Mixing-weight entropy concavity near the Gaussian below one-half signal

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Abstract

We establish a Gaussian neighborhood of mixing-weight entropy concavity for symmetric variance-one inputs below one-half retained signal. For each prescribed upper signal fraction less than one half, an explicit uniform relative-density radius guarantees concavity, strictly for non-Gaussian inputs, for every mixing weight and every smaller positive signal fraction. This gives a negative answer, on that signal range, to Congyi Luo's question about near-Gaussian counterexample sequences, including sequences with strong log-concavity and any prescribed finite Gaussian moment matching. The proof controls arbitrary bounded even perturbations through an infinite-mode Hermite estimate that absorbs nonlinear curvature errors relative to the quadratic energy. Complementary finite-jet estimates yield a common all-weight restoration threshold for families with a common absolute-moment envelope and a positive finite Hermite discrepancy, as well as eventual concavity for each fixed symmetric non-Gaussian input with all moments finite. They also force every fixed Gaussian-moment discrepancy of a family with nonnegative small-signal curvature to decay faster than every fixed algebraic power of the signal fraction. A uniform log-concave-class endpoint estimate and exactly moment-matched exponential-tail examples distinguish the resulting restoration regimes.

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1 Introduction and main result

Let X, Y be independent copies of a centered variance-one random variable. For an independent standard Gaussian G, write

$$X_q = \sqrt{q}\, X + \sqrt{1-q}\, G, \qquad 0 \le q \le 1.$$

Given independent copies U_q, V_q of X_q, set

$$F_X(q,t) = h(\sqrt{t}\, U_q + \sqrt{1-t}\, V_q), \qquad 0 \le t \le 1,$$

where h is differential entropy with natural logarithms. Equivalently the output is

$$Z_{q,t} = \sqrt{qt}\, X + \sqrt{q(1-t)}\, Y + \sqrt{1-q}\, G. \tag{1}$$

The variance is one, and $F_X(q,t) = F_X(q,1-t)$.

The central question is whether nonconcavity can persist arbitrarily close to the Gaussian after a fixed amount of Gaussian smoothing. Luo [6, Problem 10.2] asks, for a fixed signal fraction $q \in (0, 4/7]$ and an integer $K \geq 1$, whether there exist symmetric strongly log-concave variance-one densities f_j matching the Gaussian moments through order $2K$ such that

$$\|f_j/\varphi - 1\|_{C_b^m} \longrightarrow 0 \quad \text{for every fixed } m \geq 0,$$

while $F_{X_j}(q, \cdot)$ fails concavity for every j. Here C_b^m controls the function and its derivatives through order m uniformly, and φ is the standard Gaussian density. The question is a local form of the entropy-concavity problem of Ball, Nayar and Tkocz, with its formulation and attribution taken from Luo.

We answer this question negatively throughout $0 < q < 1/2$. More precisely, for every prescribed upper signal bound $Q < 1/2$ we construct an explicit positive radius in the uniform relative-density norm on which concavity holds for every mixing weight and every $0 < q \leq Q$, strictly for non-Gaussian inputs. The theorem allows arbitrary bounded even relative perturbations, including infinitely many Hermite modes. Its proof controls the nonlinear entropy terms relative to the actual quadratic Hermite energy, with a Gaussian smoothing margin that is uniform through the weight endpoints.

Write

$$\varphi(x) = (2\pi)^{-1/2} e^{-x^2/2}, \qquad T_a e_n = a^n e_n, \qquad N e_n = n e_n,$$

where $e_n = \mathrm{He}_n / \sqrt{n!}$ is the normalized probabilists' Hermite basis of $L^2(\varphi\, dx)$. All unqualified norms and inner products below use this Gaussian measure.

Theorem 1.1 (Gaussian-neighborhood concavity)**.** *For every $Q \in (0, 1/2)$ there is an explicit $\delta_Q > 0$ such that the following holds. Suppose g is real, even and measurable,*

$$\|g\|_\infty \leq \delta_Q, \qquad \int g\varphi = \int x^2 g\varphi = 0,$$

and X has density $\varphi(1+g)$. For every $0 < q \leq Q$, the second weight derivative extends continuously to $[0, 1]$ and

$$\partial_t^2 F_X(q, t) \leq -\frac{1}{16} \|N L(q, t)\|_2^2, \qquad L(q, t) = T_{\sqrt{qt}} g + T_{\sqrt{q(1-t)}} g. \tag{2}$$

Thus $F_X(q, \cdot)$ is strictly concave unless $g = 0$ almost everywhere. The hypothesis allows measurable bounded relative perturbations.

An explicit sufficient radius is as follows:

$$\begin{aligned} c &= \frac{1 + (2Q)^{-1/2}}{2}, & r &= \frac{c-1}{4}, & \rho &= \frac{1+2r}{c}, \\ K &= \frac{\sqrt{\rho}}{(1-\sqrt{\rho})^2}, & A &= \frac{K}{8} + 60 + \frac{1}{8r^2}, & \delta_Q &= \min\left\{\frac{1}{12}, \frac{8}{K}, \frac{1}{16A}\right\}. \end{aligned} \tag{3}$$

This explicit sufficient radius depends on Q and decreases to zero as $Q \uparrow 1/2$.

Corollary 1.2 (Answer to the near-Gaussian counterexample question)**.** *Fix $q \in (0, 1/2)$. There is no sequence of symmetric variance-one densities f_j with $\|f_j/\varphi - 1\|_\infty \to 0$ for which every $F_{X_j}(q, \cdot)$ fails concavity. The same statement holds after imposing strong log-concavity, any prescribed finite Gaussian moment matching, or convergence of all fixed-order relative-density derivatives.*

1.1 Relation to prior work and proof mechanism

Luo [6] constructs asymmetric counterexamples for every positive signal fraction and symmetric strongly log-concave counterexamples for $q > 4/7$, with arbitrarily close Gaussian-relative approximation and prescribed finite Gaussian moment matching. Theorem 1.1 and Corollary 1.2 identify a positive relative neighborhood in the low-signal symmetric regime. The additional regularity and moment-matching requirements in Problem 10.2 are compatible with the theorem: uniform relative-density proximity alone places every sufficiently late member of such a sequence within its concavity neighborhood.

