Entropy curvature under Gaussian smoothing: restoration and moment-matched obstructions

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Luminary Oddities

Abstract

For every fixed retained signal $q \in (0,1)$ and every prescribed finite Gaussian moment matching order, we construct smooth strongly log-concave symmetric inputs approaching the Gaussian whose three-copy entropy fails concavity at input-dependent interior weights tending to a face. The construction matches the fourth moment exactly and uses a higher-order face expansion with a controlled differentiated remainder. In contrast, for $m \geq 3$ copies and each $Q < m^{-1/(m-1)}$, one bounded Gaussian-relative neighborhood has a negative definite entropy Hessian in every tangent direction at the exact barycentre for non-Gaussian inputs, simultaneously for all $0 < q \leq Q$. This bound holds at the barycentre itself. For two copies and every retained signal below one half, an explicit bounded Gaussian-relative neighborhood has strict all-weight concavity; this answers Luo's Problem 10.2 negatively on $0 < q < 1/2$. We also weaken eventual two-copy restoration to finite moments: if the first differing Gaussian moment has even order $n \geq 4$, absolute moment $2n+2$ suffices for strict all-weight concavity at small signal; one common moment bound of order $2N+4$ suffices under a positive finite Hermite discrepancy through degree N. All proofs are included, together with degree-by-degree Gaussian-boundary estimates, whole-class small-signal endpoint restoration and moment-matched exponential-tail examples, which separate the pointwise, family-uniform, barycentric and near-face regimes.

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1 Introduction and main result

Let X, Y be independent copies of a centered variance-one random variable. For an independent standard Gaussian G, write

$$X_q = \sqrt{q}\, X + \sqrt{1-q}\, G, \qquad 0 \leq q \leq 1.$$

Given independent copies U_q, V_q of X_q, set

$$F_X(q,t) = h(\sqrt{t}\, U_q + \sqrt{1-t}\, V_q), \qquad 0 \leq t \leq 1,$$

where h is differential entropy with natural logarithms. Equivalently the output is

$$Z_{q,t} = \sqrt{qt}\, X + \sqrt{q(1-t)}\, Y + \sqrt{1-q}\, G. \tag{1}$$

The variance is one, and $F_X(q,t) = F_X(q,1-t)$.

The new results distinguish restoration at balanced weights from failure near faces, even after arbitrarily high finite Gaussian moment matching. They also replace all-moment assumptions in

eventual two-copy restoration by explicit sufficient finite moments. The barycentric theorem is infinitesimal; it is compatible with the near-face obstructions.

Write

$$\varphi(x) = (2\pi)^{-1/2} e^{-x^2/2}, \qquad T_a e_n = a^n e_n, \qquad N e_n = n e_n,$$

where $e_n = \mathrm{He}_n / \sqrt{n!}$ is the normalized probabilists' Hermite basis of $L^2(\varphi\, dx)$. All unqualified norms and inner products below use this Gaussian measure.

For $m \geq 3$ independent copies $X_1, \ldots, X_m$ and an independent standard Gaussian G, define on the probability simplex

$$\mathcal{F}^{(m)}_{q,g}(t) = h\left(\sum_{i=1}^{m} \sqrt{q t_i}\, X_i + \sqrt{1-q}\, G \right), \qquad X_i \sim \varphi(1+g).$$

Write $u = (1/m, \ldots, 1/m)$ for its barycentre.

Theorem 1.1 (Moment-matched three-copy obstructions)**.** *Fix $q \in (0,1)$ and an integer $K \geq 2$. There exist even smooth strongly log-concave variance-one densities $f_j = \varphi(1+g_j)$ whose Gaussian moments match exactly through order $2K$, with $\|g_j\|_{C^r_b} \to 0$ for every fixed integer $r \geq 0$, but whose three-copy entropy is not concave on the simplex. More precisely, there are $\epsilon_j \downarrow 0$ for which its second derivative in direction $(1,-1,0)$ at $(\epsilon_j, \epsilon_j, 1-2\epsilon_j)$ is strictly positive. The input sequence and weights may depend on fixed q, K; no common face distance or barycentric counterexample is asserted.*

Here $\|g\|_{C^r_b}$ controls g and its derivatives through order r uniformly. The theorem imposes arbitrarily high finite Gaussian matching, not agreement of the entire moment sequence.

Theorem 1.2 (Multicopy barycentric Hessian)**.** *Fix $m \geq 3$ and $0 < Q < m^{-1/(m-1)}$. There is an explicit $\delta_{m,Q} > 0$ such that every real even measurable g satisfying*

$$\|g\|_\infty \leq \delta_{m,Q}, \qquad \int g\varphi = \int x^2 g\varphi = 0$$

has, for all $0 < q \leq Q$ and all real tangent vectors v with $\sum_i v_i = 0$,

$$\operatorname{Hess} \mathcal{F}^{(m)}_{q,g}(u)[v,v] \leq -\frac{m}{16} \|v\|^2_{\mathbb{R}^m} \|N L_q\|^2_2, \qquad L_q = m T_{\sqrt{q/m}} g. \tag{2}$$

The Hessian is negative definite on the tangent space unless $g = 0$ almost everywhere. The assertion is at the exact barycentre; it does not supply a common surrounding weight region or a sharp signal threshold.

For three copies the sufficient range is $Q < 1/\sqrt{3}$. The pointwise Hessian conclusion is compatible with failures near faces: these are different points of the simplex. For each fixed input and signal, Hessian continuity also gives an input-dependent open neighborhood of the barycentre. No uniform such neighborhood is asserted here.

1.1 Two-copy local background

The central question is whether nonconcavity can persist arbitrarily close to the Gaussian after a fixed amount of Gaussian smoothing. Luo [7, Problem 10.2] asks, for a fixed signal fraction $q \in (0, 4/7]$ and an integer $K \geq 1$, whether there exist symmetric strongly log-concave variance-one densities f_j matching the Gaussian moments through order $2K$ such that

$$\|f_j/\varphi - 1\|_{C^m_b} \longrightarrow 0 \quad \text{for every fixed } m \geq 0,$$

while $F_{X_j}(q,\cdot)$ fails concavity for every j. Here C_b^m controls the function and its derivatives through order m uniformly, and φ is the standard Gaussian density. The question is a local form of the entropy-concavity problem of Ball, Nayar and Tkocz, with its formulation and attribution taken from Luo.

We give a self-contained negative answer on $0 < q < 1/2$ as a background theorem for the restoration results below. More precisely, for every prescribed upper signal bound $Q < 1/2$ we construct an explicit positive radius in the uniform relative-density norm on which concavity holds for every mixing weight and every $0 < q \le Q$, strictly for non-Gaussian inputs. The theorem allows arbitrary bounded even relative perturbations, including infinitely many Hermite modes. Its proof controls the nonlinear entropy terms relative to the actual quadratic Hermite energy, with a Gaussian smoothing margin that is uniform through the weight endpoints. The extensions developed here replace all-moment assumptions by sufficient finite moments and distinguish the multicopy barycentric Hessian from moment-matched near-face failure.

Theorem 1.3 (Gaussian-neighborhood concavity)**.** *For every $Q \in (0,1/2)$ there is an explicit $\delta_Q > 0$ such that the following holds. Suppose g is real, even and measurable,*

$$\|g\|_\infty \le \delta_Q, \qquad \int g\varphi = \int x^2 g\varphi = 0,$$

and X has density $\varphi(1+g)$. For every $0 < q \le Q$, the second weight derivative extends continuously to $[0,1]$ and

$$\partial_t^2 F_X(q,t) \le -\frac{1}{16}\,\|NL(q,t)\|_2^2\,, \qquad L(q,t) = T_{\sqrt{qt}}g + T_{\sqrt{q(1-t)}}g. \tag{3}$$

Thus $F_X(q,\cdot)$ is strictly concave unless $g = 0$ almost everywhere. The hypothesis allows measurable bounded relative perturbations.

An explicit sufficient radius is as follows:

$$\begin{aligned} c &= \frac{1+(2Q)^{-1/2}}{2}, & r &= \frac{c-1}{4}, & \rho &= \frac{1+2r}{c}, \\ K &= \frac{\sqrt{\rho}}{(1-\sqrt{\rho})^2}, & A &= \frac{K}{8} + 60 + \frac{1}{8r^2}, & \delta_Q &= \min\left\{\frac{1}{12}, \frac{8}{K}, \frac{1}{16A}\right\}. \end{aligned} \tag{4}$$

This explicit sufficient radius depends on Q and decreases to zero as $Q \uparrow 1/2$.

Corollary 1.4 (Answer to the near-Gaussian counterexample question)**.** *Fix $q \in (0,1/2)$. There is no sequence of symmetric variance-one densities f_j with $\|f_j/\varphi - 1\|_\infty \to 0$ for which every $F_{X_j}(q,\cdot)$ fails concavity. The same statement holds after imposing strong log-concavity, any prescribed finite Gaussian moment matching, or convergence of all fixed-order relative-density derivatives.*

1.2 Relation to prior work and proof mechanism

Luo [7] constructs asymmetric counterexamples for every positive signal fraction and symmetric strongly log-concave counterexamples for $q > 4/7$, with arbitrarily close Gaussian-relative approximation and prescribed finite Gaussian moment matching. Theorem 1.3 and Corollary 1.4 identify a positive relative neighborhood in the low-signal symmetric regime. The additional regularity and moment-matching requirements in Problem 10.2 are compatible with the theorem: uniform relative-density proximity alone places every sufficiently late member of such a sequence within its concavity neighborhood.

