Local comparison counting for linear extensions and a bounded-range balance refinement

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Luminary Oddities

Abstract

We determine the smallest positive probability of a specified order of three vertices in a uniformly random linear extension of a finite poset with maximum incomparability degree at most D. Writing $c_j = \binom{2j}{j}$, its reciprocal is

$$M_D = \frac{c_D}{2} + \frac{\lfloor D^2/4 \rfloor + D}{2D} c_{\lfloor D/2 \rfloor} c_{\lceil D/2 \rceil} \qquad (D \geq 2).$$

Two-chain bridge posets attain this bound. In particular $M_D \sim 4^D/(4\pi)$. The proof first conditions on comparisons that remain unresolved after the desired order is imposed. This reduces the extremal problem to an event realized by a single extension. A structural reduction then leaves two chains and a middle vertex; an exact finite difference locates the extremal insertion threshold. For four marked vertices the exact reciprocal is $\binom{2D}{D}^2$. More generally, writing $B_D = \binom{2D}{D}$, the exact reciprocal is B_D^r for $2r$ marked vertices and $M_D B_D^{r-1}$ for $2r+1 \geq 3$ marked vertices, for every $D \geq 2$. For even mark counts we characterize equality and determine the second smallest positive probability. Excluding saturated two-chain incomparability components gives a further exact formula for every mark count. Thus the optimal exponential denominator rate for fixed m is $4^{D\lfloor m/2 \rfloor}$. Combining the sharp triple estimate with the current quantitative balance analysis gives a bounded-range increment asymptotic to $(4\pi/(5+3\sqrt{5}))4^{-D}$ above the classical BFT constant for posets with bounded incomparability degree. This replaces the increment $(D+1)^{-3(D+1)}/12$ of Aires, Chan, Pak and Panova, which they sharpen to $(D+1)^{-3(D+1)}/(5+3\sqrt{5})$, by one of order 4^{-D}, and the exact marked-order floors sharpen their bound $(D+1)^{-m(D+1)}$ for events specified by comparisons among m elements.

1 Definitions and the sharp denominator

Let $P = (V, <_P)$ be a finite poset. A linear extension is a total order on V containing every relation of P. Write $e(P)$ for the number of extensions and use their uniform distribution. Define

$$\mathrm{inc}(P) = \max_{x \in V} |\{y \in V : y \parallel_P x\}|.$$

This is a bound on the number of incomparable neighbors of each vertex, not a width or cardinality bound. A specified marked order is *consistent* if some linear extension realizes it. Adding consistent comparisons always means taking their transitive closure.

The sharp global pair-extremum and its two-block shuffle examples have classical antecedents in Fishburn's work [1, 2]. We include the local pair argument needed here. The triple theorem below concerns an extremum under a local incomparability-degree restriction. The balance application below explicitly separates the external BFT/ACP inputs from the new finite-range deduction.

Theorem 1. *For an integer $D \geq 2$, put $c_j = \binom{2j}{j}$ and*

$$M_D = \frac{c_D}{2} + \frac{\lfloor D^2/4 \rfloor + D}{2D} c_{\lfloor D/2 \rfloor} c_{\lceil D/2 \rceil}.$$

Every consistent order $a < b < c$ of three distinct marked vertices in a finite poset P with $\mathrm{inc}(P) \leq D$ satisfies

$$\mathbb{P}_P(a < b < c) \geq M_D^{-1}.$$

For every $D \geq 2$ equality is attained by an explicitly described bridge poset. Moreover $M_D \sim 4^D/(4\pi)$. At $D = 0$ the sharp denominator is $M_0 = 1$; at $D = 1$ it is $M_1 = 2$.

The proof gives explicit attainers. A positive event may originally have comparable marked vertices; the theorem includes this case. Theorem 9 extends this exact extremum to every number of marks: the sharp denominators are products of the pair denominator B_D and at most one triple denominator M_D. The bounded insertion estimate and a classical entropy argument give this extension in every degree.

2 The local pair bound and exact conditioning

For a vertex x, write $x{\downarrow} = \{u : u \leq_P x\}$ and $x{\uparrow} = \{u : x \leq_P u\}$.

Lemma 2. *If $x \parallel_P y$, let*

$$A = (x{\downarrow}) \setminus (y{\downarrow}), \qquad B = (y{\uparrow}) \setminus (x{\uparrow}).$$

Then $\mathbb{P}_P(x < y) \geq \binom{|A|+|B|}{|A|}^{-1}$. In particular, if $\mathrm{inc}(P) \leq D$, every consistent pair orientation has probability at least B_D^{-1}, where $B_D = \binom{2D}{D}$.

Proof. The sets A, B are disjoint. In any extension keep the positions occupied by $A \cup B$ and every outside vertex fixed. Refill these positions with its A subsequence followed by its B subsequence, preserving internal orders. Thus A moves weakly left and B weakly right. No relation runs from B to A, since such a relation would imply $y <_P x$.

For completeness, suppose an outside vertex w precedes some $u \in A$ in P. Then $w \leq_P x$ and $w \notin A$ imply $w \leq_P y$. Hence w precedes every member of B. Moving u into an earlier B position cannot cross w. Dually an outside successor of a member of B follows every member of A, so moving B right cannot cross it. The remaining outside relations are preserved by the directions of movement. The output is therefore an extension with $x < y$.

The output and the original binary A/B membership pattern recover the input. Each output has at most $\binom{|A|+|B|}{|A|}$ preimages. Finally $A \subseteq \{u : u \parallel_P y\}$ and $B \subseteq \{u : u \parallel_P x\}$, so both sizes are at most D. A forced orientation has probability one. □

Lemma 3 (Finite conditioning)**.** *Let E be a consistent event given by adding comparisons to P, and let Q be the resulting augmented poset. If u, v are incomparable in Q, at least one of $P_+ = P + (u < v)$ and $P_- = P + (v < u)$ has*

$$0 < \mathbb{P}_{P_\pm}(E) \leq \mathbb{P}_P(E).$$

Incomparability degrees cannot increase under this operation.