Gaussian scale mixtures have entropy concavity [4]. Abu-Ajamieh [2] derives the exact symmetric endpoint-curvature identity, establishes endpoint estimates and a neighborhood for an auxiliary endpoint inequality, and proves an all-weight theorem for a specified fourth-Hermite perturbation family. Our local theorem treats a bounded infinite-dimensional relative-density neighborhood. The proof combines classical Gaussian/Hermite coordinates [1], conditional-expectation contraction, a complex-weight smoothing margin, and number-operator estimates to absorb the full nonlinear curvature error. Gaussian-channel identities have further classical antecedents [5].

1.2 Complementary restoration results

The Gaussian-relative theorem describes one neighborhood topology. The results below describe how all-weight concavity is restored under a common moment envelope or for a fixed input, and give a whole-class endpoint estimate. Together with exact moment-matched tail examples, they explain the distinct roles of proximity, symmetry and retained signal.

A complementary finite-Hermite criterion permits a common threshold even when the first non-Gaussian moment varies across the input family.

Theorem 1.3 (Finite Hermite discrepancy)**.** *Let $\mathcal{C}$ be a family of symmetric variance-one laws with a uniform absolute-moment envelope: for each integer $m \geq 1$,*

$$\sup_{X \in \mathcal{C}} \mathbb{E}|X|^m \leq B_m < \infty.$$

Put $a_n(X) = \mathbb{E}e_n(X)$. Suppose that for some even $N \geq 4$ and $\varepsilon > 0$,

$$\sum_{\substack{4 \leq n \leq N \\ n \ even}} a_n(X)^2 \geq \varepsilon^2 \qquad (X \in \mathcal{C}). \tag{4}$$

There is $q_{\mathcal{C},N,\varepsilon} > 0$ such that every $X \in \mathcal{C}$ has strict all-weight entropy concavity for every $0 < q \leq q_{\mathcal{C},N,\varepsilon}$, including continuous second derivatives at both endpoints. The family may include discrete laws.

The finite-jet estimate also quantifies the Gaussian boundary of a family with nonnegative curvature.

Corollary 1.4 (Quantitative Gaussian boundary)**.** *Let $\mathcal{C}$ be a family of symmetric variance-one laws with the common absolute-moment envelope in Theorem 1.3. For every fixed even $N \geq 4$, there are $K_N < \infty$ and $q_N > 0$ such that, for every $X \in \mathcal{C}$ and $0 < q \leq q_N$,*

$$(\exists t \in [0,1] : F_{X,tt}(q,t) \geq 0) \quad \Longrightarrow \quad \sum_{\substack{4 \leq n \leq N \\ n \text{ even}}} a_n(X)^2 q^n \leq K_N q^{N+1}. \tag{5}$$

Consequently, for each fixed polynomial P and every $A > 0$, there are $K_{P,A} < \infty$ and $q_{P,A} > 0$ such that the same antecedent, with $0 < q \le q_{P,A}$, implies

$$|\mathbb{E}P(X) - \mathbb{E}P(G)| \le K_{P,A} q^A.$$

The constants are uniform over $\mathcal{C}$ for each fixed P, A. In particular, along any sequence $X_j \in \mathcal{C}$, $q_j \downarrow 0$, for which $F_{X_j}(q_j, \cdot)$ fails concavity, every fixed polynomial moment discrepancy is $O(q_j^A)$ for every fixed $A > 0$.

The proof in Section 5 fixes the jet order before choosing the signal interval. Thus the conclusion gives a degree-by-degree necessary condition on a changing family. For symmetric variance-one log-concave laws, their common moment envelope also gives Gaussian-only limiting behavior (Corollary 5.1).

Theorem 1.5 (Eventual concavity for each input)**.** *Let X be symmetric about zero, with $\mathbb{E}X^2 = 1$ and $\mathbb{E}|X|^m < \infty$ for every integer $m \ge 1$. If X is non-Gaussian, there is $q_X > 0$ such that $\partial_t^2 F_X(q,t) < 0$ for every $0 < q \le q_X$ and every $t \in [0,1]$, with continuous second derivatives at both endpoints. If $n = 2k \ge 4$ is the first order at which the moments of X differ from Gaussian moments and κ_n is its nonzero nth cumulant, then*

$$F_X(q,t) = h(\varphi) - \frac{\kappa_n^2}{2n!} q^n \left[t^k + (1-t)^k\right]^2 + o(q^n) \tag{6}$$

in $C^2([0,1])$ as $q \downarrow 0$.

This includes every fixed symmetric variance-one log-concave input and discrete symmetric inputs with all moments finite; the threshold is attached to the law.

The three all-weight regimes have distinct useful inputs: an explicit relative-density neighborhood, a fixed non-Gaussian law, or a positive finite Hermite discrepancy under a common moment envelope. Corollary 1.4 quantifies the changing near-Gaussian boundary. Appendix B supplies exactly moment-matched exponential-tail examples distinguishing these topologies.

Theorem 1.6 (Whole-class endpoint restoration)**.** *Let X have a symmetric log-concave density and variance one. If $0 < q \le 1/20$ and p is the density of X_q, then*

$$\partial_t^2 F_X(q,0) = \partial_t^2 F_X(q,1) \le -\frac{1}{128} \mathbb{E}\left[\left(1 + (\log p)''(X_q)\right)^2\right]. \tag{7}$$

The endpoint derivatives are continuous one-sided extensions. The bound is strict unless X is Gaussian.

This last theorem uses the endpoint identity from [2], supplemented by uniform small-signal estimates. The conclusion concerns the two endpoint derivatives.

1.3 Organization of the proofs

Section 2 proves the central Gaussian-neighborhood theorem and its answer to Problem 10.2 directly. Sections 3–5 develop Gaussian-gain regularity, fixed-input eventual concavity, and the family-uniform finite-Hermite and Gaussian-boundary results. Appendix A proves the uniform endpoint estimate, using the credited endpoint identity. Appendix B constructs the exactly moment-matched exponential tails that distinguish moment proximity from bounded Gaussian-relative proximity.