An external unrefereed candidate revised on November 11, 2026 [5] claims a uniform bounded relative-density neighborhood for the entire two-copy range $0 < q \le 4/7$, using a critical Gaussian interpolation inequality. It credits the conditional Hermite and entropy-energy scheme used in the below-half proof here. We retain that external attribution and its unrefereed status; none of its critical smoothing theorem is a premise of the proofs below. Its Section 7 constructs multicopy failures near faces, while Section 9.3 asks about barycentric and interior survival. Our barycentric theorem concerns the exact Hessian with an explicit sufficient signal range. Theorem 1.1 extends the face mechanism to arbitrary finite Gaussian moment matching, including exact fourth matching, where that candidate's leading fourth-cumulant term vanishes and its Sections 7.4(9) and 9.3 treat the question separately. The barycentric and face results are thus complementary: one fixes the weight at the centre, the other lets it approach a face.

Gaussian scale mixtures have entropy concavity [4]. Abu-Ajamieh [2] derives the exact symmetric endpoint-curvature identity, establishes endpoint estimates and a neighborhood for an auxiliary endpoint inequality, and proves an all-weight theorem for a specified fourth-Hermite perturbation family. Our local theorem treats a bounded infinite-dimensional relative-density neighborhood. The proof combines classical Gaussian/Hermite coordinates [1], conditional-expectation contraction, a complex-weight smoothing margin, and number-operator estimates to absorb the full nonlinear curvature error. Gaussian-channel identities have further classical antecedents [6].

1.3 Complementary restoration results

The Gaussian-relative theorem describes one neighborhood topology. The results below describe how all-weight concavity is restored under a common moment envelope or for a fixed input, and give a whole-class endpoint estimate. Together with exact moment-matched tail examples, they explain the distinct roles of proximity, symmetry and retained signal.

A complementary finite-Hermite criterion permits a common threshold under one sufficiently high moment bound, even when the first non-Gaussian moment varies across the input family.

Theorem 1.5 (Finite Hermite discrepancy)**.** *Fix an even $N \ge 4$, $B < \infty$ and $\varepsilon > 0$. Let $\mathcal{C}$ be a family of symmetric variance-one laws satisfying*

$$\sup_{X \in \mathcal{C}} \mathbb{E}|X|^{2N+4} \le B.$$

Put $a_n(X) = \mathbb{E}e_n(X)$ and suppose that

$$\sum_{\substack{4 \le n \le N \\ n \ even}} a_n(X)^2 \ge \varepsilon^2 \qquad (X \in \mathcal{C}). \tag{5}$$

There is $q_{B,N,\varepsilon} > 0$ such that every $X \in \mathcal{C}$ has strict all-weight entropy concavity for every $0 < q \le q_{B,N,\varepsilon}$, including continuous second derivatives at both endpoints. The family may include discrete laws and laws with some higher moments infinite.

The finite-jet estimate also quantifies the Gaussian boundary of a family with nonnegative curvature.

Corollary 1.6 (Quantitative Gaussian boundary)**.** *Let $\mathcal{C}$ be a family of symmetric variance-one laws with a common absolute-moment envelope: $\sup_{X \in \mathcal{C}} \mathbb{E}|X|^j \le B_j < \infty$ for every integer $j \ge 1$. For every fixed even $N \ge 4$, there are $K_N < \infty$ and $q_N > 0$ such that, for every $X \in \mathcal{C}$ and $0 < q \le q_N$,*

$$(\exists t \in [0,1] : F_{X,tt}(q,t) \ge 0) \quad \Longrightarrow \quad \sum_{\substack{4 \le n \le N \\ n \text{ even}}} a_n(X)^2 q^n \le K_N q^{N+1}. \tag{6}$$

Consequently, for each fixed polynomial P and every $A > 0$, there are $K_{P,A} < \infty$ and $q_{P,A} > 0$ such that the same antecedent, with $0 < q \le q_{P,A}$, implies

$$|\mathbb{E}P(X) - \mathbb{E}P(G)| \le K_{P,A} q^A.$$

The constants are uniform over $\mathcal{C}$ for each fixed P, A. In particular, along any sequence $X_j \in \mathcal{C}$, $q_j \downarrow 0$, for which $F_{X_j}(q_j, \cdot)$ fails concavity, every fixed polynomial moment discrepancy is $O(q_j^A)$ for every fixed $A > 0$.

The proof in Section 5 fixes the jet order before choosing the signal interval. Thus the conclusion gives a degree-by-degree necessary condition on a changing family. For symmetric variance-one log-concave laws, their common moment envelope also gives Gaussian-only limiting behavior (Corollary 5.1).

Theorem 1.7 (Finite-moment eventual concavity)**.** *Let X be symmetric with variance one. Suppose that for some even $n = 2k \ge 4$, its moments below order n equal the Gaussian moments, $\kappa_n = \mathbb{E}X^n - \mathbb{E}G^n \ne 0$, and $\mathbb{E}|X|^{2n+2} < \infty$. There is $q_X > 0$ such that $\partial_t^2 F_X(q,t) < 0$ for every $0 < q \le q_X$ and every $t \in [0,1]$, with continuous second derivatives at both endpoints. Moreover*

$$F_X(q,t) = h(\varphi) - \frac{\kappa_n^2}{2n!} q^n \left[t^k + (1-t)^k\right]^2 + o(q^n) \tag{7}$$

in $C^2([0,1])$ as $q \downarrow 0$. In particular every fixed symmetric non-Gaussian variance-one law with all absolute moments finite has such an eventual threshold.

This includes every fixed symmetric variance-one log-concave input and discrete symmetric inputs with all moments finite, but also genuine polynomial-tail laws. The threshold is attached to the law. The stated finite moment orders are sufficient; no minimal-moment claim is made.

The three all-weight regimes have distinct useful inputs: an explicit relative-density neighborhood, a fixed non-Gaussian law, or a positive finite Hermite discrepancy under a common moment envelope. Corollary 1.6 quantifies the changing near-Gaussian boundary. Appendix B supplies exactly moment-matched exponential-tail examples distinguishing these topologies.

Theorem 1.8 (Whole-class endpoint restoration)**.** *Let X have a symmetric log-concave density and variance one. If $0 < q \le 1/20$ and p is the density of X_q, then*

$$\partial_t^2 F_X(q,0) = \partial_t^2 F_X(q,1) \le -\frac{1}{128} \mathbb{E}\left[\left(1 + (\log p)''(X_q)\right)^2\right]. \tag{8}$$

The endpoint derivatives are continuous one-sided extensions. The bound is strict unless X is Gaussian.

This last theorem uses the endpoint identity from [2], supplemented by uniform small-signal estimates. The conclusion concerns the two endpoint derivatives.

1.4 Organization of the proofs

Section 2 proves the below-half Gaussian-neighborhood theorem directly. Sections 3–5 develop finite-moment Gaussian-gain regularity, fixed-input eventual concavity, and family-uniform finite-Hermite and Gaussian-boundary results. Section 6 proves the multicopy barycentric Hessian bound, including the active-subset interpolation and explicit radius. Section 7 proves the higher-moment face obstruction, its differentiated remainder and the bounded moment-corrected inputs. Appendix A proves the uniform endpoint estimate, using the credited endpoint identity. Appendix B constructs the exactly moment-matched exponential tails that distinguish moment proximity from bounded Gaussian-relative proximity.

2 Proof of Gaussian-neighborhood concavity

We prove Theorem 1.3. Fix $Q < 1/2$ and the constants (4), and put $\delta = \|g\|_\infty \le \delta_Q$. The conditions on g imply that its Hermite coefficients a_n vanish except at even orders $n \ge 4$. The output density from (1) is $\varphi(1+R)$, where

$$R = L + M, \qquad L = T_{\sqrt{qt}}g + T_{\sqrt{q(1-t)}}g, \tag{9}$$

and

$$M(x) = \mathbb{E}[g(G_1)g(G_2) \mid \sqrt{qt}G_1 + \sqrt{q(1-t)}G_2 + \sqrt{1-q}G_3 = x].$$

The G_i here are independent standard Gaussians. Hence $\|R\|_\infty \le 2\delta + \delta^2 \le 3\delta =: \eta \le 1/4$ on the real weight interval. The output has mass one and variance one, so

$$F_{tt} = -\langle R'', \log(1+R)\rangle - \int \frac{(R')^2}{1+R}\varphi. \tag{10}$$

Primes in this section denote weight derivatives unless otherwise specified. We justify this differentiation below after deriving the required Hilbert-space regularity.

2.1 Linear energy

For $n = 2k \ge 4$ the multiplier of a_n in L is

$$l_n = q^k S_k(t), \qquad S_k(t) = t^k + (1-t)^k.$$

Log-convexity of power sums and $S_1/S_0 = 1/2$ show $S_k \ge S_{k-1}/2$ and $S_k \ge S_{k-2}/4$; also $S_{k-2} \ge S_k$. Thus

$$|l_n'| \le n l_n, \qquad \frac{n^2}{8} l_n \le l_n'' \le n^2 l_n.$$

With $D = \|NL\|_2$, we obtain

$$\langle L'', L\rangle \ge D^2/8, \qquad \left\|N^{-1}L''\right\|_2 \le D, \qquad \|L\|_2 \le D/4. \tag{11}$$

The inverse is restricted to positive Hermite degrees. If $g \ne 0$, then $D > 0$ at every positive q and every weight.