Proof. Both orientations extend to a linear extension of Q, since u, v are incomparable there. Thus both refined events are positive. The extensions of P partition according to the orientation of u, v, giving exactly

$$\mathbb{P}_P(E) = \frac{e(P_+)\mathbb{P}_{P_+}(E) + e(P_-)\mathbb{P}_{P_-}(E)}{e(P_+) + e(P_-)}.$$

Choose a branch whose value is no larger than this weighted average. Adding relations only removes incomparable pairs. In particular the refined law is uniform on a poset's extensions; this is not conditioning on an arbitrary subset of extensions. □

3 A uniquely realized triple order

Lemma 4. *Suppose $a \parallel_P b$ and $b \parallel_P c$, and exactly one extension realizes $a < b < c$. Then P is the union of a chain ending at a, the vertex b, and a chain beginning at c, with possible extra forward relations between the two chains. An original relation $a <_P c$ is permitted.*

Proof. The poset $Q = P + (a < b < c)$ has a unique extension and hence is total: every incomparable pair in a finite poset admits both orders in linear extensions. There is no vertex z strictly between a and b in Q. A path from a to z cannot use an added edge, since such an edge would pass through b or c. Likewise a path from z to b cannot use an added edge: using $a < b$ would require $z \le_Q a$, and using $b < c$ would require $c \le_Q b$. Thus $a <_P z <_P b$, a contradiction. The same argument excludes vertices strictly between b and c.

All relations among the vertices at or below a in Q were already in P: a path using an added edge first reaches b, above all these vertices. They form a P-chain. Dually the vertices at or above c form a P-chain. These chains and b exhaust V. Any cross-chain relation points from the first chain to the second. Relations from the first chain to b form a prefix and relations from b to the second chain form a suffix. The opposite orientations would contradict the order of Q. □

Lemma 5. *Under Lemma 4, if $\mathrm{inc}(P) \le D$, then $e(P)$ is at most the extension count of a bridge with two chains of length $D-1$ and relations*

$$x_l < b < y_{l+2}, \qquad 0 \le l \le D-2.$$

Here the poset is generated by these relations and the two chains, including all their transitive consequences. At $l = 0$ the lower relation is absent; at $l = D-2$ the upper relation is absent. In particular $D = 2$ gives a three-element antichain. These bridges have exactly one extension with $x_{D-1} < b < y_1$ and actual maximum incomparability degree D.

Proof. Write the two chains as $X = (x_1 < \cdots < x_s = a)$ and $Y = (c = y_1 < \cdots < y_t)$. Let u be the number of X vertices below b, v the number of Y vertices not above b, p the number of X vertices below y_1, and q the number of Y vertices not above x_s. Then $s - u \ge 1$, $v \ge 1$, while $p = s$ or $q = 0$ is allowed. The degrees of b, c, a, respectively, give

$$s - u + v \le D, \qquad s - p + 1 \le D, \qquad q + 1 \le D.$$

Delete the first $\min(u, p)$ vertices of X. They form a forced ordinal prefix, lying below b, every Y vertex, and every remaining X vertex. Delete also all Y vertices after position $\max(v, q)$; they form a forced ordinal suffix by the dual argument. These deletions preserve the extension count exactly. The remaining chain lengths satisfy

$$s - \min(u, p) = \max(s - u, s - p) \le D - 1, \qquad \max(v, q) \le D - 1.$$

The number incomparable with b is unchanged.

Replace the remaining poset by the transitive closure generated solely by the two chains and their relations to b. Each generator was already a relation, so the extension count can only increase. Cross-chain relations forced through b are retained; only other cross-chain relations are discarded. The incomparability degree is still at most D: a chain vertex sees at most the entire other chain and b, while the degree of b has not changed. Enlarge X at its bottom

by new vertices declared below b, and enlarge Y at its top by new vertices declared above b, until both have length $D-1$. Each old extension injects into a new one by placing the new X vertices first and the new Y vertices last. Degrees are still at most D.

Let $h \geq 1$ be the incomparable tail length of the enlarged X and put $l = D-1-h$. The incomparable prefix length of Y is at most $D-h = l+1$, by the degree of b. Remove upper relations from b until that prefix length is exactly $l+1$. This gives the displayed bridge and can only increase its extension count. Here $0 \leq l \leq D-2$. The endpoint x_{D-1} is incomparable with all of Y and with b, so its degree is D; the dual endpoint has the same degree. The degree of b is $h+(l+1) = D$. Its marked favorable order forces all of X, then b, then all of Y, so is unique. □

4 Exact bridge counting

Set $N = D-1$ and, for $0 \leq i, j \leq N$, define

$$A_N(i,j) = \binom{i+j}{i}\binom{2N-i-j}{N-i}.$$

The bridge in Lemma 5 has extension count

$$T(D,l) = \sum_{i=l}^{N}\sum_{j=0}^{l+1} A_N(i,j). \tag{1}$$

Indeed i, j count the X, Y vertices before b. The two binomial factors count the prefix and suffix interleavings. Use the convention that binomial coefficients outside their support are zero.

Lemma 6. *For $0 \leq l \leq N-2$,*

$$T(D,l+1) - T(D,l) = \frac{N-2l-2}{N+1} A_N(l,l+2). \tag{2}$$

Consequently the maximum of $T(D,l)$ occurs at $l = \lfloor (D-2)/2 \rfloor$, with a second tied central maximum exactly when D is odd.

Proof. Every row and column of A_N has sum $B = \binom{2N+1}{N}$. For a row with index i, count words containing $N+1$ letters X and N letters Y according to the number j of Y letters before their $(i+1)$st X. Removing this distinguished X leaves exactly the two factors of $A_N(i,j)$. Let $H_l = \sum_{j=l+2}^{N} A_N(l,j)$. Removing and adding rectangle boundaries in (1), using the common row sum and the symmetries $A_N(i,j) = A_N(j,i) = A_N(N-i, N-j)$, gives

$$T(D,l+1) - T(D,l) = H_l - H_{N-l-2}.$$

In the same uniform word model, H_l/B is the probability that the first $2l+2$ letters contain at least $l+2$ letters Y. Reversing the word shows that H_{N-l-2}/B is the probability that the first $2l+3$ letters contain at most l letters Y.