2 Proof of Gaussian-neighborhood concavity

We prove Theorem 1.1. Fix $Q < 1/2$ and the constants (3), and put $\delta = \|g\|_\infty \le \delta_Q$. The conditions on g imply that its Hermite coefficients a_n vanish except at even orders $n \ge 4$. The output density from (1) is $\varphi(1+R)$, where

$$R = L + M, \qquad L = T_{\sqrt{qt}}g + T_{\sqrt{q(1-t)}}g, \tag{8}$$

and

$$M(x) = \mathbb{E}[g(G_1)g(G_2) \mid \sqrt{qt}G_1 + \sqrt{q(1-t)}G_2 + \sqrt{1-q}G_3 = x].$$

The G_i here are independent standard Gaussians. Hence $\|R\|_\infty \le 2\delta + \delta^2 \le 3\delta =: \eta \le 1/4$ on the real weight interval. The output has mass one and variance one, so

$$F_{tt} = -\langle R'', \log(1+R)\rangle - \int \frac{(R')^2}{1+R}\varphi. \tag{9}$$

Primes in this section denote weight derivatives unless otherwise specified. We justify this differentiation below after deriving the required Hilbert-space regularity.

2.1 Linear energy

For $n = 2k \ge 4$ the multiplier of a_n in L is

$$l_n = q^k S_k(t), \qquad S_k(t) = t^k + (1-t)^k.$$

Log-convexity of power sums and $S_1/S_0 = 1/2$ show $S_k \ge S_{k-1}/2$ and $S_k \ge S_{k-2}/4$; also $S_{k-2} \ge S_k$. Thus

$$|l_n'| \le n l_n, \qquad \frac{n^2}{8} l_n \le l_n'' \le n^2 l_n.$$

With $D = \|NL\|_2$, we obtain

$$\langle L'', L\rangle \ge D^2/8, \qquad \left\|N^{-1}L''\right\|_2 \le D, \qquad \|L\|_2 \le D/4. \tag{10}$$

The inverse is restricted to positive Hermite degrees. If $g \ne 0$, then $D > 0$ at every positive q and every weight.

2.2 Bilinear term and a smoothing margin

Gaussian convolution in Hermite coordinates [1] gives

$$M_n(q,z) = q^{n/2} \sum_{j+k=n} a_j a_k \binom{n}{j}^{1/2} z^{j/2}(1-z)^{k/2}. \tag{11}$$

Only even $j, k \ge 4$ occur. Let g_{abs} be the $L^2(\varphi)$ function with coefficients $|a_n|$; it need not be bounded, but $\|g_{\mathrm{abs}}\|_2 \le \delta$.

Fix $t \in [0,1]$ and $|z - t| \le r$. Set $a = |z|$, $b = |1-z|$, $P = q(a+b)$, $w = a/(a+b)$ and $\bar{P} = cq$. Our choice of constants gives

$$P \le q(1+2r) < \bar{P} < 1, \qquad P/\bar{P} \le \rho < 1, \tag{12}$$
$$\bar{P}^2 = c^2 q^2 \le c^2 Q q < q/2.$$

Absolute coefficient majorization in (11) reduces the norm to the real bilinear expression $B(P, w; g_{\mathrm{abs}}, g_{\mathrm{abs}})$. For any $h \in L^2(\varphi)$,

$$\|B(P, w; h, h)\|_2 \le \left\|T_{\sqrt{P}} h\right\|_2^2. \tag{13}$$

Indeed smooth two independent inputs separately and condition their product on their weighted Gaussian sum; L^2 contraction and independence give (13). For two inputs, the same conditioning argument bounds the bilinear map by the product of their smoothed L^2 norms. Applying this bound to differences of finite Hermite truncations gives L^2 convergence and extends the identity and inequality to L^2 inputs.

Spectral Cauchy–Schwarz gives

$$\left\|T_{\sqrt{\bar{P}}} g_{\mathrm{abs}}\right\|_2^2 = \sum a_n^2 \bar{P}^n \le \delta \left\|T_{\bar{P}} g_{\mathrm{abs}}\right\|_2.$$

By (12) and $S_k(t) \ge 2^{1-k}$,

$$\left\|T_{\bar{P}} g_{\mathrm{abs}}\right\|_2 \le \left\|T_{\sqrt{q/2}} g\right\|_2 \le \|L\|_2 / 2.$$

Consequently $\|M(q, z)\|_2 \le \delta \|L\|_2 / 2$ throughout this disk. For NM, split off the output multiplier $(P/\bar{P})^{n/2}$; its number-operator cost is at most

$$\sup_{n \ge 0} n\rho^{n/2} \le \sum_{n \ge 1} n(\sqrt{\rho})^n = K.$$

It follows that $\|NM(q, z)\|_2 \le K\delta \|L\|_2 / 2$. Strictness in (12) leaves a slightly larger disk. The same majorants, with the exponentially decaying output multiplier, show normal convergence of the coefficient series and all fixed number-operator derivatives there. Hence M is Hilbert-valued analytic. Cauchy's estimate on radius r and (10) yield

$$\|M\|_2 \le \delta D/8, \qquad \|NM\|_2 \le K\delta D/8, \qquad \|M''\|_2 \le \delta D/(4r^2). \tag{14}$$

The linear term is analytic too, since $q|z|$ and $q|1 - z|$ are below $\bar{P} < 1$. All estimates are uniform in $t \in [0, 1]$ and $0 < q \le Q$.

2.3 A bounded-function estimate

For smooth bounded real u with $Nu \in L^2(\varphi)$,

$$\left\|(u_x)^2\right\|_2 \le 3 \|u\|_\infty \|Nu\|_2. \tag{15}$$

To prove it, put $B = \int u_x^4 \varphi$. Gaussian integration by parts gives

$$\int Nu\, u(u_x)^2 \varphi = B + 2 \int u(u_x)^2 u_{xx} \varphi.$$

Cauchy–Schwarz and the Gaussian Bochner identity $\|u_{xx}\|_2 \le \|Nu\|_2$ imply $B \le 3 \|u\|_\infty \sqrt{B} \|Nu\|_2$, proving the assertion. Smooth cutoffs and Gaussian Sobolev approximation justify the operator-domain version; here positive smoothing also bounds the relevant spatial derivatives.

