\Lambda = \log(4e^3 x)$, and $\frac{1}{2}\log(8e^2 N)$ are $\log E + O(1)$, $\log b + O(1)$, $\log x + O(\log\log x)$ and $\frac{1}{2}\log N + O(1)$. For the lower limits, given a large E choose $T = \log E - 3\log\log E$; then $4(T^2+1)e^T \leq E$ for E large, and $\mu_2(\mathcal{Q}_T) D^2 \geq T^2/(e^{1/T} + 1/T) = (1 - o(1)) \log^2 E$. Since $\beta(\mathcal{Q}_T) < E(\mathcal{Q}_T)$, the same family serves for β; for $L/D \leq x$ take $T = \log(x/2)$. The same choices with $\mathcal{P}_T$ give the Dirichlet limits; for the last one note that $3\beta(\mathcal{P}_T) + 2|V_\mathrm{D}| \leq 3E(\mathcal{P}_T) + 2$. The bounds on c_E are Proposition 1.7. □

5.1 Mean distance on cubic Ramanujan graphs

Lemma 5.8. *Let G be a connected d-regular graph with adjacency matrix A, and let $\mathcal{G}$ be the metric graph with unit edges. If θ is the largest eigenvalue of A smaller than d, then $\mu_2(\mathcal{G}) \geq \arccos(\theta/d)^2$.*

Proof. Let μ be an eigenvalue in $(0, \pi^2)$ with eigenfunction f, and put $k = \sqrt{\mu} \in (0, \pi)$. On an edge from a to b, parametrized by $x \in [0,1]$ from a, $f(x) = \bigl(f(a)\sin k(1-x) + f(b)\sin kx\bigr)/\sin k$, so the derivative at a in the direction of the edge is $k\bigl(f(b) - f(a)\cos k\bigr)/\sin k$. The Kirchhoff condition at a becomes $(Af)(a) = d\cos k\, f(a)$, and $f|_{\mathsf{V}} \neq 0$, since otherwise f would vanish on every edge. Thus $d\cos k$ is an eigenvalue of A different from d, so $d\cos k \leq \theta$ and $\mu \geq \arccos(\theta/d)^2$. Eigenvalues at least π^2 satisfy the bound trivially. □

Lemma 5.9. *Let $\mathcal{G}$ be a connected 3-regular graph with unit edges and total length L. Then every ball satisfies $|\{y : \operatorname{dist}(x,y) < r\}| < 9 \cdot 2^r$ for $r > 0$, and $\rho \geq r\,(1 - 9 \cdot 2^r/L)$.*

Proof. Let v be an endpoint of the edge of x with $\operatorname{dist}(x,v) \leq \frac{1}{2}$, and for y with $\operatorname{dist}(x,y) < r$ let u be an endpoint of the edge of y with $\operatorname{dist}(y,u) \leq \frac{1}{2}$. Then $\operatorname{dist}(v,u) < r+1$, and between vertices the metric distance equals the graph distance, so u is among the fewer than $3 \cdot 2^{r+1}$ vertices within graph distance $\lfloor r+1 \rfloor$ of v. Each such u accounts for at most three half-edges, of total length $\frac{3}{2}$, which gives the first bound. Then $\frac{1}{L}\int_{\mathcal{G}} \operatorname{dist}(x,y)\,dy \geq r\,(1 - 9 \cdot 2^r/L)$ for every x, and averaging over x gives the second. □