Put $k = l+1$ and $w = N-k$. Thus

$$H_l = \sum_{j \geq k+1} \binom{2k}{j}\binom{2w+1}{N-j}, \qquad H_{N-l-2} = \sum_{j \leq k-1} \binom{2k+1}{j}\binom{2w}{N-j}.$$

Apply Pascal's identity to the odd-length binomials, and in the first sum replace j by $2k-j$, using complement symmetry in the even-length binomials. Their difference becomes

$$\sum_{j=0}^{k-1}\left[\binom{2k}{j}\binom{2w}{N-j+1} - \binom{2k}{j-1}\binom{2w}{N-j}\right]. \tag{3}$$

First suppose $k \le N/2$, so the following active-support divisions are valid. Set $f_j = \binom{2k}{j}\binom{2w}{N-j}$ and $f_{-1} = 0$. The jth summand of (3) equals

$$(N-2k)f_j \frac{2k+1-2j}{(N-j+1)(2k-j+1)}.$$

Since

$$\frac{f_{j-1}}{f_j} = \frac{j(N-2k+j)}{(N-j+1)(2k-j+1)},$$

and the denominator minus the numerator is $(N+1)(2k+1-2j)$, the sum in (3) telescopes to $(N-2k)f_{k-1}/(N+1)$. This is (2).

Reversing both chains and interchanging their names gives $T(D,l) = T(D, N-l-1)$. This extends the difference identity to the other half, and proves the claimed maximum by the sign of $N-2l-2$. When $N=1$ there is only one threshold and no difference to prove. □

Lemma 7. *Writing $c_j = \binom{2j}{j}$ and $k = l+1$, one has*

$$T(D,l) = \frac{c_D}{2} + \frac{k(D-k)+D}{2D} c_k c_{D-k}. \tag{4}$$

In particular its maximum is M_D from Theorem 1.

Proof. At $l = 0$, b occupies either of the first two positions relative to the Y chain. Each relative order allows $\binom{2D-1}{D-1}$ interleavings with X. Hence $T(D,0) = 2\binom{2D-1}{D-1} = c_D$. Define, with $k+q = D$,

$$F_k = \frac{kq+D}{2D} c_k c_q.$$

Then $F_1 = c_D/2$. Expressing $F_{k+1} - F_k$ in terms of $c_k c_{q-1}/D$ gives the bracket

$$\frac{((k+1)(q-1)+D)(2k+1)}{k+1} - \frac{(kq+D)(2q-1)}{q}.$$

After multiplication by $(k+1)q$, this bracket has numerator

$$k\left[q^2 - (k+2)q + (k+1)\right] = k(q-1)(q-k-1).$$

Also

$$A_{D-1}(k-1, k+1) = c_k c_{q-1} \frac{k(q-1)}{(k+1)q}.$$

Thus $F_{k+1} - F_k$ equals the difference in Lemma 6. Starting with $F_1 = c_D/2$ proves (4) for all thresholds. At a central threshold $k(D-k) = \lfloor D^2/4 \rfloor$, giving the stated maximum. □

5 Proof of the sharp universal theorem

Proof of Theorem 1. Let $E = \{a < b < c\}$ be consistent. If $a <_P b$ or $b <_P c$, it is a single pair event and Lemma 2 gives reciprocal probability at most $B_D = c_D = T(D,0) \le M_D$. Otherwise the adjacent marked pairs are incomparable. Put $Q = P + (a < b < c)$. If Q is not total, choose a pair incomparable in Q and apply Lemma 3, retaining the branch with no larger event probability. If an adjacent marked comparison becomes forced, use the pair case and stop. Otherwise repeat. Each step makes at least one previously incomparable pair comparable, so this procedure terminates after finitely many steps.

If it does not stop in the pair case, it ends with an event realized by exactly one extension. Lemmas 4 and 5 show that its reciprocal probability is at most $T(D,l)$ for some l, hence at

most M_D. Throughout, the incomparability bound did not increase and the event probability did not increase. Thus the original probability is at least M_D^{-1}. A central bridge in Lemma 5 attains exactly this value.

For $D = 0$ every positive marked order is forced. For $D = 1$, the middle marked vertex cannot be incomparable with both other marked vertices. Consistency therefore forces an adjacent comparison, and the pair bound gives $1/2$. An antichain of two vertices followed by a third vertex attains $1/2$.

Finally Stirling's formula gives

$$c_D \sim \frac{4^D}{\sqrt{\pi D}}, \qquad c_{\lfloor D/2 \rfloor} c_{\lceil D/2 \rceil} \sim \frac{2\,4^D}{\pi D}.$$

The coefficient of the latter product is $D/8 + O(1)$, while the former term is smaller by a factor of order $D^{-1/2}$. Hence $M_D \sim 4^D/(4\pi)$. □

Corollary 8. *There is no fixed positive constant c for which all positive marked triple orders in degree at most D have probability at least c/B_D.*

Proof. The attaining bridges have reciprocal M_D, and Stirling's formula gives $M_D/B_D \sim \sqrt{D}/(4\sqrt{\pi}) \to \infty$. □

6 Exact probabilities for every number of marks

Write $B = B_D = \binom{2D}{D}$ and $M = M_D$ for $D \geq 2$, and define

$$Q_1 = 1, \qquad Q_{2r} = B^r \ (r \geq 1), \qquad Q_{2r+1} = MB^{r-1} \ (r \geq 1).$$

Theorem 9. *For every pair of integers $D \geq 2$ and $m \geq 1$, the exact smallest positive probability of a specified order of m distinct marked vertices, over finite posets of incomparability degree at most D, is Q_m^{-1}. For even m it is attained by an ordinal sum of sharp pair gadgets; for odd $m \geq 3$, by one sharp triple bridge and sharp pair gadgets. For degree one the exact probability is $2^{-\lfloor m/2 \rfloor}$; for degree zero it is one.*

The four-mark denominator is therefore exactly B_D^2. For each fixed m, the optimal exponential denominator rate is $4^{D\lfloor m/2 \rfloor}$. These conclusions hold without a cutoff relating m to D. Section 7 characterizes even-order equality and gives exact probabilities after saturated pair components are excluded.