Set $\psi(v) = \log(1 + v) - v$. The chain rule gives

$$N\psi(R) = \psi'(R) NR - \psi''(R)(R_x)^2.$$

Using (15) and $\|R\|_\infty \le \eta \le 1/4$,

$$\|N\psi(R)\|_2 \le \left[\frac{\eta}{1 - \eta} + \frac{3\eta}{(1 - \eta)^2}\right] \|NR\|_2 \le 30\delta(1 + K\delta/8)D. \tag{16}$$

2.4 Curvature sign and regularity

The density ratio lies in $[1-\eta, 1+\eta]$. Hilbert-valued C^2 regularity of R and the bounded second derivative of $u \log u$ on this interval justify (9): scalar Taylor's formula and L^2 convergence of difference quotients give the differentiated integrals. The resulting formula is continuous to the endpoints. Spatial regularity from smoothing and the preceding number-operator estimates justify self-adjointness and the chain rule used in (16).

Drop the last, nonpositive term of (9) and decompose $\log(1+R) = L + M + \psi(R)$. The error terms satisfy

$$\begin{aligned} |\langle L'', M\rangle| &\le K\delta D^2/8, \\ |\langle L'', \psi(R)\rangle| &\le 30\delta(1 + K\delta/8)D^2 \le 60\delta D^2, \\ |\langle M'', \log(1+R)\rangle| &\le \frac{\delta D^2}{4r^2}\frac{1/4 + \delta/8}{1 - 3\delta} \le \frac{\delta D^2}{8r^2}. \end{aligned}$$

Here $\delta \le 8/K$ in the second estimate, and $\delta \le 1/12$ gives $(1/4 + \delta/8)/(1 - 3\delta) \le 25/72 < 1/2$ in the third. Therefore

$$F_{tt} \le -D^2/8 + A\delta D^2 \le -D^2/16.$$

This proves Theorem 1.1. Corollary 1.2 follows by taking $Q = q$ and observing that every sufficiently large member of the proposed sequence satisfies its relative-density hypothesis.

3 A Gaussian-gain regularity lemma

The following argument applies under the hypotheses of Theorem 1.5.

Lemma 3.1. *Let X, Y be independent copies of a symmetric variance-one random variable with all absolute moments finite. For $H = (h_1, h_2)$ sufficiently near zero put*

$$\sigma(H) = \sqrt{1 - h_1^2 - h_2^2}, \quad Z_H = h_1 X + h_2 Y + \sigma(H) G, \quad K(H) = h(Z_H).$$

Then K is smooth near zero and even in each coordinate. The function

$$J(a, b) = K(\sqrt{a}, \sqrt{b}), \qquad a, b \ge 0,$$

is smooth up to both axes on a sufficiently small positive quadrant.

Proof. Work on a closed gain disk with σ bounded away from zero. The density is

$$p_H(z) = \mathbb{E}\varphi_{\sigma(H)}(z - h_1 X - h_2 Y).$$

For each finite multiindex α, Gaussian-kernel differentiation gives

$$\frac{\partial_H^\alpha p_H(z)}{p_H(z)} = \mathbb{E}[P_{\alpha,H}(X, Y, G) \mid Z_H = z], \tag{17}$$

where $P_{\alpha,H}$ is a polynomial with coefficients bounded on the disk. This follows by replacing the standardized kernel residual by G in the joint construction. On compact z intervals differentiation under the expectation follows from finite input moments and bounded Gaussian-kernel derivatives.

Choose R with $\Pr(|X| \le R, |Y| \le R) > 0$. Restricting the kernel expectation to this event yields uniform constants

$$p_H(z) \ge c e^{-C(1+z^2)}, \qquad p_H(z) \le C', \qquad |\log p_H(z)| \le C''(1 + z^2).$$

Every formal derivative of $\int p_H \log p_H$ is a finite sum of integrals of a density derivative times $1+\log p_H$, or of p_H times products of ratios from (17). Conditional Hölder and Jensen inequalities bound these terms in each fixed $L^s(p_H)$ by finitely many moments of X, Y, G, uniformly on the disk. The logarithm is handled by its quadratic bound and moments of Z_H. Higher moments and Hölder's inequality make the integral tails over $|Z_H| > L$ vanish uniformly as $L \to \infty$. On compact spatial intervals all terms are continuous with positive denominators. Differentiation under the entropy integral to every fixed order, and continuity of the derivatives, follow by this compact-interval and uniform-tail decomposition. The same proof applies to $D(p_H\|\varphi)$ because $\log\varphi$ is quadratic.

Symmetry gives evenness in each gain. To handle square roots at the axes, if E is a smooth even function and $a = h^2$, then for $j \geq 1$

$$\frac{d^j}{da^j} E(\sqrt{a}) = \frac{1}{2^{2j-1}(j-1)!} \int_0^1 (1-u^2)^{j-1} E^{(2j)}(uh)\, du. \tag{18}$$

For $j = 1$ this is the fundamental theorem of calculus applied to $E'(h)$; induction follows by differentiation and integration by parts. The formula extends continuously to $h = 0$, with value $j!E^{(2j)}(0)/(2j)!$. Applying it in both coordinates proves all the asserted mixed-derivative extensions. Taylor formulas on the quadrant follow by integrating along segments contained in it. □

4 Eventual all-weight concavity for each input

We now prove Theorem 1.5. The Gaussian moment sequence is determinate: its even moments obey $(2m-1)!! \leq (2m)^m$, so the Carleman sum diverges. A non-Gaussian symmetric all-moment input therefore has a first differing even moment $n = 2k \geq 4$. Its cumulants of orders $3, \ldots, n-1$ vanish, and $\kappa_n = \mathbb{E}X^n - \mathbb{E}G^n \neq 0$.

Use the gain model from Lemma 3.1. For its characteristic function,

$$\mathbb{E}e^{i\xi Z_H} = e^{-\xi^2/2}[e^{h_1^2\xi^2/2}\chi_X(h_1\xi)][e^{h_2^2\xi^2/2}\chi_X(h_2\xi)].$$

Finite moment differentiation shows that the density gain jet has no nonconstant term below total degree n, and its degree-n term is

$$p_{[n]}(H, z) = \frac{\kappa_n}{n!}(h_1^n + h_2^n)\varphi(z)\,\mathrm{He}_n(z). \tag{19}$$

This identifies a finite Taylor jet, not a convergent cumulant series. All density derivatives at zero are Gaussian times polynomials, so Fourier-transform uniqueness verifies the density identification directly.

