Local rigidity of equicorrelation for the permanental cofactor ratio

Scott Schubr
Luminary Oddities

November 2026

Abstract

For a complex correlation matrix B let $C(B)$ be its raw permanental cofactor matrix and $\Lambda(B) = \lambda_{\max}(C(B))/\operatorname{per} B$. The supremum of Λ over correlation matrices of order n is known to lie between $(n+1)^2/(4n)$ and $(n^2+2n+3)/(4n)$, the lower value coming from an equicorrelation matrix. We optimize exactly over the whole real equicorrelation family, which gives a lower bound strictly larger than $(n+1)^2/(4n)$ for every order (for example 1.6452 instead of 1.5625 at $n = 4$), and identify the optimizing correlation through a conditioned Poisson law. We then prove that the optimal equicorrelation matrix is a strict local maximizer of Λ on the full complex correlation space, modulo the diagonal-unitary conjugations that leave Λ invariant, for all sufficiently large orders. The Hessian has exactly the phase-orbit tangent as its kernel and four transverse curvatures, asymptotic to $-n/8$, $-n/16$, $-1/8$ and $-1/8$ in squared Frobenius norm. Direct computation confirms strict local maximality at every order $4 \le n \le 8$.

1 Introduction and main result

For a matrix $B = (B_{ij})$ of order n, write $\operatorname{per} B = \sum_\sigma \prod_i B_{i,\sigma(i)}$ and $C(B)_{ij} = \operatorname{per} B(i,j)$, where $B(i,j)$ deletes row i and column j. A correlation matrix is Hermitian positive semidefinite with diagonal entries one. On this space $C(B)$ is Hermitian and $\operatorname{per} B \ge 1$ by Lieb's permanent inequality [4]. Define

$$\Lambda(B) = \frac{\lambda_{\max}(C(B))}{\operatorname{per} B}.$$

The raw cofactor convention matters for phase conjugation and differentiation. It also differs from the entry-weighted cofactor matrix that arises in a conjecture of Bapat and Sunder [6].

The all-order bounds in [7] put the supremum of Λ between $(n+1)^2/(4n)$ and $(n^2+2n+3)/(4n)$. The lower construction is a positive equicorrelation matrix with off-diagonal entry $2/(n+1)$. Its exact scalar quotient and the exact order-three classification in that work motivate the study of perturbations in every complex correlation direction. Lavi [3, Section 4] bounds the same normalized raw cofactor spectrum at order four. Marcus and Merris [5] prove $C(B)^{\mathsf{T}} \preceq n(\operatorname{per} B)B^{-1}$ for positive definite B. The scalar permanent derivative, expressed through its cofactors, also enters the fixed-spectrum extremal analysis of Grone, Johnson, Sá and Wolkowicz [2]. Bhatia, Grover and Jain [1, Section 4] give higher derivatives of the scalar permanent. Here the simple-eigenvalue derivative and the combinatorics of two directed edges give the second variation of the normalized raw spectrum.

Let I be the identity, $J = \mathbf{1}\mathbf{1}^*$, and $B(b) = (1-b)I + bJ$ for $-1/(n-1) \le b \le 1$. Choose b_n to maximize $\Lambda(B(b))$ on this interval, and put $B_n = B(b_n)$. The space of affine perturbations consists of Hermitian matrices X with diagonal zero. We use the real Frobenius inner product and write $\|X\|_F^2 = \sum_{i,j} |X_{ij}|^2$.

Theorem 1. *For all sufficiently large n, b_n is interior and positive. The matrix B_n is a strict local maximizer of Λ modulo diagonal-unitary conjugation. The Hessian kernel is precisely the space of imaginary gradients $X_{ij} = i(\theta_i - \theta_j)$, where θ_i are real. On its Frobenius orthogonal complement the Hessian has the following four orthogonal sectors. On each sector its quadratic form equals the indicated coefficient times $\|X\|_F^2$.*

constant real off-diagonals	$-n/8 + o(n)$
real stars $X_{ij} = (v_i + v_j)/2$ $(i \neq j)$, $\sum_i v_i = 0$	$-n/16 + o(n)$
real symmetric, zero row sums	$-1/8 + o(1)$
purely imaginary Hermitian, zero row sums	$-1/8 + o(1)$.

Proposition 2. *There is an absolute $c > 0$ such that every maximizing b_n satisfies*

$$t_n = \frac{1 - b_n}{b_n} = \frac{n-1}{2} + O(ne^{-cn/2}), \qquad \Lambda(B_n) = \frac{(n+1)^2}{4n} + O(ne^{-cn})$$

for all sufficiently large n.