2.2 Bilinear term and a smoothing margin

Gaussian convolution in Hermite coordinates [1] gives

$$M_n(q,z) = q^{n/2} \sum_{j+k=n} a_j a_k \binom{n}{j}^{1/2} z^{j/2}(1-z)^{k/2}. \tag{12}$$

Only even $j, k \ge 4$ occur. Let g_{abs} be the $L^2(\varphi)$ function with coefficients $|a_n|$; it need not be bounded, but $\|g_{\mathrm{abs}}\|_2 \le \delta$.

Fix $t \in [0,1]$ and $|z - t| \le r$. Set $a = |z|$, $b = |1-z|$, $P = q(a+b)$, $w = a/(a+b)$ and $\bar{P} = cq$. Our choice of constants gives

$$P \le q(1+2r) < \bar{P} < 1, \qquad P/\bar{P} \le \rho < 1, \tag{13}$$
$$\bar{P}^2 = c^2 q^2 \le c^2 Q q < q/2.$$

Absolute coefficient majorization in (12) reduces the norm to the real bilinear expression $B(P, w; g_{\mathrm{abs}}, g_{\mathrm{abs}})$. For any $h \in L^2(\varphi)$,

$$\|B(P, w; h, h)\|_2 \le \left\|T_{\sqrt{P}} h\right\|_2^2. \tag{14}$$

Indeed smooth two independent inputs separately and condition their product on their weighted Gaussian sum; L^2 contraction and independence give (14). For two inputs, the same conditioning argument bounds the bilinear map by the product of their smoothed L^2 norms. Applying this bound to differences of finite Hermite truncations gives L^2 convergence and extends the identity and inequality to L^2 inputs.

Spectral Cauchy–Schwarz gives

$$\left\|T_{\sqrt{\bar{P}}} g_{\mathrm{abs}}\right\|_2^2 = \sum a_n^2 \bar{P}^n \le \delta \left\|T_{\bar{P}} g_{\mathrm{abs}}\right\|_2.$$

By (13) and $S_k(t) \ge 2^{1-k}$,

$$\left\|T_{\bar{P}} g_{\mathrm{abs}}\right\|_2 \le \left\|T_{\sqrt{q/2}} g\right\|_2 \le \|L\|_2 / 2.$$

Consequently $\|M(q, z)\|_2 \le \delta \|L\|_2 / 2$ throughout this disk. For NM, split off the output multiplier $(P/\bar{P})^{n/2}$; its number-operator cost is at most

$$\sup_{n \ge 0} n\rho^{n/2} \le \sum_{n \ge 1} n(\sqrt{\rho})^n = K.$$

It follows that $\|NM(q, z)\|_2 \le K\delta \|L\|_2 / 2$. Strictness in (13) leaves a slightly larger disk. The same majorants, with the exponentially decaying output multiplier, show normal convergence of the coefficient series and all fixed number-operator derivatives there. Hence M is Hilbert-valued analytic. Cauchy's estimate on radius r and (11) yield

$$\|M\|_2 \le \delta D/8, \qquad \|NM\|_2 \le K\delta D/8, \qquad \|M''\|_2 \le \delta D/(4r^2). \tag{15}$$

The linear term is analytic too, since $q|z|$ and $q|1 - z|$ are below $\bar{P} < 1$. All estimates are uniform in $t \in [0, 1]$ and $0 < q \le Q$.

2.3 A bounded-function estimate

For smooth bounded real u with $Nu \in L^2(\varphi)$,

$$\left\|(u_x)^2\right\|_2 \le 3 \|u\|_\infty \|Nu\|_2. \tag{16}$$

To prove it, put $B = \int u_x^4 \varphi$. Gaussian integration by parts gives

$$\int Nu\, u(u_x)^2 \varphi = B + 2 \int u(u_x)^2 u_{xx} \varphi.$$

Cauchy–Schwarz and the Gaussian Bochner identity $\|u_{xx}\|_2 \le \|Nu\|_2$ imply $B \le 3 \|u\|_\infty \sqrt{B} \|Nu\|_2$, proving the assertion. Smooth cutoffs and Gaussian Sobolev approximation justify the operator-domain version; here positive smoothing also bounds the relevant spatial derivatives.

Set $\psi(v) = \log(1 + v) - v$. The chain rule gives

$$N\psi(R) = \psi'(R) NR - \psi''(R)(R_x)^2.$$

Using (16) and $\|R\|_\infty \le \eta \le 1/4$,

$$\|N\psi(R)\|_2 \le \left[\frac{\eta}{1 - \eta} + \frac{3\eta}{(1 - \eta)^2}\right] \|NR\|_2 \le 30\delta(1 + K\delta/8)D. \tag{17}$$

2.4 Curvature sign and regularity

The density ratio lies in $[1-\eta, 1+\eta]$. Hilbert-valued C^2 regularity of R and the bounded second derivative of $u \log u$ on this interval justify (10): scalar Taylor's formula and L^2 convergence of difference quotients give the differentiated integrals. The resulting formula is continuous to the endpoints. Spatial regularity from smoothing and the preceding number-operator estimates justify self-adjointness and the chain rule used in (17).

Drop the last, nonpositive term of (10) and decompose $\log(1+R) = L + M + \psi(R)$. The error terms satisfy

$$\begin{aligned} |\langle L'', M\rangle| &\le K\delta D^2/8, \\ |\langle L'', \psi(R)\rangle| &\le 30\delta(1 + K\delta/8)D^2 \le 60\delta D^2, \\ |\langle M'', \log(1+R)\rangle| &\le \frac{\delta D^2}{4r^2}\frac{1/4 + \delta/8}{1 - 3\delta} \le \frac{\delta D^2}{8r^2}. \end{aligned}$$

Here $\delta \le 8/K$ in the second estimate, and $\delta \le 1/12$ gives $(1/4 + \delta/8)/(1 - 3\delta) \le 25/72 < 1/2$ in the third. Therefore

$$F_{tt} \le -D^2/8 + A\delta D^2 \le -D^2/16.$$

This proves Theorem 1.3. Corollary 1.4 follows by taking $Q = q$ and observing that every sufficiently large member of the proposed sequence satisfies its relative-density hypothesis.

3 Finite-moment Gaussian-gain regularity

Lemma 3.1. *Let $k \ge 1$ be an integer and suppose $\mathbb{E}|X|^{k+2} < \infty$. For independent copies X, Y and an independent standard Gaussian G, put, for $H = (h_1, h_2)$ with $|H| < 1$,*

$$\sigma(H) = \sqrt{1 - h_1^2 - h_2^2}, \qquad Z_H = h_1 X + h_2 Y + \sigma(H) G, \qquad K(H) = h(Z_H).$$

Then K is C^k. On a fixed compact gain disk its derivatives through order k have common bounds under a common $(k+2)$nd absolute-moment bound and a common positive bounded-input mass. For centered variance-one laws the latter mass is uniform. If X is symmetric, K is even in each coordinate. When $k = 2s$, the function $J(a, b) = K(\sqrt{a}, \sqrt{b})$ is C^s up to the axes of a small positive quadrant, with the corresponding common derivative bounds.

Proof. Work on a fixed closed disk where σ stays positive. The density is

$$p_H(z) = \mathbb{E}\varphi_{\sigma(H)}(z - h_1 X - h_2 Y).$$

For a gain multiindex α, $j = |\alpha| \le k$, kernel differentiation gives

$$\frac{\partial_H^\alpha p_H(z)}{p_H(z)} = \mathbb{E}[P_{\alpha,H}(X, Y, G) \mid Z_H = z], \tag{18}$$

where the coefficients are smooth and bounded on the disk, the input degree in (X, Y) is at most j, and the Gaussian-residual degree is at most $2j$. To verify these separate degree bounds, let $u = (z - h_1 X - h_2 Y)/\sigma$. One kernel derivative divided by the kernel is

$$X_i u/\sigma + (\sigma_i/\sigma)(u^2 - 1), \qquad X_1 = X, \quad X_2 = Y.$$

A derivative of a previously formed polynomial also uses $\partial_{h_i} u = -X_i/\sigma - (\sigma_i/\sigma)u$. Each step adds at most one input degree and two residual degrees. Replacing the residual by the actual independent G gives (18). On compact spatial intervals differentiation under the expectation is dominated by at most j input factors, since Gaussian-polynomial kernel derivatives are bounded.

A positive-mass bounded rectangle in (X, Y) gives constants, uniform on the gain disk, such that

$$p_H(z) \geq ce^{-C(1+z^2)}, \qquad p_H(z) \leq C', \qquad |\log p_H(z)| \leq C''(1+z^2).$$

For variance-one centered inputs, $\Pr(|X| \leq 2) \geq 3/4$, so these constants can be common to the family.

Every formal entropy derivative of order j is a sum of a term $\partial_H^\alpha p_H(1 + \log p_H)$ and terms $p_H \prod_i r_{\beta_i,H}$, where $r_\beta = (\partial_H^\beta p_H)/p_H$, $b_i = |\beta_i| \geq 1$ and $\sum_i b_i = j$. Conditional Jensen and weighted Young give

$$\prod_i |r_{\beta_i,H}| \leq \sum_i \frac{b_i}{j} |r_{\beta_i,H}|^{j/b_i} \leq \mathbb{E}\left[\sum_i \frac{b_i}{j} |P_{\beta_i,H}|^{j/b_i} \,\middle|\, Z_H = z\right].$$

Their joint majorant has input degree at most j and Gaussian degree at most $2j$. The logarithmic term has a majorant

$$C(1 + |X| + |Y|)^{j+2}(1 + |G|)^{2j+2}.$$

This is integrable by independence and the $(k+2)$nd input moment. Uniformly on the gain disk, $|Z_H| > R$ implies $C_1(1 + |X| + |Y| + |G|) > R$. Dominated convergence for this fixed joint law therefore removes all spatial tails uniformly in H. On compact spatial sets the density derivatives are continuous and denominators are positive.