Write $D(H) = D(p_H\|\varphi)$. By Lemma 3.1, all its finite derivatives can be computed under the integral. At $p = \varphi$ the integrand jet is

$$p\log(p/\varphi) = (p-\varphi) + \frac{(p-\varphi)^2}{2\varphi} + \text{terms of at least third order in } p-\varphi.$$

This is used only for finite gain derivatives at zero. The integrated linear term vanishes by mass conservation. Three nonzero density derivatives have total gain degree at least $3n$, so through degree $2n$ the only nonlinear contribution is the square of (19). Since $\int \mathrm{He}_n^2\,\varphi = n!$,

$$D(H) = \frac{\kappa_n^2}{2n!}(h_1^n + h_2^n)^2 + R_0(H). \tag{20}$$

Here R_0 has zero gain jet through total degree $2n$. The output has mean zero and variance one, hence $K(H) = h(\varphi) - D(H)$.

In squared gains $a = h_1^2, b = h_2^2$, the even-square formula (18) identifies Taylor coefficients at the origin and gives

$$J(a,b) = h(\varphi) - \frac{\kappa_n^2}{2n!}(a^k + b^k)^2 + R(a,b).$$

The Taylor polynomial of R through total degree n vanishes. Taylor's theorem for its derivatives yields

$$\partial^\alpha R(a,b) = o\big((a+b)^{n-|\alpha|}\big), \qquad |\alpha| \le 2,$$

uniformly over positive-quadrant directions, including the axes. Substituting $a = qt, b = q(1-t)$ proves (6) in $C^2([0,1])$: two weight derivatives introduce $q^2(R_{aa} - 2R_{ab} + R_{bb})$.

For $S_k(t) = t^k + (1-t)^k$ and $k \ge 2$,

$$S_k \ge 2^{1-k}, \quad S_k'' \ge k(k-1)2^{3-k}, \quad (S_k^2)'' \ge k(k-1)2^{5-2k} =: c_k > 0.$$

For $k = 2$ the second bound uses $S_0 = 2$. The uniform little-oh remainder permits choosing $q_X > 0$ so that

$$\partial_t^2 F_X(q,t) \le -\frac{\kappa_n^2 c_k}{4n!} q^n < 0$$

for all $0 < q \le q_X$ and all weights. This proves Theorem 1.5. Log-concave laws have exponential tails, hence all absolute moments, giving the stated corollary for that class.

5 Uniform finite-moment restoration and the Gaussian boundary

We prove Theorem 1.3 by a relative finite entropy-jet estimate that allows changing lower cumulants, then extract its quantitative Gaussian boundary.

Proof. Uniform finite derivatives. On the common gain disk $|H| \le 1/2$, Chebyshev's inequality gives $\Pr(|X| \le 2) \ge 3/4$. Thus the positive kernel lower bound in Lemma 3.1 is uniform, as are the density upper bound and quadratic logarithm bound. Its conditional-polynomial derivative formulas and the common moment envelope bound every fixed-order entropy derivative uniformly. Formula (18) transfers these bounds to squared-gain derivatives on a common small quadrant. If $P_{X,N}$ denotes the degree-N Taylor polynomial of J_X, then

$$|\partial^\alpha (J_X - P_{X,N})(a,b)| \le C_N (a+b)^{N+1-|\alpha|}, \qquad |\alpha| \le 2. \tag{21}$$

Here and below constants may depend on finitely many B_m and on N. The integral Taylor formula applied to the derivatives uses uniform derivatives through degree $N+1$. Substitution $a = qt, b = q(1-t)$ bounds the remainder and each of its first two weight derivatives by a common constant times q^{N+1}.

The finite entropy jet. Only a formal finite jet at zero is used. By Gaussian-kernel differentiation, the relative-density jet is $1 + L + M$ with positive-degree Hermite coefficients

$$L_n = a_n(X)(a^{n/2} + b^{n/2}),$$

$$M_n = \sum_{\substack{j+k=n \\ j,k \ge 4 \ \text{even}}} a_j(X) a_k(X) \binom{n}{j}^{1/2} a^{j/2} b^{k/2}.$$

Each symbol $a_n(X)$ carries squared-gain degree $n/2$. Mass conservation removes the linear relative-entropy term. Hermite orthogonality makes its quadratic contribution from L alone, through total squared-gain degree N, exactly

$$\frac{1}{2}\sum_{\substack{4\le n\le N\\ n \text{ even}}} a_n(X)^2(a^{n/2}+b^{n/2})^2. \tag{22}$$

Every other nonzero monomial in this finite relative-entropy jet has at least three input factors, because LM has three, M^2 has four, and higher entropy powers have at least three. Such a term is a numerical constant times

$$\left(\prod_{\ell=1}^{m} a_{n_\ell}(X)\right) a^r b^s, \quad m\ge 3, \quad n_\ell \ge 4 \text{ even}, \quad r+s=\frac{1}{2}\sum_{\ell=1}^{m} n_\ell \le N. \tag{23}$$

Index conservation follows directly from the density jet and multiplication; integration of Gaussian polynomials only changes the numerical constants. There are finitely many such monomials, and r, s are nonnegative integers. These statements compute finite derivatives justified above.

Relative control of nonlinear terms. Set

$$E_{X,N}(q)=\sum_{\substack{4\le n\le N\\ n \text{ even}}} a_n(X)^2 q^n.$$

Order the indices in (23) as $n_1 \le n_2 \le \cdots \le n_m$. Necessarily $n_2 \le N$: otherwise already $n_2+n_3 > 2N$, contrary to index conservation. Thus the two smallest factors belong to $E_{X,N}$. For $0<q\le 1$,

$$|a_{n_1}a_{n_2}|q^{(n_1+n_2)/2} \le \frac{1}{2}\left(a_{n_1}^2 q^{n_1}+a_{n_2}^2 q^{n_2}\right).$$

The remaining coefficients have common bounds. Their factor is q^β with $\beta\ge 2$, so it is bounded above by q^2 for $0<q\le 1$. The second derivative of $t^r(1-t)^s$ is bounded on $[0,1]$ by a monomial-dependent constant. Summing the finite monomial list therefore bounds the absolute nonlinear contribution to weight curvature by $C_N q^2 E_{X,N}(q)$, uniformly through the endpoints.

