Left-Sector Mellin Certification of a Weyl-Blind Twisting Mode

Scott Schubr
Luminary Oddities

October 23, 2026

Abstract

We certify a complex Mellin invariant of a fixed connected step junction on a ray with negative real part. Positive-axis exponential flatness is propagated through an exterior sector using a classical Phragmén–Lindelöf multiplier, with explicit constants. A left-sector contour then suppresses the reciprocal-Gamma conditioning of the Mellin formula. The application proves that the original fifth logarithmic harmonic of inverse twist speed, invisible to the leading Weyl profile, is detected by the lower heat profile of the same fixed Euclidean tube. The noncommuting DD/DN aperture inverse is enclosed by arbitrary trial columns, complete residuals and infinite input tails. Sixty-seven complex nodes, a tiny endpoint calculation, every infinite quadrature tail and the stated floating arithmetic errors are included. The certified junction Mellin modulus is below $3.198 \cdot 10^{-9}$, while cancellation requires modulus above $1.18 \cdot 10^{-8}$; the fifth heat multiplier has modulus greater than $2.16 \cdot 10^{-7}$. The general propagation mechanism is classical; the contribution is its explicit connected-junction certification and this original-mode separation. A standalone replay suite accompanies the proof.

Keywords: relative heat trace; Mellin transform; sectorial resolvent; twisted tube; validated computation; inverse spectrum.

1 The fixed geometry and the precise new conclusion

Let

$$\omega = \operatorname{int}\big(\overline{(1,3)\times(0,1)} \cup \overline{(3,\sqrt{11})\times(0,2)}\big) \tag{1.1}$$

be a polar section. Its aperture $r = 3$, $0 < \vartheta < 1$, is interior. A Euclidean tube end is parametrized by

$$(a, r, \vartheta) \longmapsto (a, r\cos(\vartheta + \Theta(a)), r\sin(\vartheta + \Theta(a))), \qquad (r, \vartheta) \in \omega.$$

The speed $b(a) = \Theta'(a)$ is smooth, positive and strictly increasing outside a bounded core. Its inverse in the absolute axial coordinate is, for sufficiently large y,

$$R_\epsilon(y) = y\{1 + \epsilon\cos(5\tau\log y)\}, \qquad \tau = \frac{\pi}{\log 2}, \qquad |\epsilon| < \frac{1}{\sqrt{1+25\tau^2}}. \tag{1.2}$$

The strict bound implies $g > 0$ and $g + g' > 0$ for $g(u) = 1 + \epsilon\cos(5\tau u)$. The same fixed polar section is used through the bounded core, with a Dirichlet cap as in the preceding heat theorem [1]. Changing a bounded core-volume normalization changes only a constant in the lower heat profile below.

Theorem 1.1 (Fifth-mode detection)**.** *Under the arithmetic model specified in Section 6, the source certificate proves the following statement for the Dirichlet tube* (1.1)–(1.2)*. Put* $z_5 = 1 + 5i\tau$*. Its actual one-end heat trace has the expansion*

$$\operatorname{Tr} e^{-H_\epsilon/T^2} = \frac{T^4}{6\pi} + T^3\left\{C_{\mathrm{core}} + \epsilon\,\mathrm{Re}\big(C_5 e^{5i\tau\log T}\big)\right\} + o(T^3), \qquad |C_5| > 2.16\cdot 10^{-7}. \tag{1.3}$$

The leading profile is independent of ϵ, while the full eigenvalue multiset determines ϵ. In particular two distinct admissible amplitudes in this specified one-end family cannot be isospectral. The same conclusion holds when two identical specified ends carry the same amplitude.

The global two-profile expansion and its coefficient formula are inherited from [1, Theorem 1.1 and the connected-step formula]. Our new assertion is a certified separation of that formula from zero at z_5, using a left-sector deformation. We prove the connected resolvent and all numerical error bounds needed for that separation.