Lemma 10. *For all positive i, j, one has $Q_i Q_j \leq Q_{i+j}$.*

Proof. The endpoint triple bridge has count B, so $M \geq B$. Ignoring all restrictions on the position of its middle vertex bounds every bridge count by $(2D-1)\binom{2D-2}{D-1} = DB/2$. Also $B \geq D^2 + 1$, by counting a fixed D-subset of a $2D$-set and its D^2 distinct one-element exchanges. Consequently

$$B \leq M \leq DB/2, \qquad M^2 \leq B^3, \qquad M \leq B^2.$$

For two even indices, or an even index and an odd index at least three, the product assertion is an equality. For two odd indices at least three it is exactly $M^2 \leq B^3$. The singleton cases use respectively $1 \leq B$, $B \leq M$, and $M \leq B^2$. □

Lemma 11 (Bounded insertion)**.** *For any vertex v of a finite poset R with* $\mathrm{inc}(R) \leq D$,

$$e(R) \leq (D+1)e(R - v).$$

Proof. Fix an extension of the induced deletion $R - v$. Legal insertion slots for v lie after its last predecessor and before its first successor, with list endpoints used when either set is empty. If these boundaries are inconsistent there are no legal slots. Otherwise every vertex between them is incomparable with v: another predecessor or successor there would contradict the chosen boundary. There are at most D such vertices, hence at most $D + 1$ slots. Restriction of an extension of R determines its extension of $R - v$ and its insertion slot, proving the inequality. □

Lemma 12. *Let P be a finite poset with $\mathrm{inc}(P) \le D$, where $D \ge 2$ is an integer, and let $m \ge 4$ be an integer. Suppose the specified order of m marked vertices has exactly one favorable extension and all adjacent marked pairs are incomparable. After removing forced ordinal ends, the poset has two outside chains of length at most $D - 1$ and the $m - 2$ middle marks. Its extension count is at most Q_m.*

Proof. Let $Q = P + (x_1 < \cdots < x_m)$, which is total. No vertex can lie strictly between adjacent marks in Q: a path from x_i to an intervening vertex cannot use an added marked arrow without already reaching x_{i+1} or later, and the dual path cannot use one either. Both relations would therefore have been in P, contradicting $x_i \parallel x_{i+1}$. Thus Q has the form

$$X, x_2, \ldots, x_{m-1}, Y,$$

where X ends at x_1, Y starts at x_m, and both are chains in P.

Remove the common initial segment of X below all later marks, and the common terminal segment of Y above all earlier marks. They are respectively below and above every remaining vertex, so their removal preserves extension counts exactly. The marked endpoints survive. For each later mark x_j, the X vertices not below it form a tail of incomparable vertices. The mark also has an adjacent incomparable marked neighbor outside X: use x_3 when $j = 2$ and x_{j-1} otherwise. Every such tail has length at most $D - 1$, so their union, the residual X, does too. The dual argument bounds the residual Y by $D - 1$. Let R be the actual induced residual poset. Its incomparability degree is still at most D.

Delete the $m-2$ middle marked vertices from R. The remaining induced poset consists of its two chains, possibly with cross-relations; ignoring those cross-relations only for this final count gives at most

$$C_D = \binom{2D-2}{D-1} = \frac{DB}{2(2D-1)}$$

extensions. Iterated application of Lemma 11, always to actual induced subposets of R, gives

$$e(P) = e(R) \le (D+1)^{m-2} C_D. \tag{5}$$

The counting relaxation is not used as an intermediate reinsertion poset.

For $D \ge 3$, $B \ge (D+1)^2$: at $D = 3$ this is $20 \ge 16$; for $D \ge 4$, central maximality gives $B \ge \binom{2D}{2} = D(2D-1)$, and the difference from $(D+1)^2$ is $D^2 - 3D - 1 > 0$. For the even base $m = 4$, therefore, $(D+1)^2 C_D \le B^2 = Q_4$. For the odd base $m = 5$ it suffices to prove $(D+1)^3 C_D \le B^2 \le MB$. The first inequality is equivalent to $(D+1)^3 \le 2(2D-1)B/D$. Since $B \ge \binom{2D}{3} = (2/3)D(D-1)(2D-1)$, the ratio of its right side to its left side is at least

$$\frac{4}{3}\frac{D-1}{D+1}\left(\frac{2D-1}{D+1}\right)^2 \ge \frac{4}{3}\cdot\frac{1}{2}\left(\frac{5}{4}\right)^2 = \frac{25}{24} > 1.$$

Both ratios increase with $D \ge 3$. Increasing m by two multiplies the right side of (5) by $(D+1)^2 \le B$, while Q_m is multiplied by B. This proves every parity case for $D \ge 3$.

For $D = 2$, the residual poset has at most m vertices, and Lemma 13 below bounds its extension count by $6^{m/3}$. For $m \ge 4$, $m/3 \le \lfloor m/2 \rfloor$, and $M_2 = B_2 = 6$. Thus this too is at most Q_m. □

The remaining degree-two estimate is the classical Brégman inequality specialized to allowed rank intervals. We give its finite entropy proof, following Radhakrishnan's random-row method [7], so that no external certificate or unproved counting premise is needed.

Lemma 13. *If a finite poset R has d_v vertices incomparable with v, then*

$$\log e(R) \le \sum_{v \in R} \frac{\log((d_v+1)!)}{d_v+1}.$$

In particular, degree at most two gives $e(R) \le 6^{|R|/3}$.

Proof. For a finite random variable Z with probabilities p_z, define $H(Z) = -\sum_{p_z>0} p_z \log p_z$. Conditional entropy is the average of this expression over the conditioning values. The chain rule $H(Z_1,\dots,Z_n) = \sum_i H(Z_i \mid Z_1,\dots,Z_{i-1})$ follows by factoring each joint probability into its conditional probabilities and taking the expectation of its negative logarithm. A variable supported on at most r values has entropy at most $\log r$: Jensen's inequality gives $\sum p_z \log(1/p_z) \le \log \sum p_z(1/p_z) \le \log r$.

Let a_v be the number of strict predecessors of v and put $r_v = d_v + 1$. Every extension rank of v lies in $I_v = \{a_v+1,\dots,a_v+r_v\}$, since all predecessors precede v and all successors follow it. Let σ be a uniform extension rank permutation, and independently choose a uniform reveal order of the vertices. For each fixed reveal order, the chain rule expresses $H(\sigma) = \log e(R)$ as a sum of conditional rank entropies. After the previously revealed ranks are fixed, the rank of v is supported on its still-unused columns in I_v, so its conditional entropy is at most the logarithm of their number.