Coercivity. For $n=2k$, Section 4 gives $(S_k^2)'' \ge k(k-1)2^{5-2k} > 0$. Let

$$c_N=\frac{1}{2}\min_{2\le k\le N/2} k(k-1)2^{5-2k} > 0.$$

Since entropy is Gaussian entropy minus relative entropy, equations (21)–(22) imply

$$F_{tt}(q,t) \le -c_N E_{X,N}(q) + C_N q^2 E_{X,N}(q) + C_N q^{N+1}.$$

By (4), $E_{X,N}(q)\ge \varepsilon^2 q^N$. Choose a common positive threshold below the gain-domain bound and below one so that $C_N q^2 \le c_N/4$ and $C_N q \le c_N\varepsilon^2/4$ at and below it. Then

$$F_{tt}(q,t) \le -\frac{c_N}{2}E_{X,N}(q) < 0,$$

uniformly over the entire family and all weights, proving the theorem. □

Proof of Corollary 1.4. Fix an even $N\ge 4$. The preceding proof establishes the uniform curvature inequality

$$F_{X,tt}(q,t) \le -c_N E_{X,N}(q) + C_N q^2 E_{X,N}(q) + C_N q^{N+1}$$

on a common small-signal interval under the moment envelope, with $c_N > 0$. Choose that interval smaller so that $C_N q^2 \le c_N/2$. At a point of nonnegative curvature,

$$E_{X,N}(q) \le \frac{2C_N}{c_N} q^{N+1}.$$

Every summand is nonnegative, so for even $4 \le n \le N$,

$$|a_n(X)| \le \left(\frac{2C_N}{c_N}\right)^{1/2} q^{(N+1-n)/2}. \tag{24}$$

For each fixed even $n \ge 4$ and $A > 0$, choose one even $N \ge \max\{4, n, n + 2A - 1\}$ and take $q \le 1$ as well. Then (24) is $O(q^A)$ uniformly over $\mathcal{C}$. The moments of degrees zero, one and two already have their Gaussian values, and symmetry fixes every odd moment. Each fixed polynomial is a finite linear combination of Hermite polynomials. Taking the smallest of the finitely many signal thresholds and summing their constants proves the polynomial assertion. Finally Gaussian-gain regularity makes the weight function C^2 on the closed interval in this regime. Its failure of concavity supplies an interior point of positive second derivative. Applying the same estimate at that point proves the sequence assertion. □

Corollary 5.1 (Gaussian-only limiting obstruction)**.** *Let $\mathcal{L}$ be the class of symmetric variance-one log-concave laws on $\mathbb{R}$, and let $\mathcal{U}$ be any weakly open neighborhood of the standard Gaussian law γ. There is $q_{\mathcal{U}} > 0$ for which every $X \in \mathcal{L} \setminus \mathcal{U}$ has strict all-weight entropy concavity whenever $0 < q \le q_{\mathcal{U}}$. Consequently, if $X_j \in \mathcal{L}$, $q_j \downarrow 0$, and $F_{X_j}(q_j, \cdot)$ fails concavity for every j, then X_j converges weakly to γ and in every fixed absolute moment.*

Proof. The log-concave moment estimate $\|X\|_p \le p/2$ for $p \ge 2$ [7] supplies a common moment envelope. These laws are tight; uniform higher moments preserve variance and every fixed moment under weak convergence. Log-concavity and symmetry are closed under weak limits [8], and preserved variance excludes a degenerate limit. Thus $\mathcal{L}$ is weakly compact and all fixed Hermite moments are continuous on it.

The compact set $\mathcal{L} \setminus \mathcal{U}$ excludes γ. For each of its members, some even Hermite moment of order at least four is nonzero, by Gaussian moment determinacy. The positive loci of the increasing continuous functions $\sum_{4 \le n \le N,\, n \text{ even}} a_n(X)^2$ cover this compact set. A finite subcover gives a common even N, and the positive minimum of the resulting function gives (4). Theorem 1.3 supplies the uniform threshold. If a proposed bad sequence did not converge weakly to γ, a subsequence would stay outside one weak Gaussian neighborhood and contradict this threshold. Uniform higher moments then give convergence of every fixed absolute moment. □

Fixing a first differing moment order and a positive cumulant gap is one special case of Theorem 1.3. The finite-jet criterion also permits lower moments to vary and lower cumulants to vanish. Corollary 1.4 supplies the quantitative necessary condition as the discrepancy shrinks, degree by degree.

6 Restoration regimes and moment topology

Theorem B.1 supplies a concrete distinction between the two neighborhoods. Its laws match any fixed finite set of Gaussian moments exactly and approach the Gaussian in total variation and every fixed absolute moment, while their positive-signal smoothings retain exponential tails and an unbounded Gaussian-relative density ratio. This makes the moment and relative-density regimes mathematically complementary.

A Uniform endpoint restoration for log-concave inputs

Fix a symmetric variance-one log-concave input and $0 < q \le 1/20$. Let $U = X_q$, with density p, score $\rho = (\log p)'$, and put

$$g = -\rho', \quad s = \rho + x, \quad A = \mathbb{E}(g(U) - 1)^2, \quad \delta = \mathbb{E}s(U)^2, \quad I = \mathbb{E}\rho(U)^2, \quad K = \mathbb{E}g(U)^2.$$

The letters g, δ, K in this section are score quantities, distinct from the relative-density notation in Section 2.

Gaussian convolution preserves log-concavity, and the posterior-variance identity for the Gaussian kernel gives

$$0 \le g \le M := \frac{1}{1-q}. \tag{25}$$

Indeed $(\log p)'' = -1/(1-q) + q\,\mathrm{Var}(X \mid U)/(1-q)^2$, while log-concavity supplies the opposite bound. The score is odd and $|\rho(x)| \le M|x|$.