Insert smooth compact spatial cutoffs into the entropy integral. The truncated integrals are C^k; their derivatives through order k converge uniformly on the gain disk by the preceding tail bounds. The order-zero tails use the same quadratic logarithm and second moments. Uniform derivative convergence along coordinate segments proves actual C^k regularity and identifies the derivatives with their integrals. Full-expectation bounds without a tail indicator give the claimed family-uniform derivative bounds. No uniform integrability of a family's highest moments is asserted.

Symmetry gives evenness in each gain. If E is a smooth even function and $a = h^2$, the finite-order formula

$$\frac{d^j}{da^j} E(\sqrt{a}) = \frac{1}{2^{2j-1}(j-1)!} \int_0^1 (1-u^2)^{j-1} E^{(2j)}(uh)\, du \tag{19}$$

holds whenever E is C^{2j}. It follows for $j = 1$ by the fundamental theorem of calculus and then by induction and integration by parts. At $a = 0$ it equals $j!E^{(2j)}(0)/(2j)!$. Applying it in both coordinates uses total gain order at most twice the squared-gain order, and proves the C^s conclusion, including mixed derivatives on the axes. Taylor formulas follow along segments in the positive quadrant. □

4 Eventual all-weight concavity for each input

We prove the finite-moment assertion first. Its specified first differing even moment is $n = 2k \geq 4$; the cumulants of orders $3, \ldots, n-1$ vanish and $\kappa_n = \mathbb{E}X^n - \mathbb{E}G^n \neq 0$. Lemma 3.1 with gain order $2n$ gives $K \in C^{2n}$ and squared-gain regularity $J \in C^n$, using only moment $2n+2$.

Use the gain model from Lemma 3.1. For its characteristic function,

$$\mathbb{E}e^{i\xi Z_H} = e^{-\xi^2/2}[e^{h_1^2\xi^2/2}\chi_X(h_1\xi)][e^{h_2^2\xi^2/2}\chi_X(h_2\xi)].$$

Finite moment differentiation shows that the density gain jet has no nonconstant term below total degree n, and its degree-n term is

$$p_{[n]}(H,z) = \frac{\kappa_n}{n!}(h_1^n + h_2^n)\varphi(z)\,\mathrm{He}_n(z). \tag{20}$$

This identifies a finite Taylor jet, not a convergent cumulant series. All density derivatives used in this finite jet are Gaussian times polynomials, so Fourier-transform uniqueness verifies the density identification directly.

Write $D(H) = D(p_H\|\varphi)$. By Lemma 3.1, its derivatives through total gain order $2n$ can be computed under the integral. At $p = \varphi$ the integrand jet is

$$p\log(p/\varphi) = (p-\varphi) + \frac{(p-\varphi)^2}{2\varphi} + \text{terms of at least third order in } p-\varphi.$$

This is used only for finite gain derivatives at zero. The integrated linear term vanishes by mass conservation. Three nonzero density derivatives have total gain degree at least $3n$, so through degree $2n$ the only nonlinear contribution is the square of (20). Since $\int \mathrm{He}_n^2\,\varphi = n!$,

$$D(H) = \frac{\kappa_n^2}{2n!}(h_1^n + h_2^n)^2 + R_0(H). \tag{21}$$

Here R_0 has zero gain jet through total degree $2n$. The output has mean zero and variance one, hence $K(H) = h(\varphi) - D(H)$.

In squared gains $a = h_1^2, b = h_2^2$, the even-square formula (19) identifies Taylor coefficients at the origin and gives

$$J(a,b) = h(\varphi) - \frac{\kappa_n^2}{2n!}(a^k + b^k)^2 + R(a,b).$$

The Taylor polynomial of R through total degree n vanishes. Taylor's theorem for its derivatives yields

$$\partial^\alpha R(a,b) = o\big((a+b)^{n-|\alpha|}\big), \qquad |\alpha| \le 2,$$

uniformly over positive-quadrant directions, including the axes. Substituting $a = qt, b = q(1-t)$ proves (7) in $C^2([0,1])$: two weight derivatives introduce $q^2(R_{aa} - 2R_{ab} + R_{bb})$.

For $S_k(t) = t^k + (1-t)^k$ and $k \ge 2$,

$$S_k \ge 2^{1-k}, \quad S_k'' \ge k(k-1)2^{3-k}, \quad (S_k^2)'' \ge k(k-1)2^{5-2k} =: c_k > 0.$$

For $k = 2$ the second bound uses $S_0 = 2$. The uniform little-oh remainder permits choosing $q_X > 0$ so that

$$\partial_t^2 F_X(q,t) \le -\frac{\kappa_n^2 c_k}{4n!} q^n < 0$$

for all $0 < q \le q_X$ and all weights. This proves Theorem 1.7. For the all-moment assertion, Gaussian moments are determinate: $(2j-1)!! \le (2j)^j$ makes the Carleman sum diverge. A non-Gaussian symmetric all-moment law therefore has a first differing even moment, so the finite-moment theorem applies. Log-concave laws have exponential tails and hence all absolute moments.

For a genuinely finite-moment example, consider

$$f(x) = \frac{256}{189\pi}(1 + x^2/9)^{-6}.$$

The beta integral gives mass one, variance one and fourth moment $27/7$, so $\kappa_4 = 6/7$. Its tail is proportional to $|x|^{-12}$, giving finite tenth absolute moment and infinite eleventh absolute moment. Thus the theorem with $n = 4$ applies, although the all-moment hypothesis does not. This standardized Student law is not globally log-concave and is not a whole log-concave-class example.

5 Uniform finite-moment restoration and the Gaussian boundary

We prove Theorem 1.5 by a relative finite entropy-jet estimate that allows changing lower cumulants, then extract its quantitative Gaussian boundary.

Proof. Uniform finite derivatives. On the common gain disk $|H| \le 1/2$, Chebyshev's inequality gives $\Pr(|X| \le 2) \ge 3/4$. Thus the positive kernel lower bound in Lemma 3.1 is uniform, as are the density upper bound and quadratic logarithm bound. The single moment bound $2N+4$ in Theorem 1.5 bounds all lower moments by Lyapunov. Lemma 3.1 at gain order $2N+2$ gives uniform entropy derivative bounds through that order. Formula (19) transfers them to squared-gain derivatives through order $N+1$ on a common small quadrant. If $P_{X,N}$ denotes the degree-N Taylor polynomial of J_X, then

$$|\partial^\alpha (J_X - P_{X,N})(a,b)| \le C_N (a+b)^{N+1-|\alpha|}, \qquad |\alpha| \le 2. \tag{22}$$

Here and below constants depend only on B and N. The integral Taylor formula applied to the derivatives uses uniform bounds through degree $N+1$, not a common modulus of continuity of the highest derivative across the family. Substitution $a = qt, b = q(1-t)$ bounds the remainder and each of its first two weight derivatives by a common constant times q^{N+1}.

The finite entropy jet. Only a formal finite jet at zero is used. By Gaussian-kernel differentiation, the relative-density jet is $1 + L + M$ with positive-degree Hermite coefficients

$$\begin{aligned} L_n &= a_n(X)(a^{n/2} + b^{n/2}), \\ M_n &= \sum_{\substack{j+k=n \\ j,k \ge 4 \text{ even}}} a_j(X) a_k(X) \binom{n}{j}^{1/2} a^{j/2} b^{k/2}. \end{aligned}$$

Each symbol $a_n(X)$ carries squared-gain degree $n/2$. Mass conservation removes the linear relative-entropy term. Hermite orthogonality makes its quadratic contribution from L alone, through total squared-gain degree N, exactly

$$\frac{1}{2} \sum_{\substack{4 \le n \le N \\ n \text{ even}}} a_n(X)^2 (a^{n/2} + b^{n/2})^2. \tag{23}$$

Every other nonzero monomial in this finite relative-entropy jet has at least three input factors, because LM has three, M^2 has four, and higher entropy powers have at least three. Such a term is a numerical constant times

$$\left(\prod_{\ell=1}^{m} a_{n_\ell}(X) \right) a^r b^s, \quad m \ge 3, \quad n_\ell \ge 4 \text{ even}, \quad r + s = \frac{1}{2} \sum_{\ell=1}^{m} n_\ell \le N. \tag{24}$$

Index conservation follows directly from the density jet and multiplication; integration of Gaussian polynomials only changes the numerical constants. There are finitely many such monomials, and r, s are nonnegative integers. These statements compute finite derivatives justified above.

Relative control of nonlinear terms. Set

$$E_{X,N}(q) = \sum_{\substack{4 \le n \le N \\ n \text{ even}}} a_n(X)^2 q^n.$$

Order the indices in (24) as $n_1 \le n_2 \le \cdots \le n_m$. Necessarily $n_2 \le N$: otherwise already $n_2 + n_3 > 2N$, contrary to index conservation. Thus the two smallest factors belong to $E_{X,N}$. For $0 < q \le 1$,

$$|a_{n_1} a_{n_2}| q^{(n_1+n_2)/2} \le \frac{1}{2}\left(a_{n_1}^2 q^{n_1} + a_{n_2}^2 q^{n_2}\right).$$

The remaining coefficients have common bounds. Their factor is q^β with $\beta \ge 2$, so it is bounded above by q^2 for $0 < q \le 1$. The second derivative of $t^r(1-t)^s$ is bounded on $[0,1]$ by a monomial-dependent constant. Summing the finite monomial list therefore bounds the absolute nonlinear contribution to weight curvature by $C_N q^2 E_{X,N}(q)$, uniformly through the endpoints.