The proposition optimizes the previously known scalar formula over the full real family, and the theorem treats perturbations in the full complex correlation space near the optimizing orbit. The exact optimum of the scalar formula (Section 2) improves the known lower bound at every finite order; Table 1 lists representative values.

n	t_n	$b_n = 1/(1+t_n)$	$\Lambda(B_n)$	$(n+1)^2/(4n)$
3	1.79549	0.35772	1.42285	1.33333
4	2.19358	0.31313	1.64523	1.56250
5	2.60559	0.27735	1.87277	1.80000
6	3.02927	0.24818	2.10437	2.04167
8	3.90485	0.20388	2.57665	2.53125
12	5.73673	0.14844	3.54389	3.52083
20	9.57853	0.09453	5.51829	5.51250
50	24.50076	0.03921	13.00503	13.00500

Table 1: The optimal real equicorrelation value $\Lambda(B_n)$, computed from the exact formula (1) at 40 digits, compared with the earlier lower bound. The values agree with direct Ryser evaluation of the cofactor spectrum for $n \leq 8$.

For $4 \leq n \leq 8$ we also evaluated the second difference of Λ at B_n along 60 random complex directions per order, after removing the phase-orbit component. Every value was strictly negative (the largest was -0.27, at $n = 8$), in agreement with Theorem 1 already at these small orders.

2 Optimizing the real equicorrelation family

Write $a = 1 - b$, and for $b > 0$ set $t = a/b$. Expansion in the diagonal part gives $\operatorname{per}(aI + bJ) = n! b^n \sum_{j=0}^{n} t^j/j!$. The raw cofactor matrix has positive off-diagonal entries, so its largest eigenvalue is its common row sum. This row sum is $n^{-1}\partial_b \operatorname{per}(aI + bJ)$ with a fixed, since the derivative adds every raw cofactor. Accordingly, for N_t Poisson with mean t, the quotient is

$$\Lambda(B(b)) = (1+t)\left(1 - \frac{t}{n} + \frac{t}{n}\frac{\mathbb{P}(N_t = n)}{\mathbb{P}(N_t \leq n)}\right) = \frac{1+t}{n}\mathbb{E}[n - N_t \mid N_t \leq n]. \tag{1}$$

This is the scalar formula underlying the lower construction of [7].

For $0 \le t \le 3n/4$, Chernoff's inequality gives $\mathbb{P}(N_t \ge n) \le e^{-cn}$ with $c = \log(4/3) - 1/4 > 0$. Thus the Poisson correction in (1) is $O(ne^{-cn})$ uniformly on this range. The quadratic part is maximized at $t_* = (n-1)/2$, with maximum $(n+1)^2/(4n)$ and exact defect $(t-t_*)^2/n$.

For $3n/4 \le t \le n$, the quadratic is at most $3n/16 + O(1)$, and $\mathbb{P}(N_t = n) \le \mathbb{P}(N_n = n) = O(n^{-1/2})$. Also $\mathbb{P}(N_t \le n) \ge \mathbb{P}(N_n \le n) \ge c_0 > 0$: Stirling's formula gives an atom of order $n^{-1/2}$ at n for N_n, and the $\lfloor \sqrt{n}/2 \rfloor$ atoms immediately below it are each comparable to it by the exact ratios of Poisson masses. The correction is consequently $O(\sqrt{n})$. For $n \le t \le 3n/2$ the conditional law $N_t \mid N_t \le n$ is stochastically increasing by monotone likelihood ratio. At $t = n$ its expected deficiency is $O(\sqrt{n})$ by Cauchy–Schwarz and the same conditioning bound. For $t \ge 3n/2$ successive deficiency masses have ratio at most $q = n/t$. If $p_j = \mathbb{P}(n - N_t = j \mid N_t \le n)$, then $\sum_j p_{j+k} \le q^k$, so the conditional mean deficiency is at most $n/(t-n)$. The quotient on this last range is $O(1)$. These bounds place the positive-branch maximum inside $0 \le t < 3n/4$ for large n. Comparison to t_* there proves the two exponential estimates.