The exact symmetric endpoint identity from [2, Theorem 3.1] is

$$F_{tt}(q, 0) = -\frac{A}{2} - \frac{\kappa_4(U)}{12}B, \qquad B = \mathbb{E}g^2 - \frac{1}{3}\mathbb{E}\rho^4. \tag{26}$$

We first justify its use without original input smoothness. Put $t = u^2$ in (1) and allow $|u| \le 1/4$. The Gaussian-kernel argument of Lemma 3.1 applies to the smooth gain functions $\sqrt{q}\sqrt{1-u^2}$, $\sqrt{q}\,u$, with fixed positive noise variance. It proves $H(u) = F_X(q, u^2)$ is C^4 and even. Hence

$$\lim_{t \downarrow 0} F_{tt}(q, t) = H^{(4)}(0)/12.$$

The fourth-order density expansion is

$$\begin{aligned} p_u &= p + c_2 u^2 + c_4 u^4 + o(u^4), \\ c_2 &= (p + xp' + p'')/2, \\ 2c_4 &= 3p/4 + 5xp'/4 + x^2 p''/4 + 3p''/2 + xp'''/2 + \mathbb{E}U^4\, p''''/12. \end{aligned}$$

Differentiating entropy gives $F_{tt}(q, 0) = -\int 2c_4 \log p - \int c_2^2/p$. The Gaussian fourth-moment portion is the normalized Ornstein–Uhlenbeck curvature $-A/2$; the remaining term is $-\kappa_4(U) \int p'''' \log p/12$. Two integrations by parts and $\mathbb{E}[\rho^2\rho'] = -\mathbb{E}\rho^4/3$ give (26). This reproduces the cited identity and its OU decomposition. Smooth cutoffs justify these integrations: the conditional-polynomial derivative bounds control spatial derivatives, g is bounded, ρ is at most linear, and all input moments are finite.

We use the standard one-dimensional Poincaré bound $C_P(X) \le 12\,\mathrm{Var}(X)$ for log-concave measures [3]. Tensorization applied to $f(\sqrt{q}X + \sqrt{1-q}G)$ gives

$$C_P(U) \le qC_P(X) + (1-q) \le 1 + 11q =: C \le 31/20 < 8/5.$$

Integration by parts gives

$$I = \mathbb{E}g = 1 + \delta, \qquad \mathbb{E}[Us(U)] = 0, \qquad K = 1 + A + 2\delta.$$

Poincaré applied to the odd function s yields $\delta \le CA$. Also $g^2 \le Mg$, and $M \le 2, I \ge 1$, so

$$A = K - 2I + 1 \le (M-2)I + 1 \le M - 1 \le 1/19 < 1/16,$$

and

$$K \le 1 + (1+2C)A \le 116/95 < 5/4. \tag{27}$$

Write κ_j for cumulants. The identity $\mathbb{E}[U^3\rho(U)] = -3$ implies

$$\kappa_4(U) = \mathbb{E}[(U^3 - 3U)s(U)].$$

Cauchy–Schwarz gives

$$|\kappa_4(U)| \le \sqrt{\mathbb{E}[(U^3-3U)^2]CA}.$$

The exact moment expression is

$$\mathbb{E}[(U^3-3U)^2] = 6 + 9\kappa_4(U) + \kappa_6(U).$$

Cumulant addition gives $\kappa_4(U) = q^2\kappa_4(X)$ and $\kappa_6(U) = q^3\kappa_6(X)$. The moment bound $\|X\|_p \le p/2$ for $p \ge 2$ [7] gives $\mathbb{E}X^4 \le 16$, $\mathbb{E}X^6 \le 729$, hence $\kappa_4(X) \le 13$ and $\kappa_6(X) \le 759$. Therefore

$$\mathbb{E}[(U^3-3U)^2] \le 6 + 117q^2 + 759q^3 < 7, \quad |\kappa_4(U)| < \frac{27}{8}\sqrt{A}.$$

For clarity we reproduce the centered correction estimate from the endpoint framework [2]. Let $D_0 = g - \rho^2$, with $\mathbb{E}D_0 = 0$. Integration by parts gives $\mathbb{E}[g\rho^2] = \mathbb{E}\rho^4/3$, so

$$B = \mathbb{E}[(g-1)D_0], \qquad \mathbb{E}D_0^2 = 2K - B, \qquad B^2 \le A(2K-B).$$

Consequently, for $A > 0$,

$$\frac{|B|}{\sqrt{A}} \le \frac{\sqrt{A} + \sqrt{A+8K}}{2} < \frac{7}{4},$$

by $A < 1/16$ and (27). Combining the two bounds gives $|\kappa_4(U)B| < (189/32)A$. Equation (26) now yields

$$F_{tt}(q,0) \le -\frac{6 - 189/32}{12}A = -A/128.$$

If $A = 0$, smooth positivity implies $(\log p)'' = -1$ everywhere, and centering and normalization give $p = \varphi$. Dividing characteristic functions by the nonzero Gaussian factor shows X is Gaussian. Otherwise the bound is strict. The symmetry in the mixing weight supplies the other endpoint, completing Theorem 1.6.

B Moment-matched exponential tails

The Gaussian-relative and weak/moment neighborhoods are distinct. A tangent-tail potential and compactly supported moment corrections give the following explicit separating family.

Theorem B.1. *For every integer $k \ge 1$ there are symmetric variance-one log-concave densities f_R, with $R \to \infty$, whose Gaussian moments agree exactly through order $2k$, and whose first unmatched cumulant satisfies*

$$\kappa_{2k+2}(f_R) \sim 2\varphi(R)R^{2k-1} > 0.$$

Their laws converge to the Gaussian in total variation and in every fixed absolute moment. Nevertheless, for each member and every $q \in (0,1]$, the density of its Gaussian smoothing has an unbounded ratio to φ and infinite chi-square divergence from φ.