Coercivity. For $n = 2k$, Section 4 gives $(S_k^2)'' \ge k(k-1)2^{5-2k} > 0$. Let

$$c_N = \frac{1}{2} \min_{2 \le k \le N/2} k(k-1)2^{5-2k} > 0.$$

Since entropy is Gaussian entropy minus relative entropy, equations (22)–(23) imply

$$F_{tt}(q,t) \le -c_N E_{X,N}(q) + C_N q^2 E_{X,N}(q) + C_N q^{N+1}.$$

By (5), $E_{X,N}(q) \ge \varepsilon^2 q^N$. Choose a common positive threshold below the gain-domain bound and below one so that $C_N q^2 \le c_N/4$ and $C_N q \le c_N \varepsilon^2/4$ at and below it. Then

$$F_{tt}(q,t) \le -\frac{c_N}{2} E_{X,N}(q) < 0,$$

uniformly over the entire family and all weights, proving the theorem. □

Proof of Corollary 1.6. Fix an even $N \ge 4$. The preceding proof establishes the uniform curvature inequality

$$F_{X,tt}(q,t) \le -c_N E_{X,N}(q) + C_N q^2 E_{X,N}(q) + C_N q^{N+1}$$

on a common small-signal interval under the moment envelope, with $c_N > 0$. Choose that interval smaller so that $C_N q^2 \le c_N/2$. At a point of nonnegative curvature,

$$E_{X,N}(q) \le \frac{2C_N}{c_N} q^{N+1}.$$

Every summand is nonnegative, so for even $4 \le n \le N$,

$$|a_n(X)| \le \left(\frac{2C_N}{c_N}\right)^{1/2} q^{(N+1-n)/2}. \tag{25}$$

For each fixed even $n \ge 4$ and $A > 0$, choose one even $N \ge \max\{4, n, n + 2A - 1\}$ and take $q \le 1$ as well. Then (25) is $O(q^A)$ uniformly over $\mathcal{C}$. The moments of degrees zero, one and two already have their Gaussian values, and symmetry fixes every odd moment. Each fixed polynomial is a finite linear combination of Hermite polynomials. Taking the smallest of the finitely many signal thresholds and summing their constants proves the polynomial assertion. Finally Gaussian-gain regularity makes the weight function C^2 on the closed interval in this regime. Its failure of concavity supplies an interior point of positive second derivative. Applying the same estimate at that point proves the sequence assertion. □

Corollary 5.1 (Gaussian-only limiting obstruction)**.** *Let $\mathcal{L}$ be the class of symmetric variance-one log-concave laws on $\mathbb{R}$, and let $\mathcal{U}$ be any weakly open neighborhood of the standard Gaussian law γ. There is $q_{\mathcal{U}} > 0$ for which every $X \in \mathcal{L} \setminus \mathcal{U}$ has strict all-weight entropy concavity whenever $0 < q \le q_{\mathcal{U}}$. Consequently, if $X_j \in \mathcal{L}$, $q_j \downarrow 0$, and $F_{X_j}(q_j, \cdot)$ fails concavity for every j, then X_j converges weakly to γ and in every fixed absolute moment.*

Proof. The log-concave moment estimate $\|X\|_p \le p/2$ for $p \ge 2$ [9] supplies a common moment envelope. These laws are tight; uniform higher moments preserve variance and every fixed moment under weak convergence. Log-concavity and symmetry are closed under weak limits [10], and preserved variance excludes a degenerate limit. Thus $\mathcal{L}$ is weakly compact and all fixed Hermite moments are continuous on it.

The compact set $\mathcal{L} \setminus \mathcal{U}$ excludes γ. For each of its members, some even Hermite moment of order at least four is nonzero, by Gaussian moment determinacy. The positive loci of the increasing continuous functions $\sum_{4\le n\le N,\, n \text{ even}} a_n(X)^2$ cover this compact set. A finite subcover gives a common even N, and the positive minimum of the resulting function gives (5). Theorem 1.5 supplies the uniform threshold. If a proposed bad sequence did not converge weakly to γ, a subsequence would stay outside one weak Gaussian neighborhood and contradict this threshold. Uniform higher moments then give convergence of every fixed absolute moment. □

Fixing a first differing moment order and a positive cumulant gap is one special case of Theorem 1.5. The finite-jet criterion also permits lower moments to vary and lower cumulants to vanish. Corollary 1.6 supplies the quantitative necessary condition as the discrepancy shrinks, degree by degree.

6 The multicopy Hessian at the barycentre

We prove Theorem 1.2 with all active subset sizes retained in the smoothing estimate. The constants in this section are distinct from those of the two-copy local proof. Put

$$c_0 = \min_{2\le j\le m} \left[\frac{m^{j-1}}{j^j Q^{j-1}}\right]^{1/j}, \quad c = \frac{1+c_0}{2}, \quad r = \frac{(c-1)\sqrt{2}}{4m},$$
$$\rho = \frac{1+mr/\sqrt{2}}{c}, \quad K = \frac{\sqrt{\rho}}{(1-\sqrt{\rho})^2}, \quad S = \frac{2^m-m-1}{m},$$
$$a = 2^m-1, \quad A = \frac{mKS}{16} + \frac{7ma}{2} + \frac{S}{3r^2}, \quad \delta_{m,Q} = \min\left\{1, \frac{1}{4a}, \frac{4}{KS}, \frac{1}{S}, \frac{m}{16A}\right\}. \tag{26}$$

These are finite positive constants. Indeed $x^{x/(x-1)}$ increases strictly for $x > 1$, since its logarithmic derivative is $(x-1-\log x)/(x-1)^2 > 0$. Hence the smallest of $m/j^{j/(j-1)}$, $2 \le j \le m$, is $m^{-1/(m-1)}$. Our hypothesis implies $c_0 > 1$, and simultaneously

$$c^j Q^{j-1} j^j / m^{j-1} < 1 \qquad (2 \le j \le m). \tag{27}$$

The minimum over j is essential; controlling only the last subset size after choosing a large c would not imply the other inequalities.

6.1 Linear energy and the conditional subset expansion

Normalize v to Euclidean norm one, with $\sum_i v_i = 0$, and let $t_i(s) = 1/m + sv_i$. Write $\delta = \|g\|_\infty$. For real s sufficiently near zero these are positive simplex weights. Conditioning independent Gaussian input factors gives output density $\varphi(1+R_s)$, where

$$R_s = L_s + M_s, \qquad L_s = \sum_i T_{\sqrt{q t_i(s)}} g,$$

and M_s is the sum over all subsets $H \subseteq [m]$ with $|H| \geq 2$ of the conditional products of the g factors in H. The Hermite coefficient of one such product, at total degree $n = \sum_{i \in H} n_i$, is

$$q^{n/2} \sum_{\sum n_i = n} \left(\frac{n!}{\prod_{i \in H} n_i!} \right)^{1/2} \prod_{i \in H} a_{n_i} t_i^{n_i/2}. \tag{28}$$

Every active n_i is even and at least four. The formula follows by conditioning products of normalized Hermites on their Gaussian sum; it also follows by multiplying their Gaussian generating functions. On real simplex weights,

$$\|R_s\|_\infty \leq (1+\delta)^m - 1 \leq a\delta =: \eta \leq 1/4. \tag{29}$$

The second inequality uses $\delta^j \leq \delta$ for $\delta \leq 1$. The output retains mass one and variance one.

At $s = 0$, put $L = L_0 = mT_{\sqrt{q/m}} g$ and $D = \|NL\|_2$. For $n = 2k \geq 4$, the multiplier of a_n in L_s is $l_n(s) = q^k \sum_i (1/m + s v_i)^k$, so

$$l_n'(0) = 0, \qquad l_n''(0)/l_n(0) = mk(k-1).$$

Therefore

$$\langle L'', L \rangle \geq mD^2/8, \qquad \left\| N^{-1} L'' \right\|_2 \leq mD/4, \qquad \|L\|_2 \leq D/4. \tag{30}$$

Primes here denote s derivatives at zero. The inverse is restricted to positive Hermite degrees. A nonzero g gives $D > 0$ for every $q > 0$.

6.2 Higher-product interpolation and analytic completion

For complex z with $|z| \leq r$ set $z_i = 1/m + z v_i$. For a subset H of size j, define

$$P_H = q \sum_{i \in H} |z_i|, \qquad \overline{P}_j = cqj/m.$$

Cauchy–Schwarz on the coordinates of v and (27) give

$$\frac{P_H}{\overline{P}_j} \leq \frac{1 + mr/\sqrt{j}}{c} \leq \rho = \frac{c+3}{4c} < 1, \tag{31}$$

$$\overline{P}_j^{\,j} < q/m, \qquad \overline{P}_j < 1. \tag{32}$$

If $P_H > 0$, normalize $w_i = |z_i| / \sum_{k \in H} |z_k|$. Absolute coefficient majorization in (28) reduces the subset to a real Gaussian product at total signal P_H and these weights, with inputs g_{abs}. This is the L^2 function with Hermite coefficients $|a_n|$; its norm is at most δ, but it need not be bounded. If $P_H = 0$, the positive-degree subset is zero and no normalized weights are needed.