Fix a full extension σ. Its r_v columns in I_v are occupied by r_v distinct vertices including v. In the independent reveal order, the relative position of v among these vertices is uniform. Hence the number of unused allowed columns when v is reached is uniform on $\{1,\dots,r_v\}$. Average the support bounds over this reveal order and then over σ. The contribution of v is at most $\sum_{j=1}^{r_v} \log j/r_v = \log(r_v!)/r_v$, proving the formula. For $r_v \le 3$, the largest value of $(r_v!)^{1/r_v}$ is $6^{1/3}$, which gives the final assertion. □

Proof of Theorem 9. Fix $D \ge 2$ and induct on m, using the sure one-mark event, the pair bound, and Theorem 1 as bases. Suppose $m \ge 4$. If $x_i <_P x_{i+1}$ is forced, the target event is the intersection of the left i-mark order and the right $(m-i)$-mark order. Their cross connection is forced, and all other cross comparisons follow transitively. Apply the induction floor to the left order, then to the right order in the poset augmented by the left comparisons. The full target witnesses positivity and degree never increases. The probability is at least $1/(Q_i Q_{m-i}) \ge 1/Q_m$ by Lemma 10.

Otherwise apply Lemma 3 repeatedly to a pair incomparable in the poset with the target order imposed. Both favorable branches remain positive; retain one of no larger probability. At every step check again for a forced adjacent marked comparison and use the preceding case if one appears. Since every step adds a new relation, the process is finite. If it ends with the target-augmented poset total and no adjacent marked comparison forced, Lemma 12 bounds its extension count by Q_m. Its favorable extension is unique, so it has probability at least $1/Q_m$. The same bound then holds in every preceding selected poset and in the original one. This completes induction.

For attainment, the pair gadget consists of two independent D-element chains marked at the top of the first and bottom of the second. It has B_D extensions and one favorable pair order. An ordinal sum of r such gadgets has no cross-block incomparabilities, multiplies counts, and has one favorable $2r$-mark order. For odd $m = 2r+1 \ge 3$, use one sharp triple bridge and $r-1$ pair gadgets. Their count is $M_D B_D^{r-1}$ and the favorable order is again unique. Thus both floors are attained.

For degree one, the incomparability graph is a matching. If $u \parallel v$, every other vertex lies either below both or above both: the two contrary orientations would force a relation between u, v. Hence the poset is an ordinal sum of singleton and two-element-antichain blocks. Uniform extensions independently swap each two-element block. A positive m-mark order fixes at most $\lfloor m/2 \rfloor$ such swaps, giving the stated lower bound. That many fully marked antichain blocks, and one marked singleton when needed, attain it. Degree zero is total and immediate. □

For fixed $m = 2r$, the minimum is asymptotic to $(\pi D)^{r/2} 4^{-rD}$. For fixed $m = 2r + 1 \geq 3$, it is asymptotic to $4\pi(\pi D)^{(r-1)/2} 4^{-rD}$. Both follow directly from the exact formulas and Stirling's formula. A positive event specified by comparisons among m marked vertices contains at least one consistent total marked order; therefore the same lower floor applies to that event as well.

7 Saturated components and the next probability

Call an incomparability component a *sharp pair block* if its induced poset consists of two independent D-element chains. Its favorable marked pair consists of the top of one chain followed by the bottom of the other. Every vertex outside this component lies uniformly below or above it: the comparison direction cannot change across an incomparable edge. Thus each sharp pair block is an ordinal-sum factor.

Lemma 14 (Full fibers and saturation)**.** *Let $D \geq 2$ be an integer and put $B = B_D$. For any consistent pair order $x < y$ in a finite poset P with $\mathrm{inc}(P) \leq D$, either (x, y) is the favorable pair of a sharp pair block, with probability $1/B$, or its probability is at least $1/(B-1)$. A sharp pair block present after adding consistent comparisons was already present, with the same chain orders and marked endpoints, before those comparisons were added.*

Proof. Use the sets A, C of Lemma 2, writing C for its second set. The stable-partition map fixes every favorable extension, because all of A precedes all of C whenever $x < y$. Each fiber is encoded by a binary shuffle of the fixed internal subsequences. If either size is less than D, there are at most $B - 1$ such words. If both sizes equal D and some fiber contains all B words, every cross pair occurs in both orientations among its preimages. Hence every member of A is incomparable with every member of C. These D cross incomparabilities exhaust every vertex's degree allowance. All same-side and outside comparisons therefore exist: A, C are chains forming an isolated $K_{D,D}$ component, with x top of A and y bottom of C. This ordinal block has B shuffles and precisely one favorable shuffle. If it is absent, each favorable fiber has at most $B - 1$ preimages, proving the alternative. Forced favorable comparisons have probability one.

Finally suppose an augmentation has such a block. Its D opposite-side incomparabilities per vertex were also present in the original poset. They already exhausted the original degree bound, so all internal and outside comparisons existed originally, with the same directions. This proves the last assertion. □

The missing-fiber argument uses the local degree bound. The ambient extension count can grow arbitrarily, so integrality of that count alone would not give the displayed isolation interval.

Theorem 15 (Excluding sharp pair blocks)**.** *Fix an integer $D \geq 2$, and put $b = B_D - 1$ and $M = M_D$. Among finite posets of incomparability degree at most D with no $K_{D,D}$ component in their incomparability graph, the exact smallest positive probability of a specified order of m distinct marks is R_m^{-1}, where*

$$R_1 = 1, \qquad R_{2r} = b^r, \qquad R_{2r+1} = Mb^{r-1} \quad (r \geq 1).$$

Every displayed value is attained.

Proof. The class is closed under adding consistent comparisons, by Lemma 14. From Lemma 10, $B \le M \le DB/2$ and $D^2 \le B - 1 = b$, where $B = B_D$. Since $D \le b$ and $B \le 2b$, these give

$$b \le M \le b^2, \qquad M^2 \le bB^2/4 \le b^3.$$

Consequently $R_iR_j \le R_{i+j}$ for every pair of positive indices, including singleton indices, by the same parity calculation as before. The pair base is Lemma 14; the triple base is Theorem 1.