Proof. Let $V_R(x) = x^2/2$ for $|x| \le R$ and $V_R(x) = R|x| - R^2/2$ otherwise, and let $b_R = (2\pi)^{-1/2} e^{-V_R}$. This potential is even, convex and C^1. For fixed integer $j \ge 0$, put $B_{j,R} = \int x^{2j}(b_R - \varphi)$ and $A_R = B_{0,R}$. Substitution $x = R + y/R$ in both tails gives

$$\begin{aligned} B_{j,R} &= 2\varphi(R) R^{2j-1} \int_0^\infty e^{-y} \left(1 + \frac{y}{R^2}\right)^{2j} \left(1 - e^{-y^2/(2R^2)}\right) dy \\ &\sim 2\varphi(R) R^{2j-3}. \end{aligned}$$

Dominated convergence after multiplying the integral by R^2 uses the majorant $e^{-y}(1+y)^{2j} y^2/2$ for $R \ge 1$; the limiting integral is one. Thus $A_R \sim 2\varphi(R) R^{-3}$. If $m_j = (2j-1)!!$, the moment deviations of the normalized base law are

$$d_{j,R} = \frac{B_{j,R} - m_j A_R}{1 + A_R} \sim 2\varphi(R) R^{2j-3} \quad (j \ge 1).$$

Choose an even smooth cutoff χ_a, equal to one on $[-a, a]$ and zero outside $[-a-1, a+1]$, and set $\psi_j(x) = \chi_a(x) x^{2j}$ for $j = 1, \dots, k$. For sufficiently large fixed a, the matrix

$$C_{ij} = \operatorname{Cov}_\gamma(G^{2i}, \psi_j(G)), \qquad 1 \le i, j \le k,$$

is invertible. Indeed it converges as $a \to \infty$ to the positive-definite covariance matrix of $G^2, \dots, G^{2k}$: a nonconstant polynomial cannot be almost surely constant under a positive Gaussian density. Freeze such controls.

For $R > a + 1$ tilt the base density by $e^{-\sum \theta_j \psi_j}$ and normalize, and let $M_R(\theta)$ be its vector of moment deviations of orders $2, \dots, 2k$. Then $M_R(0) = d_R$ and

$$(DM_R(\theta))_{ij} = -\operatorname{Cov}_{R,\theta}(X^{2i}, \psi_j(X)).$$

Boundedness of the controls and the base moment estimates imply uniform convergence of these derivatives on a fixed small parameter ball to those for the corresponding tilted Gaussian, and give uniform bounds on second derivatives. In particular $DM_R(0) \to -C$ has uniformly bounded inverse. Writing $M_R(\theta) = d_R + J_R\theta + E_R(\theta)$, the map

$$\theta \longmapsto -J_R^{-1}(d_R + E_R(\theta))$$

is a contraction of a ball of radius $C_0 \|d_R\|$ into itself for large R. Its fixed point θ_R gives exact moment matching and

$$\|\theta_R\| = O(\varphi(R) R^{2k-3}).$$

This is the usual finite-dimensional implicit-function correction, with a uniform inverse and quadratic remainder.

The corrected potential is $W_R = V_R + \sum \theta_{j,R} \psi_j$. On the fixed control support, $V_R'' = 1$ and $W_R'' \ge 1/2$ for large R; outside it the potential equals the convex V_R. The smooth controls have vanishing value and first derivative at their support boundary, so W_R is globally convex. The density is therefore log-concave, even, and already has variance one by exact matching. Its tails retain rate R.

For every fixed moment order, the correction changes its value by $O(\|\theta_R\|)$, by the bounded-control covariance formula along the parameter segment. At order $2k+2$, this is smaller by R^{-2} than the base deviation $2\varphi(R) R^{2k-1}$. The stated first-unmatched-moment asymptotic follows; because all lower moments match, that deviation is also the cumulant. The tilt converges uniformly to

one. The same tail-integral argument works for every fixed nonnegative absolute power and proves moment and total-variation convergence to the Gaussian.

Finally the corrected density has an exact positive exponential tail $c_R e^{-Rx}$ for large positive x. In its Gaussian convolution at $q < 1$, restrict the Gaussian summand to absolute value at most one. For all sufficiently large y this yields

$$p_{R,q}(y) \geq C_{R,q} e^{-Ry/\sqrt{q}}.$$

For $q = 1$ the tail identity gives the same conclusion directly. Therefore $p_{R,q}/\varphi$ is unbounded, and $\int p_{R,q}^2/\varphi$ diverges by comparison with a positive constant times $\exp(y^2/2 - 2Ry/\sqrt{q})$ on a tail. □

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References

[1] E. Abbe and L. Zheng, *A coordinate system for Gaussian networks*, arXiv:1002.2271, 2010. `https://arxiv.org/abs/1002.2271`.

[2] F. Abu-Ajamieh, *Noise-Induced Restoration of Entropy Concavity Near Gaussian Equilibrium*, arXiv:2610.22386v1, 2026. `https://arxiv.org/abs/2610.22386v1`.

[3] P. Cattiaux, A. Guillin and P.-A. Zitt, *Poincaré inequalities and hitting times*, Annales de l'Institut Henri Poincaré, Probabilités et Statistiques **49** (2013), no. 1, 95–118, doi:10.1214/11-AIHP447; see equation (4.1). `https://www.numdam.org/item/10.1214/11-AIHP447.pdf`.

[4] A. Eskenazis and L. Gavalakis, *On the entropy and information of Gaussian mixtures*, Mathematika, 2024, doi:10.1112/mtk.12246.

[5] D. Guo, Y. Wu, S. Shamai and S. Verdú, *Estimation in Gaussian noise: Properties of the minimum mean-square error*, arXiv:1004.3332, 2010. `https://arxiv.org/abs/1004.3332`.

[6] C. Luo, *Near-Gaussian counterexamples to the Ball–Nayar–Tkocz entropy concavity conjecture*, arXiv:2609.11418v2, 2026. https://arxiv.org/abs/2609.11418v2.

[7] D. Murawski, *Comparing moments of real log-concave random variables*, Bernoulli **32** (2026), no. 3, 2403–2426, https://doi.org/10.3150/25-BEJ1958; arXiv:2211.05210. https://arxiv.org/abs/2211.05210.

[8] A. Saumard and J. A. Wellner, *Log-concavity and strong log-concavity: a review*, Statistical Surveys **8** (2014), 45–114. https://arxiv.org/abs/1404.5886.