For j independent smoothed Gaussian inputs and normalized nonnegative weights w, conditioning their product on the weighted Gaussian sum is an L^2 contraction. Hence, for $h \in L^2(\varphi)$,

$$\|B_j(P, w; h, \ldots, h)\|_2 \leq \left\| T_{\sqrt{P}} h \right\|_2^j. \tag{33}$$

Prove this first for finite Hermite sums by the coefficient formula. Independence gives the product L^2 norm. The same multilinear bound applied to a telescoping difference of products extends it to arbitrary L^2 inputs; no product on a single one-dimensional L^2 space is used.

All j factors can be retained in a spectral Hölder interpolation. For nonnegative x_n,

$$\sum_n x_n P^n \leq (\sum_n x_n)^{(j-1)/j} (\sum_n x_n P^{jn})^{1/j}.$$

With x_n the squared Hermite coefficients of h and $\|h\|_2 \le \delta$, raising to power $j/2$ gives

$$\left\|T_{\sqrt{P}}h\right\|_2^j \le \delta^{j-1}\left\|T_{P^{j/2}}h\right\|_2. \tag{34}$$

By (32), the real product at enlarged signal $\overline{P}_j$ is therefore bounded by

$$\delta^{j-1}\left\|T_{\overline{P}_j^{j/2}g_{\mathrm{abs}}}\right\|_2 \le \delta^{j-1}\left\|T_{\sqrt{q/m}g}\right\|_2 = \delta^{j-1}\|L\|_2/m.$$

The original signal P_H has the additional output attenuation $(P_H/\overline{P}_j)^{n/2}$. Its number-operator cost is at most $\sup_n n\rho^{n/2} \le K$. Sum the finitely many subsets and use $\delta \le 1$ to obtain, throughout the complex disk,

$$\|M_z\|_2 \le S\delta\|L\|_2, \qquad \|NM_z\|_2 \le KS\delta\|L\|_2.$$

Strictness in (31) permits a slightly larger disk. Multilinear completion controls finite-input truncation differences, and exponential output attenuation gives uniform coefficient tails, including every fixed number-operator power. The coefficient series thus converge normally in Hilbert space. Their weight exponents are integers, so zeros of complex coordinates cause no branch issue. Both M_z and L_z are Hilbert analytic. Cauchy's estimate and (30) imply

$$\|M\|_2 \le S\delta D/4, \qquad \|NM\|_2 \le KS\delta D/4, \qquad \|M''\|_2 \le S\delta D/(2r^2). \tag{35}$$

The same normal convergence on a sufficiently small complex ball of weights in $\sum z_i = 1$ gives joint parameter regularity. It is not necessary to interpret complex weights as probability laws.

6.3 Entropy curvature and relative absorption

On the positive real part of this weight neighborhood, (29) bounds the density ratio away from zero. Hilbert C^2 regularity, scalar Taylor's formula and the bounded first two derivatives of $(1+u)\log(1+u)$ on this range justify

$$(\mathcal{F}_{q,g}^{(m)})'' = -\langle R'', \log(1+R)\rangle - \int \frac{(R')^2}{1+R}\varphi.$$

Products of L^2 difference quotients converge in L^1; bounded reciprocal multipliers converge in measure and preserve these limits. This also gives continuity of mixed entropy derivatives. Positive Gaussian noise smooths the bounded input factors spatially. Its conditional Gaussian kernel has positive covariance for $q < 1$, so differentiation supplies bounded spatial derivatives through Gaussian polynomial moments. The number-operator bounds above then justify self-adjoint pairings and the Sobolev chain rule.

For $\psi(u) = \log(1+u) - u$, the same fully proved interpolation estimate (16) gives

$$\begin{aligned}\|N\psi(R)\|_2 &\le \left[\frac{\eta}{1-\eta} + \frac{3\eta}{(1-\eta)^2}\right]\|NR\|_2 \\ &\le 7a\delta(1 + KS\delta/4)D.\end{aligned}$$

Discard the nonpositive first-derivative-square term and split the logarithm as $L + M + \psi(R)$. Equations (30) and (35) give the errors

$$\begin{aligned}|\langle L'', M\rangle| &\le (mKS/16)\delta D^2, \\ |\langle L'', \psi(R)\rangle| &\le (7ma/2)\delta D^2, \\ |\langle M'', \log(1+R)\rangle| &\le S\delta D^2/(3r^2).\end{aligned}$$

For the second use $KS\delta/4 \le 1$; for the third use $S\delta \le 1$, $\eta \le 1/4$ and $\|\log(1+R)\|_2 \le (4/3)\,\|R\|_2 \le 2D/3$. Their sum is at most $A\delta D^2 \le mD^2/16$, against the restoring energy $mD^2/8$. This proves (2) for unit v, and scaling proves it for every tangent vector. Strictness follows from $D > 0$. Joint Hessian continuity and compactness of the unit tangent sphere give a neighborhood for each fixed input and signal, with no common weight-radius conclusion. This completes the theorem.

7 Higher-moment obstructions near a face

The fourth-cumulant sign mechanism has antecedents in Madiman–Nayar–Tkocz [8]; the three-copy noisy face expansion in [5, Section 7] uses that mechanism and the same density functional appearing in the endpoint identity of [2]. Its leading term vanishes under exact fourth-moment matching. We use a higher first differing moment and a differentiated finite jet to prove Theorem 1.1. The argument below supplies all expansions and sign constructions internally.

7.1 A differentiated finite jet at the face

Let $n = 2\ell \ge 6$ and let X have a bounded smooth even Gaussian-relative perturbation, variance one, and Gaussian moments through $n-2$. Fix $q \in (0,1)$ and let $U = X_q$, with density p and $\kappa_n(U) = q^\ell \kappa_n(X) \ne 0$. Set

$$J(a,b) = h(\sqrt{a}\,U_1 + \sqrt{b}\,U_2 + \sqrt{1-a-b}\,U_3).$$

Let J_G replace U_1, U_2 by independent standard Gaussians. It is a function only of $a+b$:

$$J_G(a,b) = h(\sqrt{1-a-b}\,U_3 + \sqrt{a+b}\,G).$$

Define $B_n(p) = \int p^{(n)} \log p$. We claim

$$J(a,b) - J_G(a,b) = -\frac{\kappa_n(U)}{n!}(a^\ell + b^\ell)B_n(p) + R(a,b), \tag{36}$$

$$|\partial^\alpha R(a,b)| \le C(a+b)^{\ell+1-|\alpha|}, \qquad |\alpha| \le 2, \tag{37}$$

near the origin in the positive quadrant. Constants depend on the fixed input and signal.

Here is the regularity and finite-jet justification. Put $a = h_1^2$, $b = h_2^2$. In original-input coordinates the gains are $\sqrt{q}h_1$, $\sqrt{q}h_2$, $\sqrt{q}\sqrt{1-h_1^2-h_2^2}$ and the independent Gaussian variance is the positive constant $1-q$. The Gaussian-replacement output instead has Gaussian variance $1-q+q(a+b)$. These gain and variance functions are smooth near zero and bounded away from singularity. The proof of Lemma 3.1 applies verbatim to the finite vector of three inputs: kernel derivatives have conditional polynomial ratios, a bounded-input event supplies the quadratic logarithm bound, and conditional Young bounds differentiated entropy integrals by finitely many moments of these inputs and the Gaussian. Higher moments remove their spatial tails uniformly on the gain disk. All required moments exist for the present bounded Gaussian-relative inputs. Thus the gain entropies are smooth. Evenness in the two light gains and (19) give squared-gain $C^{\ell+1}$ regularity, up to axes, using gain derivatives only through $2\ell + 2 = n + 2$.

Finite characteristic-function derivatives, rather than a convergent cumulant series, identify the first discrepancy of a light factor:

$$\chi_U(\sqrt{a}\,\xi) = e^{-a\xi^2/2}\left[1 + \frac{\kappa_n(U)}{n!}(i\xi)^n a^\ell + \text{higher finite gain jet}\right].$$

After multiplying both light factors and the heavy characteristic function, the first density difference from Gaussian replacement is

$$\frac{\kappa_n(U)}{n!}(h_1^n + h_2^n)p^{(n)}.$$

The Fourier signs agree for even n. Changing the heavy gain or the comparison baseline in this correction adds at least two gain degrees; products of two density differences start at degree $2n$. The first entropy variation is its negative pairing with $1 + \log p$. Its mass integral is zero, leaving exactly the coefficient in (36). All lower difference jets vanish. The just-proved $C^{\ell+1}$ regularity gives (37) by Taylor's theorem for derivatives of orders zero, one and two.

On $a = \epsilon + \tau$, $b = \epsilon - \tau$, the Gaussian baseline is constant in τ. Thus

$$\frac{d^2}{d\tau^2} J(\epsilon + \tau, \epsilon - \tau)\bigg|_{\tau=0} = -\frac{2\ell(\ell-1)}{n!}\kappa_n(U)B_n(p)\epsilon^{\ell-2} + O(\epsilon^{\ell-1}). \tag{38}$$

A negative product $\kappa_n(U)B_n(p)$ therefore gives strictly positive curvature at a genuine interior weight for all sufficiently small positive ϵ, for each fixed input. No sign on an entire τ segment is required to disprove concavity.

