Locating Quintic Obstructions in the Riemann Theta Product

A certified transition and minimum-degree theorem

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Luminary Oddities

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Abstract

We give a reproducible method for locating and certifying quintic obstructions for the normalized fixed-center Riemann theta product, and prove a unique transition in its odd quintic moment block on $[3.64, 3.66]$. The transition parameter lies strictly between 3.64888758188562 and 3.64888758188563. For $x \in I = [3.64, 3.66]$, below the transition, degree five is the minimum degree of a polynomial whose square has negative signed-moment value; at the transition the degree-five form is positive semidefinite with a one-dimensional nullspace; above it the form is positive definite. The proof combines a Schur-complement reduction with uniform directed interval bounds, including the complete infinite theta tails. An exact rational arithmetic replay verifies the bounds on the entire interval. An explicit integer-coefficient quintic illustrates why testing one fixed polynomial does not locate the transition of the full quintic block.

Keywords: Riemann theta density; moment matrices; positive definiteness; quintic obstruction; computer-assisted proof.

1 The fixed-center problem

Let

$$\Phi(s) = \sum_{n=1}^{\infty} (2\pi^2 n^4 e^{9s} - 3\pi n^2 e^{5s}) e^{-\pi n^2 e^{4s}}. \tag{1}$$

For $3 \le x \le 4$, put $s(x) = \frac{1}{4}\log(x/\pi)$ and

$$p_x(t) = \frac{\Phi(s(x) + t/2)\Phi(s(x) - t/2)}{\Phi(s(x))^2}. \tag{2}$$

The denominator is positive: each summand at $s(x)$ contains the positive factor $2n^2x - 3$. Normal convergence of every fixed derivative of the theta series on compact real intervals justifies the differentiations below. The function p_x is even and $p_x(0) = 1$. Define

$$m_{2j}(x) = (-1)^j p_x^{(2j)}(0) \quad (0 \le j \le 5), \qquad m_{2j+1}(x) = 0 \quad (0 \le j \le 4), \tag{3}$$

and let $\mathcal{L}_x(X^k) = m_k(x)$ on polynomials of degree at most ten. We call P an obstruction when $\mathcal{L}_x(P^2) < 0$.

Moment and positive-definiteness methods for theta kernels have a substantial history; see Csordas [1]. His associated kernels integrate products of the form $\Phi(s + t)\Phi(s - t)$ against center weights. Here the center is fixed. Hudson's theorem [2], through the Wigner representation of a non-Gaussian theta density, provides a general existence mechanism for failure at some center, not at every prescribed center and not necessarily in degree five. It does not specify a center or coefficient vector for the present theorem.

Moment-optimized polynomial-square witnesses also occur in the Wigner-negativity tests of Kot et al. [4]. The explicit theta quintic in Schubr [5] is the starting witness for the coefficient search studied here. For a related certified theta-derivative inequality using outward intervals and geometric tails, see Sra [6], Section 5.

We classify the odd quintic block on a fixed interval, including the minimum obstructing degree and the inertia at its unique transition. The proof uses a Schur-complement reduction [3]. All statements concern the fixed-center product (2).

2 Transition and minimum degree

Write

$$G = \begin{pmatrix} m_2 & m_4 \\ m_4 & m_6 \end{pmatrix}, \quad v = \begin{pmatrix} m_6 \\ m_8 \end{pmatrix}, \quad H = \begin{pmatrix} m_2 & m_4 & m_6 \\ m_4 & m_6 & m_8 \\ m_6 & m_8 & m_{10} \end{pmatrix}, \tag{4}$$

and

$$E = \begin{pmatrix} 1 & m_2 & m_4 \\ m_2 & m_4 & m_6 \\ m_4 & m_6 & m_8 \end{pmatrix}, \qquad S = m_{10} - v^{\mathsf{T}} G^{-1} v. \tag{5}$$

The inverse in (5) is justified throughout the interval by Lemma 2. For inertia we list the numbers of positive, negative, and zero eigenvalues, in that order.

Theorem 1 (Unique quintic transition)**.** *Let $I = [91/25, 183/50] = [3.64, 3.66]$. There is exactly one $x_* \in I$ with $S(x_*) = 0$, and*

$$3.64888758188562 < x_* < 3.64888758188563. \tag{6}$$

For every $x \in I$, $E(x)$ and $G(x)$ are positive definite. The complete degree-five moment matrix $M_5(x) = (m_{j+k}(x))_{j,k=0}^{5}$ has inertia

$$\begin{cases} (5,1,0), & 3.64 \le x < x_*, \\ (5,0,1), & x = x_*, \\ (6,0,0), & x_* < x \le 3.66. \end{cases}$$

Consequently:

1. *For $3.64 \le x < x_*$, degree five is the minimum degree of a real polynomial obstruction. A monic odd quintic attains this degree.*
2. *At $x = x_*$ there is no obstruction of degree at most five. The nullspace is spanned by $X^5 + a_* X^3 + b_* X$, where $(b_*, a_*)^{\mathsf{T}} = -G(x_*)^{-1} v(x_*)$.*
3. *For $x_* < x \le 3.66$, every nonzero polynomial of degree at most five has strictly positive square value.*

Lemma 2 (Uniform certified bounds)**.** *On I, the following strict lower bounds hold:*

$$\begin{aligned} m_2 > 38, \qquad & \Delta := m_2 m_6 - m_4^2 > 8{,}900{,}000, \\ m_4 - m_2^2 > 2{,}600, \qquad & \det E > 53{,}000{,}000{,}000, \\ S' > 3{,}000{,}000{,}000. & \end{aligned} \tag{7}$$

At the rational endpoints in (6)*, respectively,*

$$-0.000019 < S < -0.000018, \qquad 0.000033 < S < 0.000034. \tag{8}$$

The proof of Lemma 2, including the finite verification procedure and its full-tail estimates, occupies Section 4.

Proof of Theorem 1. Sylvester's criterion and (7) give $G > 0$ and $E > 0$. The function S is continuously differentiable on I. Its derivative is strictly positive by Lemma 2; hence it is strictly increasing. The opposite signs in (8) and the intermediate value theorem prove existence, uniqueness, and (6).

For $P = X^5 + aX^3 + bX$ and $c = (b, a)^\mathsf{T}$, expansion and completion of the square give

$$\mathcal{L}_x(P^2) = m_{10} + 2c^\mathsf{T} v + c^\mathsf{T} G c = S + (c + G^{-1}v)^\mathsf{T} G (c + G^{-1}v). \tag{9}$$

Equivalently, H is congruent to $\operatorname{diag}(G, S)$. After ordering even monomials before odd monomials, M_5 is $\operatorname{diag}(E, H)$, since every odd moment vanishes. The stated inertias follow.

Every polynomial of degree at most four splits into an even part in $\operatorname{span}\{1, X^2, X^4\}$ and an odd part in $\operatorname{span}\{X, X^3\}$. Their cross term has zero value under $\mathcal{L}_x$, while their separate matrices are $E > 0$ and $G > 0$. Thus no nonzero polynomial of degree at most four is an obstruction. When $S < 0$, choosing $c = -G^{-1}v$ in (9) gives a monic quintic obstruction. At $S = 0$ it gives the unique null direction; when $S > 0$ both parity blocks are positive definite. □

The result is local to I. At the transition, zero square value is not a negative obstruction.

3 The coefficient-search method

For a rational center in I, compute directed enclosures of the moments, G, and S. If the upper endpoint for S is negative, the optimal real coefficients are

$$b_* = \frac{m_4 m_8 - m_6^2}{\Delta}, \qquad a_* = \frac{m_4 m_6 - m_2 m_8}{\Delta}. \tag{10}$$

These follow directly from $c_* = -G^{-1}v$. Choose nearby rational coefficients, then certify their contraction using

$$\mathcal{L}_x((X^5 + aX^3 + bX)^2) = m_{10} + 2am_8 + (a^2 + 2b)m_6 + 2abm_4 + b^2 m_2. \tag{11}$$

This direct final test retains every cross term and the complete theta tails.

Lemma 3 (Certified rational selection)**.** *Suppose $S \le -\eta < 0$ and $\lambda_{\max}(G) \le M$. A rational coefficient vector $\widetilde{c}$ is an obstruction whenever $\|\widetilde{c} - c_*\|_2^2 < \eta/M$. Rounding both coefficients to the nearest multiple of a rational step h is sufficient if $Mh^2/2 < \eta$.*

Proof. Equation (9) bounds the rounded contraction by $-\eta + M\|\widetilde{c} - c_*\|_2^2$. Nearest-grid rounding makes each coordinate error at most $h/2$. □

One may take the maximum absolute row sum of an outward enclosure of G for M. Alternatively, refine (10), round, and apply (11) directly until a negative upper bound is obtained.

Corollary 4 (Termination at negative rational centers)**.** *For every rational $x \in [3.64, x_*)$, successively refining the coefficient enclosures, halving a rational grid step, and testing nearest-grid candidates eventually certifies a quintic obstruction. No such obstruction exists in I at or above x_*.*

Proof. At a fixed center below x_*, $S < 0$ and $G > 0$. Rational coefficients are dense, and nearest-grid candidates converge to c_* as both the enclosure widths and grid step tend to zero. The tail bounds in Section 4 tend to zero as the theta truncation grows, while increasing the Taylor order and arithmetic precision gives arbitrarily small finite-sum enclosure widths. Thus a strictly negative rational contraction is certified after finitely many refinements. The last assertion follows from Theorem 1. □

The supplied finite-precision program is a concrete implementation with ample margins for the reported certificates; the termination statement permits increasing truncation and precision.