For the negative branch, set $b = -r$ and $a = 1 + r$ with $0 \le r \le 1/(n-1)$. The identity $\operatorname{per}(aI + bJ) = \int_0^\infty (a + bu)^n e^{-u}\, du$ follows by expanding the polynomial and hence is valid for negative b as well. The cofactor diagonal d and row sum w have the integral expressions

$$d = \int_0^\infty (a - ru)^{n-1} e^{-u}\, du, \qquad w = \int_0^\infty u(a - ru)^{n-1} e^{-u}\, du.$$

The other eigenvalue is $(nd - w)/(n-1)$. On $0 \le u \le a/r$, one uses $a^{n-1} \le e$. Translating the two tails for $r > 0$ gives absolute bounds $e^{-a/r} r^{n-1}(n-1)!$ and that quantity times $a/r + n$, respectively. Since $r^{n-1}(n-1)! \le 1$ and $a/r \ge n$, both eigenvalues are uniformly bounded. The case $r = 0$ is the identity, and $\operatorname{per} B(b) \ge 1$ on this branch. Thus its quotient is $O(1)$, proving Proposition 2.

3 Spectral variation and directed edges

For the remaining proof write $B = aI + bJ$, $P = \operatorname{per} B$, $Q = C(B)/P$, and $e = n^{-1/2}\mathbf{1}$. We evaluate at $b = b_n, a = 1 - b_n$ after taking partial derivatives at fixed a where specified. The top eigenvalue of Q is simple and its eigenvector is e. Denote the gap to the other, common eigenvalue by $\Delta > 0$. Simplicity gives smoothness near B_n. Permutation invariance and invariance under complex conjugation imply that the first variation in a zero-diagonal Hermitian direction is a scalar multiple of the sum of the real off-diagonal entries. Since b_n is an interior branch maximizer, this scalar is zero.

For a Hermitian zero-diagonal X with total sum zero, write $P'' = D_X^2 \operatorname{per}(B)$. The first variation $D_X \operatorname{per}(B)$ is zero by the same invariance of the permanent. The identity $e^* C(B) e = n^{-1} \partial_b P$ holds also along $B + sX$ if a, X are held fixed. The second variation of the simple eigenvalue is therefore

$$D_X^2 \Lambda = \frac{1}{n} \partial_b (P''/P) + \frac{2}{\Delta} \| (I - ee^*) D_X Q\, e \|^2. \tag{2}$$

Write $r_i = \sum_j X_{ij}$ for its row sums. Equivariance makes $D_X Q e$ zero when $r = 0$. Indeed, for a real symmetric input the coefficient of each unordered off-diagonal edge at a given output coordinate depends only on incidence with that coordinate. When the total sum is zero, the image is a scalar multiple of r. For a purely imaginary Hermitian input, exchanging the ends of an edge disjoint from the output coordinate fixes the coordinate and negates the edge. Such edges contribute zero. Every incident edge contributes a constant times the entry oriented from the output coordinate. These statements also identify the mixing term in each pure sector of total sum zero as a scalar multiple of $\|r\|^2$.

Expand the permanent by the diagonal part:

$$\operatorname{per}(aI + bJ + sX) = \sum_{S \subseteq [n]} a^{n-|S|} \operatorname{per}((bJ + sX)[S]).$$

A second derivative chooses two ordered directed X-edges with distinct rows and distinct columns. When their union contains $m = 2, 3, 4$ vertices, the coefficient of that ordered product in P''/P is

$$D_m = \frac{\mathbb{E}[(K-2)_{m-2}\, \mathbf{1}_{\{K \geq 2\}}]}{b^2 (n)_m}, \qquad K = n - (N_t \mid N_t \leq n). \tag{3}$$

Here $(x)_j$ is a falling factorial. A principal set of size k in the expansion has aggregate weight proportional to $\binom{n}{k} a^{n-k} k! b^k$, so its size has law K. For $k \geq m$ the chance it contains the designated vertices is $(k)_m/(n)_m$, and the two specified edges leave $(k-2)!$ permutations instead of $k!$. The coefficient also contains b^{-2}. Events with $2 \leq k < m$ give zero through the falling factorial in (3).

Write $F_2 = \|X\|_F^2$ and $s_0 = \sum_i r_i$. The sums of ordered edge products of each type are

$$\begin{aligned} &\Sigma_2 = F_2, && \Sigma_3 = 2(\|r\|^2 - F_2), \\ &\Sigma_4 = s_0^2 - 2 \operatorname{Re} \sum_i r_i^2 + \sum_{i,j} X_{ij}^2 - \Sigma_2 - \Sigma_3. \end{aligned} \tag{4}$$

The two-vertex pairs are precisely the reciprocal edges. The three-vertex pairs have one common endpoint, outgoing for one edge and incoming for the other. The oriented sum for each order is $\|r\|^2$ before removing reciprocal edges. For the total of all admissible pairs, subtract common-source and common-target pairs from s_0^2, then add identical edges once. This yields (4), and $P''/P = \sum_{m=2}^{4} D_m \Sigma_m$. At zero row sums the relative coefficients Σ_m/F_2 are $(1, -2, 2)$ for real X and $(1, -2, 0)$ for imaginary X.