7.2 The density functional and a negative quadratic direction

For $p = \varphi(1 + r)$ with an even bounded smooth relative perturbation, $\|r\|_\infty < 1/2$, integration by parts and the logarithm expansion give

$$\begin{aligned} B_n(p) &= \mathbb{E}_\varphi[\mathrm{He}_n\, r] + Q_n(r) + \text{cubic remainder}, \\ Q_n(h) &= \int (\varphi h)^{(n)} h - \tfrac{1}{2}\mathbb{E}_\varphi[\mathrm{He}_n\, h^2]. \end{aligned} \tag{39}$$

Indeed $\int \varphi^{(n)} \log\varphi$ and $\int (\varphi r)^{(n)} \log\varphi$ vanish identically because $\log\varphi$ is quadratic and $n \geq 3$. The linear and quadratic terms left by $\log(1 + r)$ are exactly those displayed. For the later family $r = \alpha H + \beta\alpha^2 W$ with fixed bounded smooth OU images H, W, the remainder is rigorously $O(\alpha^3)$: their spatial derivatives are bounded, $(\varphi H)^{(n)}/\varphi$ and the analogous W expression have polynomial growth, and differentiated logarithms have Gaussian-polynomial integral majorants for small amplitude.

Take the ideal polynomial direction e_d with $d = n + 2$. For even n,

$$\frac{(\varphi e_d)^{(n)}}{\varphi} = \sqrt{(d+n)!/d!}\, e_{d+n}.$$

Its pairing with e_d is zero. The Hermite product formula $\mathrm{He}_d^2 = \sum_j \binom{d}{j}^2 j!\, \mathrm{He}_{2d-2j}$ gives

$$\mathbb{E}_\varphi[\mathrm{He}_n\, e_d^2] = \frac{n!d!}{(d-\ell)!(\ell!)^2} =: c_{n,d} > 0.$$

Consequently

$$Q_n(T_{\sqrt{q}} e_d) = -\tfrac{1}{2} q^d c_{n,d} < 0. \tag{40}$$

This unbounded polynomial selects an approximation direction; it is never used as a bounded relative density.

7.3 Compact moment corrections and continuity

Let $I = \{0, 2, \dots, n\}$. Choose a nonnegative even smooth compact cutoff χ positive on a nonempty open interval, and put

$$\psi_j = \chi \operatorname{He}_j, \qquad \Gamma_{ij} = \int \chi \operatorname{He}_i \operatorname{He}_j \varphi \quad (i, j \in I).$$

The matrix Γ is positive definite, since a nonzero polynomial cannot vanish on that interval, and is therefore invertible. Choose even smooth cutoffs χ_R equal to one on $[-R, R]$, zero outside $[-2R, 2R]$, and between zero and one. For $u_{R,i} = \mathbb{E}_\varphi[\operatorname{He}_i \chi_R e_d]$ define

$$v_R = \chi_R e_d - \sum_{j \in I} (\Gamma^{-1} u_R)_j \psi_j.$$

Then $\mathbb{E}_\varphi \operatorname{He}_i v_R = 0$ exactly for every $i \in I$. Also $v_R \to e_d$ in $L^2(\varphi)$: cutoff convergence follows by dominated convergence and $u_R \to 0$ by orthogonality and $d > n$. Every v_R is even, bounded, smooth and compactly supported.

For fixed $q < 1$, the map $v \mapsto Q_n(T_{\sqrt{q}} v)$ is a continuous quadratic form on $L^2(\varphi)$. To verify the needed continuity explicitly, the raising operator $A_n h = (\varphi h)^{(n)}/\varphi$ sends e_k to $\sqrt{(k+n)!/k!}\, e_{k+n}$ for even n. Multiplication by He_n is a finite sum of Hermite shifts with coefficients bounded by $C_n(1+k)^{n/2}$. After either operator is composed with $T_{\sqrt{q}}$, its exponentially attenuated coefficients are bounded, so each composition is bounded from L^2 to L^2. The pairings with $T_{\sqrt{q}} v$ in (39) are thus continuous. This does not require uniform supremum or derivative bounds on the approximating cutoff polynomials.

By (40), freeze a sufficiently large R so that $v = v_R$ has $Q := Q_n(T_{\sqrt{q}} v) < 0$. Let $e^{(n)}$ be the coordinate vector indexed by n in I, and freeze

$$w = \sum_{j \in I} (\Gamma^{-1} e^{(n)})_j \psi_j.$$

It satisfies $\mathbb{E}_\varphi \operatorname{He}_i w = 0$ for $i < n$ and $\mathbb{E}_\varphi \operatorname{He}_n w = 1$. The two directions may depend on fixed q, n, but not on the amplitude or face distance.

7.4 Actual densities and the order of choices

Choose $\beta > 0$ with $\beta q^\ell < -Q/2$. For positive α tending to zero put

$$g_\alpha = \alpha v + \beta \alpha^2 w, \qquad f_\alpha = \varphi(1 + g_\alpha).$$

All moments through $n-2$ are exactly Gaussian. The first differing cumulant is exactly

$$\kappa_n(X_\alpha) = \mathbb{E} \operatorname{He}_n(X_\alpha) = \beta \alpha^2 > 0.$$

Here $\mathbb{E}_\varphi \operatorname{He}_n w = 1$ is unnormalized; no factor $\sqrt{n!}$ is missing. The smoothed law has relative function $r_\alpha = \alpha H + \beta \alpha^2 W$, with $H = T_{\sqrt{q}} v$ and $W = T_{\sqrt{q}} w$. Equation (39) therefore gives

$$B_n(p_\alpha) = \alpha^2 (Q + \beta q^\ell) + O(\alpha^3) < 0$$

for small positive amplitude, whereas $\kappa_n(U_\alpha) = q^\ell \beta \alpha^2 > 0$.

Because v, w are fixed compact smooth functions, $\|g_\alpha\|_{C_b^r} \to 0$ for every fixed r. For small amplitude, $1 + g_\alpha \geq 1/2$ and

$$(\log f_\alpha)'' = -1 + \frac{g_\alpha''}{1+g_\alpha} - \frac{(g_\alpha')^2}{(1+g_\alpha)^2} \leq -1 + 2\left\|g_\alpha''\right\|_\infty \leq -1/2.$$

Thus these are positive smooth strongly log-concave even densities, with mass and variance exactly one. Exact Hermite matching is equivalent to ordinary moment matching because the Hermites form a triangular polynomial basis; odd moments match by symmetry.

For each fixed sufficiently small amplitude, (38) now has strictly positive leading coefficient. Choose a face distance $\epsilon(\alpha) > 0$ small enough to dominate its remainder, and also less than α and $1/4$. Taking $n = 2K + 2$ and a sequence $\alpha_j \downarrow 0$ gives Theorem 1.1, with $\epsilon_j \to 0$ (and a decreasing subsequence if necessary). The order is fixed q, K, then compact directions and β, then amplitude, then face distance. No uniform-in-amplitude remainder or fixed face weight is asserted. The equality in law between mixing copies of $U = X_q$ and adding Gaussian variance $1 - q$ after mixing copies of X identifies this J with the three-copy entropy in the theorem.

Corollary 7.1. *The nonconcavity conclusion of Theorem 1.1 holds for every fixed number of copies $m \geq 3$, at a true interior simplex point, with the same prescribed finite moment matching and Gaussian-relative convergence. The weights may depend on the input.*

Proof. For $m > 3$, embed the three-copy point with the remaining weights zero. Its positive tangent curvature is unchanged. The fixed positive Gaussian noise and the finite-vector gain argument used above give continuous weight Hessians up to these axes by the even-square formula. Therefore its positive curvature persists after sufficiently small positive extra weights are added, scaling the first three weights by their remaining total mass. All m weights are then positive. Apply this continuity argument separately to each input in the constructed sequence; no common extra-weight size is needed. □

8 Restoration regimes and moment topology

Theorem B.1 supplies a concrete distinction between the two neighborhoods. Its laws match any fixed finite set of Gaussian moments exactly and approach the Gaussian in total variation and every fixed absolute moment, while their positive-signal smoothings retain exponential tails and an unbounded Gaussian-relative density ratio. This makes the moment and relative-density regimes mathematically complementary.

A Uniform endpoint restoration for log-concave inputs

Fix a symmetric variance-one log-concave input and $0 < q \leq 1/20$. Let $U = X_q$, with density p, score $\rho = (\log p)'$, and put

$$g = -\rho', \quad s = \rho + x, \quad A = \mathbb{E}(g(U) - 1)^2, \quad \delta = \mathbb{E}s(U)^2, \quad I = \mathbb{E}\rho(U)^2, \quad K = \mathbb{E}g(U)^2.$$

The letters g, δ, K in this section are score quantities, distinct from the relative-density notation in Section 2.

Gaussian convolution preserves log-concavity, and the posterior-variance identity for the Gaussian kernel gives

$$0 \leq g \leq M := \frac{1}{1-q}. \tag{41}$$

Indeed $(\log p)'' = -1/(1-q) + q\,\mathrm{Var}(X \mid U)/(1-q)^2$, while log-concavity supplies the opposite bound. The score is odd and $|\rho(x)| \le M|x|$.