4 Full-tail certification

We give the formulas underlying the finite certificate. Every numerical bound in Lemma 2 is verified with outward intervals; decimal approximations are not used to decide signs.

4.1 Moments and center derivatives

Put

$$A_x(t) = \sum_{n\ge 1} n^2(2n^2xe^{2t} - 3)e^{-n^2x(e^{2t}-1)}e^{-(n^2-1)x}. \tag{12}$$

Substitution into (1) yields $\Phi(s(x)+t/2) = \pi e^{5s(x)-x}e^{5t/2}A_x(t)$; therefore $p_x(t) = A_x(t)A_x(-t)/A_x(0)^2$. Define the integer polynomials

$$R_0(z) = 2z - 3, \qquad R_{k+1}(z) = 2z(R_k'(z) - R_k(z)). \tag{13}$$

For $0 \le k \le 10$ let

$$B_k(x) = \sum_{n\ge 1} n^2 R_k(n^2x)e^{-(n^2-1)x}, \tag{14}$$

$$\begin{aligned} C_k(x) &= B_k'(x) \\ &= \sum_{n\ge 1} n^2\{n^2R_k'(n^2x) - (n^2-1)R_k(n^2x)\}e^{-(n^2-1)x}. \end{aligned} \tag{15}$$

Leibniz' rule gives, with $k = 2j$,

$$N_{2j} = (-1)^j \sum_{r=0}^{2j}(-1)^r\binom{2j}{r}B_rB_{2j-r}, \qquad m_{2j} = \frac{N_{2j}}{B_0^2}. \tag{16}$$

Replace each product B_rB_{k-r} by $C_rB_{k-r} + B_rC_{k-r}$ to obtain N_{2j}'. Then

$$m_{2j}' = \frac{N_{2j}'}{B_0^2} - 2m_{2j}\frac{C_0}{B_0}. \tag{17}$$

We set $m_0 = 1$ and $m_0' = 0$ exactly.

4.2 Bounds for every omitted summand

Truncate (14) and (15) at $n = 7$. Write $R_k(z) = \sum_{j=0}^{k+1} r_{k,j}z^j$ and put $q = 8$ and

$$\rho(x) = \left(\frac{9}{8}\right)^{26} e^{-17x} < 1.$$

For $x \in I$, every positive monomial majorant in either omitted series has successive-term ratio at most $\rho(x)$: its power of n is at most 26, and $(1 + 1/n)^{26}e^{-(2n+1)x} \le \rho(x)$ for $n \ge 8$. Consequently the omitted tails are bounded in absolute value by

$$T_k(x) = \frac{e^{-63x}}{1-\rho(x)}\sum_{j=0}^{k+1}|r_{k,j}|x^j 8^{2j+2}, \tag{18}$$

$$V_k(x) = \frac{e^{-63x}}{1-\rho(x)}\left\{\sum_{j=0}^{k+1}|r_{k,j}|x^j 8^{2j+4} + \sum_{j=1}^{k+1} j|r_{k,j}|x^{j-1}8^{2j+2}\right\}. \tag{19}$$

In (19) we used $n^2 - 1 \le n^2$. Adding $[-T_k, T_k]$ and $[-V_k, V_k]$ to the corresponding finite sums encloses B_k and C_k. Interval evaluation of (16) and (17) then includes all mixed-tail and tail-tail terms without a separate omission of product interactions.

For a larger truncation $N \ge 7$, replace 8 by $q = N+1$, 63 by $q^2 - 1$, 17 by $2q+1$, and $9/8$ by $(q+1)/q$. The same argument applies, and the bounds tend to zero as N grows.

4.3 Finite arithmetic and the interval cover

The interval is covered by exactly 1,000 closed cells

$$J_i = \left[\frac{91}{25} + \frac{i}{50000}, \frac{91}{25} + \frac{i+1}{50000}\right], \qquad 0 \le i < 1000. \tag{20}$$

All functions in (14)–(19) are enclosed on each whole cell. For example $e^{-\alpha J_i}$ is enclosed using its decreasing endpoint values, and polynomial evaluations use interval Horner arithmetic. Every denominator is checked to exclude zero before division.