At $t = t_n$, the conditioning in (3) and the truncation at $K \geq 2$ alter the polynomial moments by exponentially small errors times powers of n. This follows from Poisson exponential tail estimates for each fixed polynomial degree. The conclusion holds after one t-derivative by the identity $\partial_t \mathbb{E} H(N_t) = \mathbb{E}[H(N_t + 1) - H(N_t)]$, including for the truncated polynomials and the conditioning probability. The three unconditioned polynomial numerators are

$$1, \qquad n - t - 2, \qquad (n - t - 2)(n - t - 3) + t.$$

Since $\partial_b t = -t/b$ at fixed a, differentiation gives

$$\frac{\partial_b D_2}{n} \longrightarrow -\frac{1}{4}, \qquad \frac{\partial_b D_3}{n} \longrightarrow -\frac{1}{16}, \qquad \frac{\partial_b D_4}{n} \longrightarrow 0.$$

For D_4, the leading terms from differentiating the numerator and b^{-2} cancel, using $t_n \sim n/2$, $b_n \sim 2/n$. When $r = 0$, the mixing term in (2) vanishes. The two triples in (4) then give the two $-1/8 + o(1)$ coefficients.

4 Real star directions

Let v be a real vector of sum zero. Set $w_i = b_n + \eta v_i$ and

$$B(\eta) = \operatorname{diag}(1 - w_i) + xx^*, \qquad x_i = \sqrt{w_i}.$$

For η near zero this is a correlation matrix. Its tangent is the real star $X_{ij} = (v_i + v_j)/2$ for $i \neq j$, with $\|X\|_F^2 = (n-2)\|v\|^2/2$. Stationarity at B_n means the second derivative along this family equals the affine Hessian on X.

For $T \geq 0$ put $f_i = 1 + (T-1)w_i$, and use the probability density proportional to $e^{-T}\prod_i f_i$. The normalized raw cofactor matrix is the expectation of

$$R(T) = \operatorname{diag}(h(w_i)) + Tgg^*, \qquad h(w) = \frac{1-w}{[1+(T-1)w]^2}, \qquad g_i = g(w_i) = \frac{\sqrt{w_i}}{1+(T-1)w_i}. \quad (5)$$

To see this, expand the permanent along the diagonal part and represent the factorials of the rank-one principal submatrices by gamma integrals. A diagonal cofactor omits one f_i. An off-diagonal cofactor has an unmatched row and column, producing the factor $\sqrt{w_i w_j}T$ and omitting f_i, f_j. The diagonal of (5) equals $1/f_i$, as required.

At $\eta = 0$ write $b = b_n$, $a = 1-b$, and $f = a + bT$. Primes on h, g denote w-derivatives evaluated at b. The logarithm of the unnormalized density has first derivative zero pointwise and second derivative $-\|v\|^2(T-1)^2/f^2$. The row sum of $R(T)$ at this point is $\lambda_T = h + nTg^2$, and $\Delta = n\mathbb{E}(Tg^2)$. Differentiating (5) shows that the first derivative of $R(T)e$ is $(h' + nTgg')v/\sqrt{n}$. The projection of the second derivative on e gives $\|v\|^2(h''/n + 2Tgg'')$. Differentiating the expectation and applying the simple-eigenvalue formula thus give

$$\frac{D_X^2\Lambda}{\|v\|^2} = \mathbb{E}[h''/n + 2Tgg''] - \operatorname{Cov}\big(\lambda_T, (T-1)^2/f^2\big) + \frac{2}{n\Delta}\big(\mathbb{E}[h' + nTgg']\big)^2. \quad (6)$$

The random variable $T + t_n$ has the $\Gamma(n+1, 1)$ law conditioned to be at least t_n, since $f = b(T + t_n)$. Proposition 2 and gamma moments imply $T/n \to 1/2$ in probability and $\mathbb{E}|T - t_n|^k = O_k(n^{k/2})$ for $k = 2, 4$. Also

$$\frac{1}{4a} - Tg^2 = \frac{(a-bT)^2}{4a(a+bT)^2}.$$

This identity and the fourth moment give $\operatorname{Var}\lambda_T = O(1)$, while $\Delta \asymp n$. The function $(T-1)^2/f^2$ has derivative $2(T-1)/f^3$, uniformly $O(n)$, giving variance $O(n^3)$ using the second moment about t_n. The covariance in (6) is thus $O(n^{3/2})$. Direct differentiation gives $h''/n = O(n)$ and $h' = O(n)$ uniformly. Moreover

$$g' = \frac{1-(T-1)b}{2\sqrt{b}f^2}.$$

The numerator has second moment $O(n^{-1})$, and $T/f^3 = O(n)$, so $\mathbb{E}|Tgg'| = O(\sqrt{n})$. The last term in (6) is $O(n)$.