The exact symmetric endpoint identity from [2, Theorem 3.1] is

$$F_{tt}(q,0) = -\frac{A}{2} - \frac{\kappa_4(U)}{12}B, \qquad B = \mathbb{E}g^2 - \frac{1}{3}\mathbb{E}\rho^4. \tag{42}$$

We first justify its use without original input smoothness. Put $t = u^2$ in (1) and allow $|u| \le 1/4$. The Gaussian-kernel argument of Lemma 3.1 applies to the smooth gain functions $\sqrt{q}\sqrt{1-u^2}$, $\sqrt{q}\,u$, with fixed positive noise variance. It proves $H(u) = F_X(q,u^2)$ is C^4 and even. Hence

$$\lim_{t\downarrow 0} F_{tt}(q,t) = H^{(4)}(0)/12.$$

The fourth-order density expansion is

$$\begin{aligned} p_u &= p + c_2u^2 + c_4u^4 + o(u^4),\\ c_2 &= (p + xp' + p'')/2,\\ 2c_4 &= 3p/4 + 5xp'/4 + x^2p''/4 + 3p''/2 + xp'''/2 + \mathbb{E}U^4\,p''''/12. \end{aligned}$$

Differentiating entropy gives $F_{tt}(q,0) = -\int 2c_4 \log p - \int c_2^2/p$. The Gaussian fourth-moment portion is the normalized Ornstein–Uhlenbeck curvature $-A/2$; the remaining term is $-\kappa_4(U)\int p''''\log p/12$. Two integrations by parts and $\mathbb{E}[\rho^2\rho'] = -\mathbb{E}\rho^4/3$ give (42). This reproduces the cited identity and its OU decomposition. Smooth cutoffs justify these integrations: the conditional-polynomial derivative bounds control spatial derivatives, g is bounded, ρ is at most linear, and all input moments are finite.

We use the standard one-dimensional Poincaré bound $C_P(X) \le 12\,\mathrm{Var}(X)$ for log-concave measures [3]. Tensorization applied to $f(\sqrt{q}X + \sqrt{1-q}G)$ gives

$$C_P(U) \le qC_P(X) + (1-q) \le 1 + 11q =: C \le 31/20 < 8/5.$$

Integration by parts gives

$$I = \mathbb{E}g = 1 + \delta, \qquad \mathbb{E}[Us(U)] = 0, \qquad K = 1 + A + 2\delta.$$

Poincaré applied to the odd function s yields $\delta \le CA$. Also $g^2 \le Mg$, and $M \le 2, I \ge 1$, so

$$A = K - 2I + 1 \le (M-2)I + 1 \le M - 1 \le 1/19 < 1/16,$$

and

$$K \le 1 + (1+2C)A \le 116/95 < 5/4. \tag{43}$$

Write κ_j for cumulants. The identity $\mathbb{E}[U^3\rho(U)] = -3$ implies

$$\kappa_4(U) = \mathbb{E}[(U^3 - 3U)s(U)].$$

Cauchy–Schwarz gives

$$|\kappa_4(U)| \le \sqrt{\mathbb{E}[(U^3-3U)^2]CA}.$$

The exact moment expression is

$$\mathbb{E}[(U^3-3U)^2] = 6 + 9\kappa_4(U) + \kappa_6(U).$$

Cumulant addition gives $\kappa_4(U) = q^2\kappa_4(X)$ and $\kappa_6(U) = q^3\kappa_6(X)$. The moment bound $\|X\|_p \le p/2$ for $p \ge 2$ [9] gives $\mathbb{E}X^4 \le 16$, $\mathbb{E}X^6 \le 729$, hence $\kappa_4(X) \le 13$ and $\kappa_6(X) \le 759$. Therefore

$$\mathbb{E}[(U^3 - 3U)^2] \le 6 + 117q^2 + 759q^3 < 7, \quad |\kappa_4(U)| < \frac{27}{8}\sqrt{A}.$$

For clarity we reproduce the centered correction estimate from the endpoint framework [2]. Let $D_0 = g - \rho^2$, with $\mathbb{E}D_0 = 0$. Integration by parts gives $\mathbb{E}[g\rho^2] = \mathbb{E}\rho^4/3$, so

$$B = \mathbb{E}[(g-1)D_0], \qquad \mathbb{E}D_0^2 = 2K - B, \qquad B^2 \le A(2K - B).$$

Consequently, for $A > 0$,

$$\frac{|B|}{\sqrt{A}} \le \frac{\sqrt{A} + \sqrt{A + 8K}}{2} < \frac{7}{4},$$

by $A < 1/16$ and (43). Combining the two bounds gives $|\kappa_4(U)B| < (189/32)A$. Equation (42) now yields

$$F_{tt}(q, 0) \le -\frac{6 - 189/32}{12}A = -A/128.$$

If $A = 0$, smooth positivity implies $(\log p)'' = -1$ everywhere, and centering and normalization give $p = \varphi$. Dividing characteristic functions by the nonzero Gaussian factor shows X is Gaussian. Otherwise the bound is strict. The symmetry in the mixing weight supplies the other endpoint, completing Theorem 1.8.

B Moment-matched exponential tails

The Gaussian-relative and weak/moment neighborhoods are distinct. A tangent-tail potential and compactly supported moment corrections give the following explicit separating family.

Theorem B.1. *For every integer $k \ge 1$ there are symmetric variance-one log-concave densities f_R, with $R \to \infty$, whose Gaussian moments agree exactly through order $2k$, and whose first unmatched cumulant satisfies*

$$\kappa_{2k+2}(f_R) \sim 2\varphi(R)R^{2k-1} > 0.$$

Their laws converge to the Gaussian in total variation and in every fixed absolute moment. Nevertheless, for each member and every $q \in (0, 1]$, the density of its Gaussian smoothing has an unbounded ratio to φ and infinite chi-square divergence from φ.

Proof. Let $V_R(x) = x^2/2$ for $|x| \le R$ and $V_R(x) = R|x| - R^2/2$ otherwise, and let $b_R = (2\pi)^{-1/2}e^{-V_R}$. This potential is even, convex and C^1. For fixed integer $j \ge 0$, put $B_{j,R} = \int x^{2j}(b_R - \varphi)$ and $A_R = B_{0,R}$. Substitution $x = R + y/R$ in both tails gives

$$\begin{aligned} B_{j,R} &= 2\varphi(R)R^{2j-1}\int_0^\infty e^{-y}\left(1 + \frac{y}{R^2}\right)^{2j}\left(1 - e^{-y^2/(2R^2)}\right)dy \\ &\sim 2\varphi(R)R^{2j-3}. \end{aligned}$$

Dominated convergence after multiplying the integral by R^2 uses the majorant $e^{-y}(1+y)^{2j}y^2/2$ for $R \ge 1$; the limiting integral is one. Thus $A_R \sim 2\varphi(R)R^{-3}$. If $m_j = (2j-1)!!$, the moment deviations of the normalized base law are

$$d_{j,R} = \frac{B_{j,R} - m_j A_R}{1 + A_R} \sim 2\varphi(R)R^{2j-3} \quad (j \ge 1).$$

Choose an even smooth cutoff χ_a, equal to one on $[-a, a]$ and zero outside $[-a-1, a+1]$, and set $\psi_j(x) = \chi_a(x)x^{2j}$ for $j = 1, \dots, k$. For sufficiently large fixed a, the matrix

$$C_{ij} = \mathrm{Cov}_\gamma(G^{2i}, \psi_j(G)), \qquad 1 \le i, j \le k,$$

is invertible. Indeed it converges as $a \to \infty$ to the positive-definite covariance matrix of $G^2, \dots, G^{2k}$: a nonconstant polynomial cannot be almost surely constant under a positive Gaussian density. Freeze such controls.

For $R > a + 1$ tilt the base density by $e^{-\sum \theta_j \psi_j}$ and normalize, and let $M_R(\theta)$ be its vector of moment deviations of orders $2, \dots, 2k$. Then $M_R(0) = d_R$ and

$$(DM_R(\theta))_{ij} = -\mathrm{Cov}_{R,\theta}(X^{2i}, \psi_j(X)).$$

Boundedness of the controls and the base moment estimates imply uniform convergence of these derivatives on a fixed small parameter ball to those for the corresponding tilted Gaussian, and give uniform bounds on second derivatives. In particular $DM_R(0) \to -C$ has uniformly bounded inverse. Writing $M_R(\theta) = d_R + J_R\theta + E_R(\theta)$, the map

$$\theta \longmapsto -J_R^{-1}(d_R + E_R(\theta))$$

is a contraction of a ball of radius $C_0 \|d_R\|$ into itself for large R. Its fixed point θ_R gives exact moment matching and

$$\|\theta_R\| = O(\varphi(R)R^{2k-3}).$$

This is the usual finite-dimensional implicit-function correction, with a uniform inverse and quadratic remainder.

The corrected potential is $W_R = V_R + \sum \theta_{j,R}\psi_j$. On the fixed control support, $V_R'' = 1$ and $W_R'' \ge 1/2$ for large R; outside it the potential equals the convex V_R. The smooth controls have vanishing value and first derivative at their support boundary, so W_R is globally convex. The density is therefore log-concave, even, and already has variance one by exact matching. Its tails retain rate R.

For every fixed moment order, the correction changes its value by $O(\|\theta_R\|)$, by the bounded-control covariance formula along the parameter segment. At order $2k+2$, this is smaller by R^{-2} than the base deviation $2\varphi(R)R^{2k-1}$. The stated first-unmatched-moment asymptotic follows; because all lower moments match, that deviation is also the cumulant. The tilt converges uniformly to one. The same tail-integral argument works for every fixed nonnegative absolute power and proves moment and total-variation convergence to the Gaussian.

Finally the corrected density has an exact positive exponential tail $c_R e^{-Rx}$ for large positive x. In its Gaussian convolution at $q < 1$, restrict the Gaussian summand to absolute value at most one. For all sufficiently large y this yields

$$p_{R,q}(y) \ge C_{R,q} e^{-Ry/\sqrt{q}}.$$

For $q = 1$ the tail identity gives the same conclusion directly. Therefore $p_{R,q}/\varphi$ is unbounded, and $\int p_{R,q}^2/\varphi$ diverges by comparison with a positive constant times $\exp(y^2/2 - 2Ry/\sqrt{q})$ on a tail. □

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