Exponential enclosures require only rational arithmetic. If $0 \le t < 512$ and $u = t/512$, set

$$E_L(u) = \sum_{r=0}^{L} \frac{u^r}{r!}, \qquad Z_L(u) = \frac{u^{L+1}}{(L+1)!}\frac{1}{1-u/(L+2)}.$$

The positive Taylor series implies

$$\frac{1}{E_L(u) + Z_L(u)} \le e^{-u} \le \frac{1}{E_L(u)}. \tag{21}$$

Nine interval squarings give a bound for e^{-t}. Every exponential argument used here is less than 512. For the adaptive search with larger truncations, choose an integer $r \ge 0$ such that $u = t/2^r < 1$, apply (21), and use r squarings instead. For each finite truncation at a fixed rational center, increasing the Taylor order and working precision makes the arithmetic enclosure errors arbitrarily small.

The primary verifier uses 60 significant decimal digits with separate downward and upward rounding contexts, and $L = 80$. The replay uses exact fractions, rounds outward to the grid 10^{-50}, and uses $L = 100$. Both use (21); neither calls a floating-point exponential function. The replay shares the moment formulas but uses a separate interval-arithmetic implementation and the alternative determinant differentiation below.

4.4 Checking the Schur derivative

For brevity write $(u, v, w, y, z) = (m_2, m_4, m_6, m_8, m_{10})$ and

$$\Delta = uw - v^2, \qquad F = w^3 - 2vwy + uy^2.$$

The primary evaluation uses

$$S = z - F/\Delta, \qquad S' = z' - F'/\Delta + F\Delta'/\Delta^2, \tag{22}$$

where

$$\begin{aligned}\Delta' &= u'w + uw' - 2vv',\\ F' &= 3w^2w' - 2(v'wy + vw'y + vwy') + u'y^2 + 2uyy'.\end{aligned}$$

The exact-rational replay instead computes $D = \det H$ and

$$\begin{aligned}D' = {}& u'(wz - y^2) + v'(-2vz + 2wy)\\ &+ w'(uz + 2vy - 3w^2) + y'(-2uy + 2vw) + z'(uw - v^2),\end{aligned}$$

and encloses $S' = (D'\Delta - D\Delta')/\Delta^2$.

Proof of Lemma 2. Use (13) to generate the eleven integer polynomials. On each cell (20), enclose the finite sums (14) and (15), add the complete tail bounds (18) and (19), and evaluate the moments and derivatives by (16) and (17). The programs require positive lower endpoints for B_0, m_2, Δ, $m_4 - m_2^2$, $\det E$, and S' on every cell. The stored exact-rational lower bounds exceed the respective constants in (7). The cells cover I without gaps, so the resulting bounds hold throughout I.

Evaluating at the two rational centers in (6) gives the strict enclosures (8). These signs are also reproduced by the retained ordered-pair rational center verifier, which expands $A_x(t)A_x(-t)$ directly through degree ten. The supplied scripts and certificates constitute the finite arithmetic verification. □

5 An explicit witness beyond the earlier polynomial

At $x = 91/25$, direct rational certification gives

$$\mathcal{L}_x\big((X^5 - 283X^3 + 12683X)^2\big) \in [-45620067.893680, -45620067.893679],$$
$$\mathcal{L}_x\big((X^5 - 264X^3 + 10952X)^2\big) \in [41398217.303171, 41398217.303172].$$

Thus positivity of a previously chosen polynomial does not imply positivity of the quintic block. Equation (9) searches the whole coefficient plane. Below x_* its negative set is precisely

$$\{c \in \mathbb{R}^2 : (c + G^{-1}v)^{\mathsf{T}} G (c + G^{-1}v) < -S\},$$

an open ellipse. As $x \uparrow x_*$ its closure shrinks to the unique null coefficient pair. At x_* the strictly negative set is empty and the nonpositive set is that singleton; above x_* both are empty.

Corollary 5. *For every $x \in [3.64, x_*)$, p_x is not positive definite. For each such x, all sufficiently small positive h yield a negative direction in $(p_x((r-s)h))_{r,s=0}^{5}$.*

Proof. Let F_h be the forward-difference matrix with entries $(F_h)_{j,r} = h^{-j}(-1)^{j-r}\binom{j}{r}$ for $r \le j$ and zero otherwise. Repeated finite differences give

$$F_h(p_x((r-s)h))_{r,s=0}^{5} F_h^{\mathsf{T}} \longrightarrow ((-1)^k p_x^{(j+k)}(0))_{j,k=0}^{5}.$$

Writing A for this limiting matrix and $U = \operatorname{diag}(1, i, \dots, i^5)$, one has $UAU^* = M_5$, where U^* denotes the conjugate transpose. Its negative direction from Theorem 1 persists for sufficiently small h. Positive definiteness would require every finite translation Gram matrix to be positive semidefinite. □

Reproducibility

The supplement contains the uniform interval verifier, its exact-rational replay, the retained ordered-pair center verifier, and the recorded certificates. They require only Python's standard library. The README gives commands using fresh output filenames.

AI assistance

AI assisted derivation, verification code, and preparation; the author takes responsibility for the manuscript.

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