We pay the unique-event terminal case before applying conditioning. The structural stripping in Lemma 12 gives a residual poset on at most $m + 2D - 4$ vertices and the actual insertion estimate

$$e(P) \le (D+1)^{m-2}C_D, \qquad C_D = \binom{2D-2}{D-1} = \frac{DB}{2(2D-1)}.$$

Its deletion intermediates need only satisfy the degree bound; they need not remain in the component-excluded class.

For $D \ge 4$, $b \ge (D+1)^2$. At $D = 4$ this is $69 \ge 25$; for $D \ge 5$ use $B \ge D(2D-1)$ and $D^2 - 3D - 2 > 0$. Since $C_D \le b$, the even base $m = 4$ costs at most b^2. For the odd base $m = 5$, it suffices to show $(D+1)^3C_D \le B(B-1) \le Mb$. Central maximality gives $B \ge \binom{2D}{3}$, so

$$\frac{D(D+1)^3}{2(2D-1)B} \le \frac{3}{4}\frac{(D+1)^3}{(D-1)(2D-1)^2} \le \frac{125}{196} < \frac{69}{70} \le 1 - \frac{1}{B}.$$

The middle ratio decreases for $D \ge 4$, and $B \ge 70$. This proves the odd base. Increasing m by two multiplies the insertion ceiling by $(D+1)^2 \le b$, and R_m by b.

For $D = 3$, one has $b = 19, M = 20$. Lemma 13 bounds the residual count by $24^{(m+2)/4}$. Its bases are

$$24^{3/2} < 19^2, \qquad 24^{7/4} < 20 \cdot 19,$$

verified by $24^3 < 19^4$ and $24^7 < 380^4$. The two-step multiplier is $\sqrt{24} < 19$. For $D = 2$, one has $b = 5, M = 6$ and the bound is $6^{m/3}$. The bases $6^{4/3} < 25$ and $6^{5/3} < 30$ follow from $6^4 < 25^3$ and $6^5 < 30^3$; the two-step multiplier is $6^{2/3} < 5$. Thus every unique-event terminal with $m \ge 4$ costs at most R_m.

Induct on m. A forced adjacent marked comparison splits the order into two smaller positive orders. Apply the first floor, then the second in the poset augmented by the first order. The class is preserved and $R_iR_j \le R_{i+j}$ pays the product. Otherwise finite conditioning as in Lemma 3 reaches such a split or a unique-event terminal with no larger target probability. The preceding bound pays that terminal and hence the original event.

For attainment, take two independent chains $A_1 < \cdots < A_D = x$ and $y = C_1 < \cdots < C_D$ and add $A_1 < C_D$. This removes precisely the all-C-before-all-A shuffle. There are $B - 1 = b$ extensions and one favorable $x < y$ extension. The incomparability graph is $K_{D,D}$ with one edge removed, connected for $D \ge 2$, and belongs to the stated class. The sharp triple bridge has $2D - 1$ vertices, so it too belongs to the class. Ordinal sums of these pair blocks, with one triple bridge for odd m, attain the formulas. □

Theorem 16 (Even-order rigidity and the second value)**.** *Let $D \ge 2, r \ge 1$ be integers, put $B = B_D$, and let E be a consistent specified order $x_1 < \cdots < x_{2r}$ in a finite poset with* $\mathrm{inc}(P) \le D$. *Then*

$$\mathbb{P}_P(E) = B^{-r}$$

if and only if each pair (x_{2j-1}, x_{2j}) is the favorable pair of a distinct sharp pair block, with those blocks in the prescribed order. In every other case,

$$\mathbb{P}_P(E) \ge \frac{1}{B^{r-1}(B-1)}.$$

The last value is attained and is the second distinct positive probability.

Proof. Write $J_{2s} = B^{s-1}(B-1)$. The pair case is Lemma 14. The absence of the stated complete marked block decomposition persists under comparison augmentation: every sharp block of an augmentation already existed originally.

Induct on the even number of marks, using the already proved unrestricted Q floors for odd suborders. At a forced adjacent comparison split into two smaller orders. If both sizes are even, both suborders cannot attain their sharp-block decompositions unless the full target does. Indeed, backward saturation places their blocks already in P. Two disjoint favorable marked pairs cannot use the same two-chain component: its only such pairs are $(\max A, \min C)$ and $(\max C, \min A)$, and their simultaneous favorable comparisons contradict $\min A \le \max A$ and $\min C \le \max C$. Thus the blocks are distinct, and positivity orders them as required. At least one suborder therefore uses its J floor, while the other uses its Q floor. Their denominator product is J_{2r}.

If both suborders have odd size at least three, their denominator product is $M^2 B^{r-3} \le J_{2r}$, since $M \le DB/2$ and $D^2/4 \le B-1$. A singleton split, for $r \ge 2$, costs $MB^{r-2} \le J_{2r}$, using $M \le DB/2 \le B(B-1)$.

Otherwise perform finite conditioning until a forced split or a unique-event terminal. For $D \ge 3$, its insertion bound gives

$$e(P) \le (D+1)^{2r-2} C_D \le B^{r-1}(B-1),$$

using $B \ge (D+1)^2$ and $C_D \le B-1$. For $D = 2, r \ge 2$, the entropy bound gives $e(P) \le 6^{2r/3} \le 5 \cdot 6^{r-1} = J_{2r}$: the inequality holds at $r = 2$ and strengthens with r. The missing-block property persists, so the terminal gap transfers to P. This proves the lower bound in every nondecomposed case. The decomposed case has independent ordinal factors and probability exactly B^{-r}, establishing the equivalence.

For the second value, take one perturbed pair block from the attainment proof of Theorem 15 and $r-1$ sharp pair blocks in ordinal sum. Their favorable event has probability $[B^{r-1}(B-1)]^{-1}$, strictly greater than B^{-r}. □

For $D = 1$, the ordinal matching decomposition gives minimum 2^{-r} and second value $2^{-(r-1)}$ for $2r$ marks; equality fixes all r marked swaps. Excluding all sharp pair blocks in that case leaves a chain. For $D = 0$ every consistent marked order has probability one.