Proof of the lower bound in Theorem 1.6. Lemma 5.8 uses only the second largest adjacency eigenvalue, so any infinite family of 3-regular Ramanujan graphs serves. By Marcus, Spielman and Srivastava [12] there are infinitely many connected 3-regular bipartite Ramanujan graphs, that is, graphs whose adjacency eigenvalues other than ± 3 have absolute value at most $2\sqrt{2}$. For such a graph on n vertices with unit edges, Lemma 5.8 gives $\mu_2 \geq \arccos(2\sqrt{2}/3)^2$, and $E = L = 3n/2$, $\beta = n/2 + 1 \geq E/3$. Lemma 5.9 with $r = \log_2\bigl(L/(9\log_2 L)\bigr)$ gives $\rho \geq (\log_2 L - \log_2(9\log_2 L))(1 - 1/\log_2 L) = (1 - o(1))\log E/\log 2$. Hence $\mu_2\rho^2 \geq (c_0 - o(1))\log^2 E$. Finally $\rho \geq 1$ for large n, so $L/\rho \leq E$. Since the graphs of [12] are obtained by successive 2-lifts, their sizes grow by factors of 2, so the supremum over graphs with at most E edges is at least $(c_0 - o(1))\log^2 E$ for all large E. □

6 The lower side

Proof of Proposition 1.7. Every edge has length at most $2D$: its midpoint is at distance half its length from an endpoint, because a path from the midpoint to that endpoint either runs along half of the edge or passes through the other endpoint. Hence $L \leq 2DE$, with equality for a single loop. If $L \geq 2D$, the bound $\mu_2 \geq 1/(2D(L-D)) > 1/(2DL)$ of [11, (7.3)] gives $\mu_2 D^2 > D/(2L) \geq 1/(4E)$. If $L < 2D$, the bound $\mu_2 \geq \pi^2/L^2$ of [11, (4.1)], due to Nicaise, gives $\mu_2 D^2 > \pi^2/4 > 1/(4E)$.

For the upper bound, let $k \geq 1$ and join two vertices by an edge of length 1, attaching k loops of length 1 at each of them. This graph has $E = 2k+1$ edges and $D = 2$, attained by the midpoints of two loops at different vertices; two points on loops at the same vertex are at distance at most 1. The function equal to 1 on the loops at one vertex, to -1 on the loops at the other and linear on the connecting edge has mean zero, energy 4 and squared norm $2k + \frac{1}{3}$, so $\mu_2 D^2 \leq 16/(2k + \frac{1}{3}) = 16/(E - \frac{2}{3})$. For even $E = 2k+2$, subdivide the connecting edge, which gives $16/(E - \frac{5}{3})$. In both cases $\mu_2 D^2 \leq 16/(E-2)$. □

Remark 6.1. The first part of the proof combines the length–diameter lower bound of [11, (7.3)] with $L \leq 2DE$. Kennedy [10] found the sharp lower bound for μ_2 in terms of L and D, answering

a problem of [11]; inserted in place of [11, (7.3)], it gives the same order $1/E$ with a constant at least $\frac{1}{4}$.

7 Computational checks

The proofs above are complete without computation. The following scripts, included with this article, test the finite statements numerically and illustrate the asymptotic ones; each prints its output to a file of the same name ending in `.out.txt`.

- `check_lemma1d.py`: Lemma 2.1 on 60 000 random triples (step weight, s_1, s_2) (largest value of the left side 0.9954), Corollary 2.2 on 3000 random step weights (largest ratios 0.935 and 0.733), and exponential weights, whose ratios approach 1. Eigenvalues of step weights are computed exactly by a Prüfer angle that rotates by $k\ell$ on each step, and cross-checked against finite elements.
- `check_graph_bounds.py`: Lemma 3.1 and Theorems 1.2–1.3 on 150 random metric graphs with loops, parallel edges and pendant trees, using conforming finite elements, which overestimate eigenvalues, and the exact diameter; the largest ratio of a left side to its bound is 0.770.
- `check_extremal.py`: the families of Proposition 5.4 for $2 \le T \le 48$, including a finite-element check of Lemma 5.3, and the trees of Proposition 5.6 up to depth 64, together with the certificate of its proof.
- `check_expanders_dumbbell.py`: Lemma 5.8 on random cubic graphs and on the Heawood graph (a bipartite Ramanujan graph), Lemma 5.9, and the graphs of Proposition 1.7.
- `check_elementary.py`: the elementary inequalities used in Corollary 2.2 and Proposition 5.4.

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