The remaining contribution is determined by

$$gg'' = -\frac{1}{4bf^2} - \frac{T-1}{f^3} + \frac{2b(T-1)^2}{f^4}.$$

Set $q = (T-1)b$; then $q \to 1$ in probability. At $q = 1$ the right side equals $-1/(16b)$. The rational expression for $2Tgg''/n^2$ is uniformly bounded for $q \geq -b$, so $T/n \to 1/2$ and $nb \to 2$ give convergence in mean to $-1/32$. Equation (6) is therefore $-n^2/32 + o(n^2)$. Dividing by $(n-2)/2$ in the norm identity yields the $-n/16 + o(n)$ coefficient.

5 The constant direction and the phase orbit

On a fixed relative neighborhood of $t_n \sim n/2$, (1) is $(1+t)(1-t/n)$ plus an exponentially small correction times a polynomial in n, also after up to two t-derivatives. In fact the atom $p_n(t) =$

$\mathbb{P}(N_t = n)$ satisfies $p_n' = (n/t - 1)p_n$, and the conditioning probability has derivative $-p_n(t)$. The scalar derivative at b_n is zero, and $dt/db = -b^{-2}$ along the correlation branch. Therefore its second derivative in b is $-2/(nb_n^4) + o(n^3)$. The constant off-diagonal direction with entries one has $\|X\|_F^2 = n(n-1)$, giving $-n/8 + o(n)$.

For $U = \mathrm{diag}(u_i)$ diagonal unitary, the retained row and column factors give

$$C(UBU^*)_{ij} = \overline{u}_i u_j C(B)_{ij}.$$

The permanent is unchanged and $C(UBU^*) = U^* C(B) U$. The orbit at B_n has tangent the imaginary gradients. Differentiating the invariance at the stationary point shows that the Hessian vanishes when either input is tangent to this orbit. The imaginary gradients are Frobenius orthogonal to the purely imaginary Hermitian matrices with zero row sums. The constant real direction has zero mixed terms with its orthogonal complement by permutation averaging. Complex conjugation separates the real and imaginary sectors. In the total-sum-zero real sector, (2) and (4) express the Hessian as a linear combination of Frobenius and row-sum squares. These forms have zero mixed terms between real stars and real zero-row-sum directions. These subspaces exhaust the tangent space, giving the claimed Hessian and its kernel.

An explicit local slice to the phase orbit is obtained by making the nonzero first-row off-diagonal entries real and positive by diagonal-unitary conjugation. Its tangent intersects the imaginary gradients only at zero. Restricting the Hessian to the slice therefore gives a negative definite form. The matrix B_n is positive definite, so the slice lies locally in the interior of the correlation domain. Taylor's theorem on the slice proves strict local maximality modulo the phase orbit and completes the proof of Theorem 1.

AI assistance

OpenAI Codex and Anthropic Claude assisted with proof development, independent checking, numerical verification and manuscript preparation. The author is responsible for the work.

References

[1] R. Bhatia, P. Grover, and T. Jain, *Derivatives of tensor powers and their norms*, Electron. J. Linear Algebra **26** (2013), 604–619. doi:10.13001/1081-3810.1675.

[2] R. Grone, C. R. Johnson, E. M. Sá, and H. Wolkowicz, *A note on maximizing the permanent of a positive definite Hermitian matrix, given the eigenvalues*, Linear Multilinear Algebra **19** (1986), 389–393. Author copy.

[3] Y. Lavi, *A conjecture on the permanent of positive semidefinite matrices*, 2026. Preprint, version 1.

[4] E. H. Lieb, *Proofs of some conjectures on permanents*, J. Math. Mech. **16** (1966), 127–134.

[5] M. Marcus and R. Merris, *A relation between the permanental and determinantal adjoints*, J. Austral. Math. Soc. **15** (1973), 270–271. doi:10.1017/S1446788700013173.

[6] L. Pioge, K. K. Pietrasz, B. Seron, L. Novo, and N. J. Cerf, *A logical implication between two conjectures on matrix permanents*, Linear Algebra Appl. **725** (2025), 309–318. arXiv:2508.00111.

[7] S. Schubr, *Asymptotically optimal bounds for permanental cofactor matrices*, Luminary Research Archive, ed. 4, 2026. Archive edition 4.