8 A bounded-range balance consequence

For a finite nonchain poset define

$$\delta(P) = \max_{u \ne v} \min\{\mathbb{P}_P(u < v), \mathbb{P}_P(v < u)\}, \qquad C = \frac{5-\sqrt{5}}{10}, \quad K = 5 + 3\sqrt{5}.$$

Write $h(v)$ for expected rank in a uniform extension. The following application uses the quantitative BFT analysis of Aires–Chan–Pak–Panova [3, 4] and improves their small-range estimate [4, Theorem 1.3]; the triple denominator it uses is sharp.

Theorem 17. *Every finite nonchain poset of incomparability degree at most an integer $D \ge 2$ has*

$$\delta(P) > C + \frac{1}{KM_D}.$$

The added term is asymptotic to $(4\pi/K)4^{-D}$.

We specify the external inputs rather than using a changed-hypothesis version of a range-dependent theorem. A BFT triple is a nonchain triple (x, y, z) with $h(x) \le h(y) \le h(z) \le h(x) + 2$. For a connected incomparability graph, order vertices v_i by nondecreasing height, set $d_i = h(v_i) - i$ and $H = \max_i |d_i|$. The BFT selection and case analysis in [4, Lemmas 2.3 and 4.1] supply a consecutive BFT triple with an endpoint attaining H. If $\delta(P) < 67/242$, its surviving case has $x <_P z$ and $y \parallel_P x, z$; the other cases give at least $1/3$, or strictly more than $67/242$. In the surviving case [4, Lemma 2.4] supplies $R \ge 0$ satisfying

$$\delta(P) \ge C + R/K, \qquad R \ge \mathbb{P}(f(x) - f(y) \ge 2) + \mathbb{P}(f(y) - f(z) \ge 2). \tag{6}$$

Finally [4, Corollary 2.2] states that when $\delta(P) < e^{-1}$, height order $h(u) \le h(v)$ identifies $\min\{\mathbb{P}(u < v), \mathbb{P}(v < u)\}$ with $\mathbb{P}(v < u)$. These are external theorem inputs. The structural and displacement deductions below follow [4, Lemmas 3.2 and 4.2 and Section 4.3] under the new triple threshold and are reproduced for completeness.

Proof of Theorem 17. First let $D \ge 7$ and put $\eta = 1/(KM_D)$. We have $M_D \ge B_D \ge \binom{14}{7} = 3432$ and $K > 11$. Also $C < 83/300$, since $\sqrt{5} > 67/30$. Thus

$$C + \eta < \frac{83}{300} + \frac{1}{11 \cdot 3432} < \frac{67}{242} < \frac{1}{3} < e^{-1}.$$

The middle inequality follows from $67/242 - 83/300 = 7/36300 > 1/37752$.

The connected components of the incomparability graph are ordinal-sum factors: a vertex comparable with every vertex of a component must lie uniformly above or below it, since its direction cannot change across an incomparable edge. This also orders distinct components. Their internal extension laws are unchanged. Work in a nontrivial component; a two-vertex component has balance $1/2$. Assume for contradiction that $\delta(P) < C + \eta$ and select the external BFT triple above. Equation (6) gives $R < 1/M_D$. If any w admits an extension with $y < w < x$, that event has probability at least $1/M_D$ by Theorem 1, while its probability is at most R. The same contradiction excludes any extension with $z < w < y$. Thus $y < x$ forces their adjacency in every extension, and likewise $z < y$.

Moreover x is covered by z in P. Otherwise a vertex strictly between them makes their rank gap always at least two; the consistent positive event $x < y < z$ makes it at least three with positive probability. This contradicts $h(z) - h(x) \le 2$. Absence of the forbidden triple orders implies, for outside vertices,

$$w <_P y \text{ or } x <_P w, \qquad w <_P z \text{ or } y <_P w.$$

Indeed if either alternative fails, adding the corresponding three-order chain is acyclic, hence admits an extension. If a vertex outside $\{x, y, z\}$ were incomparable with y, both alternatives force $x <_P w <_P z$, contrary to the covering property. Put $L = \{w : w <_P y\}$ and $U = \{w : y <_P w\}$. The ground set partitions as $L, \{x, y, z\}, U$, with $L < U$, $L < z$, and $x < U$.

Let $p = \mathbb{P}(y < x)$, $p' = \mathbb{P}(z < y)$, and $a = \max(p, p')$. The external height-order corollary applies, so $a \le \delta(P) < 1/3$. With $m = |L| + 1$, the selected triple occupies height indices $m, m + 1, m + 2$. Let α be the expected number of L vertices after x, and β the expected number of U vertices before z. Rank counting gives

$$d_m = p - \alpha, \qquad d_{m+2} = \beta - p', \qquad d_{m+2} \le d_m.$$

The last inequality is the expected-span condition.

Take independent uniform extensions of $L' = L \cup \{x\}$ and $U' = U \cup \{z\}$. Let r be the probability that x is last in the first, and s the probability that z is first in the second. For each pair of block extensions there is one central insertion of y, one extra left insertion exactly

when x is last, and one extra right insertion exactly when z is first. The proved adjacency restrictions exclude all other insertions. Therefore

$$e(P) = e(L')e(U')(1+r+s), \qquad p = \frac{r}{1+r+s}, \quad p' = \frac{s}{1+r+s}.$$

In particular $r, s \le a/(1-2a)$. For every left extension with x not last, all central and right placements contribute at least one L vertex after x. Their weight is $1+s$, independent of the left extension. Hence

$$\alpha \ge (1-p)(1-r), \qquad \beta \ge (1-p')(1-s).$$

It follows that $\alpha, \beta \ge V(a)$, where $V(a) = (1-a)(1-3a)/(1-2a)$. Thus

$$V(a) - a \le d_{m+2} \le d_m \le a - V(a).$$

An endpoint attains H, so $H \le a - V(a)$; nonemptiness also gives $a \ge C$. An exact factorization yields

$$a - V(a) = 5(a-C)\left(1 + \frac{a-C}{1-2a}\right) \le 6(a-C),$$

because $C \le a < 1/3$ implies $(a-C)/(1-2a) \le 1-3C < 1/5$. Consequently $\delta(P) \ge C + H/6$.

The least-height vertex v_1 is minimal. A unique minimal vertex of a finite poset is below every vertex and hence isolated in its incomparability graph. There is another minimal vertex u. Delete u from an extension. Every vertex before u must be incomparable with it, so there are at most $D+1$ possible reinsertion slots, while prepending is always legal. Thus

$$H \ge h(v_1) - 1 \ge \mathbb{P}(u < v_1) \ge \mathbb{P}(u \text{ is first}) \ge \frac{1}{D+1}.$$

Since $M_D \ge B_D \ge 2^D \ge D+1$ and $K > 6$, this gives $\delta(P) \ge C + 1/[6(D+1)] > C + \eta$, a contradiction. We have proved $\delta(P) \ge C + \eta$. The left side is rational, whereas

$$C + \frac{1}{KM_D} = \frac{1}{2} - \frac{1}{4M_D} + \frac{3-2M_D}{20M_D}\sqrt{5}$$

is irrational for integer $M_D \ge 6$. Therefore the inequality is strict.

For $2 \le D \le 6$, the six-thin theorem [5] gives $\delta(P) \ge 1/3$. Since $M_D \ge 6$, $K > 11$ and $C < 7/25$, our threshold is less than $7/25 + 1/66 < 1/3$. This completes all degrees. The asymptotic increment follows from Theorem 1. □

This is a range-dependent, exponentially decreasing balance increment.

Remark 18. In the notation of [4], inc(P) is the range $\pi(P)$. Their small-range theorem [4, Theorem 1.3] gives $\delta(P) > C + (D+1)^{-3(D+1)}/12$, sharpened in [4, Remark 4.5] to $C + (D+1)^{-3(D+1)}/K$. Its proof uses the bound $(D+1)^{-m(D+1)}$ of [4, Lemma 3.1] for positive events specified by comparisons among m elements, with $m = 3$ in the structural step and $m = 2$ in the displacement step. The proof of Theorem 17 follows their argument [4, Lemmas 3.2 and 4.2 and Section 4.3] with two new inputs: the sharp triple floor of Theorem 1 in place of the case $m = 3$, and the estimate $H \ge 1/(D+1)$ in place of the case $m = 2$ in [4, (4.4)]. This replaces the increment by $1/(KM_D)$; for example at $D = 7$ it improves from about $2 \cdot 10^{-23}$ to about $2 \cdot 10^{-5}$. Separately, Theorem 9 replaces the bound of [4, Lemma 3.1] by the exact value Q_m^{-1} for every m. Finiteness is essential: the infinite Fibonacci poset has range two and, in the limiting sense of [4, Section 1.3], balance exactly C [3, Section 7].

A A supporting reversal-tail estimate

We record a complementary reversal-tail estimate for incomparable pairs, independent of the main proofs. For incomparable x, y, put $a_k = \mathbb{P}(f(x) - f(y) = k)$ for $k \geq 1$. The Kahn–Saks log-concavity theorem, as stated for arbitrary finite posets in [6, Theorem 1.2], gives $a_k^2 \geq a_{k-1}a_{k+1}$ for $k > 1$. The following elementary consequence requires checking support as well.

Lemma 19. *Put $p = \sum_{k\geq 1} a_k$, $t = \sum_{k\geq 2} a_k$, and $T_k = \sum_{j\geq k} a_j$. Then, if $0 \leq t < p$,*

$$T_k \leq p(t/p)^{k-1}, \qquad \mathbb{E}[(f(x) - f(y) - 1)_+] \leq \frac{pt}{p-t},$$

$$\mathbb{E}[(f(x) - f(y) - 1)_+^2] \leq \frac{pt(p+t)}{(p-t)^2}.$$

Proof. The positive-gap support is an interval starting at one. To see this, take an extension with y before x and gap $k \geq 2$, and let I be the intervening block. Its subset $A = \{w \in I : w <_P x\}$ is an ideal of $P[I]$. Reorder this contiguous block to put A first, respecting its induced order. If A is nonempty, its first member is incomparable with y: a relation $y < w < x$ would contradict $y \parallel x$, and $w < y$ contradicts the original extension. Swap it with y to decrease the gap by one. If A is empty, the last member of I is incomparable with x, so swap those instead. Repeat to obtain every smaller positive gap.

On this interval the ratios a_{j+1}/a_j are nonincreasing. For $k \geq 2$ and $j \geq k$ this implies $a_k a_j \geq a_{k-1}a_{j+1}$. Summing yields $a_k T_k \geq a_{k-1}T_{k+1}$, equivalently $T_k^2 \geq T_{k-1}T_{k+1}$. The tail ratios are therefore at most $T_2/T_1 = t/p$, proving the first bound. Apply the integer tail identities $\mathbb{E}Z = \sum_{j\geq 1} \mathbb{P}(Z \geq j)$ and $\mathbb{E}Z^2 = \sum_{j\geq 1}(2j-1)\mathbb{P}(Z \geq j)$ to $Z = (f(x) - f(y) - 1)_+$ and sum the geometric series. □

For a Case D triple as in (6), write the two reversal probabilities as p, p' and their gap-at-least-two tails as t, t'. Whenever $p, p' \geq p_0 > R$, the external inequality $t + t' \leq R$ and Lemma 19 give

$$\mathbb{E}[(f(x) - f(y) - 1)_+ + (f(y) - f(z) - 1)_+] \leq \frac{R}{1 - R/p_0}.$$

This estimate is independent of the degree cap.

Attribution and assistance. The global pair-extremum and shuffle examples have classical antecedents in Fishburn's work. The balance analysis uses the external BFT and current ACP case, selection and orientation theorems; the small-degree input is the six-thin theorem. The degree-two counting argument is the classical Brégman inequality in its Radhakrishnan entropy form, proved here in the needed specialization. The supporting appendix uses Kahn–Saks log-concavity. All arguments for the present local pair and sharp triple bounds are printed here; the structural and displacement deductions follow ACP's Lemmas 3.2 and 4.2 and Section 4.3 with the new threshold. AI tools assisted proof exploration and source preparation.

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