For a precise statement of the inherited reduction, let $D(t)$ be the relative heat trace of the connected step

$$\mathcal{S} = \operatorname{int}\big(\overline{(-\infty,0)\times(0,1)} \cup \overline{(0,\infty)\times(0,2)}\big)$$

against the operator with an additional Dirichlet cut at its aperture. Let

$$Z_a(t) = \sum_{n\geq 1} e^{-t(\pi n/a)^2}, \qquad E(t) = D(t) - \tfrac{1}{2}Z_1(t), \qquad \mathcal{M}_E(z) = \tfrac{1}{2}\int_0^\infty t^{(z-1)/2}E(t)\,\mathrm{d}t. \quad (1.4)$$

For $\operatorname{Re} z > 0$ these integrals converge absolutely. The global theorem in [1] gives, at every odd lattice frequency $z = 1 + ik\tau$,

$$C(z) = \frac{z}{2\sqrt{\pi}}\{AM_1(z)+3\mathcal{M}_E(z)\}, \qquad A = \tfrac{7}{2}+\sqrt{11}, \qquad M_1(z) = \frac{\Gamma((z+1)/2)\zeta(z+1)}{2\pi^{z+1}}. \quad (1.5)$$

Thus we need only exclude the particular cancellation value

$$\mathcal{M}_E(z_5) = \mathcal{T} := -\frac{A}{3}M_1(z_5). \quad (1.6)$$

The factor $2^{z+1} = -4$ at odd frequencies explains both the inherited leading blindness and the simplification (1.5).

The heat-trace predecessor settles $k = 1$; a computational companion [7] settles $k = 3$ using a right-half-plane ray. The new mechanism here is a fully justified left-sector integral with exponentially small high tail. It avoids relying on a heat Laplace integral outside its convergence half-plane. Necessary aperture identities and error bounds from the third-mode architecture are included below for a standalone certificate, with their sector constants changed explicitly. General Dirichlet-to-Neumann trace representations are classical [3].

2 Heat bounds and an explicit flatness remainder

The killed Brownian-bridge representation, or finite-box monotone limits, defines D without subtracting two infinite heat traces. The relevant identities established in [1] are

$$E \geq 0, \qquad E \text{ nonincreasing}, \qquad E(0+) = \frac{1}{36}, \qquad E(t) \leq \frac{Z_2(t) - Z_1(t)}{2}. \quad (2.1)$$

For clarity, after translating the junction vertex to $(0,0)$, its unbounded local wedge is $\{Y < 0\} \cup \{X > 0\}$. The function E counts surviving loops hitting both internal rays, confined to $-1 < Y < 1$. Scaling a loop changes only the width of this strip, proving monotonicity. The unconfined loop mass is $1/36$, equivalently the corner combination $c(3\pi/2) + c(\pi/2)$ for $c(\alpha) = (\pi^2 - \alpha^2)/(24\pi\alpha)$; see the polygon heat calculation in [2, 1].

Lemma 2.1 (Explicit loop flatness)**.** *For $t > 0$,*

$$0 \leq \tfrac{1}{36} - E(t) \leq J(t) := \frac{1}{\sqrt{\pi t}}\sum_{n\geq 1} e^{-n^2/t} \leq \frac{e^{-1/t}}{\sqrt{\pi t}(1-e^{-3/t})}. \quad (2.2)$$

Proof. A loop removed by the two horizontal walls hits $X = 0$, visits $Y = 0$, and hits at least one of $Y = 1, -1$. Discard its wedge restriction and separate the horizontal and vertical bridges. The integrated free horizontal diagonal weight of hitting zero is $1/2$. The two vertical events give, by a union bound, the integrated one-dimensional bridge weight of hitting both 0 and 1. Reflection and inclusion–exclusion give that weight as $Z_1(t) + 1/2 - (4\pi t)^{-1/2}$. The theta identity turns it into $J(t)$. Finally $n^2 \geq 1 + 3(n-1)$ bounds its sum geometrically. □

Positive integration of (2.1) gives

$$\int_0^\infty tE(t)\,\mathrm{d}t \leq \frac{1}{12}, \qquad \int_0^\infty t^2E(t)\,\mathrm{d}t \leq \frac{1}{15}, \qquad \mathcal{M}_E(1) \leq U < 0.022, \tag{2.3}$$

where a convenient explicit choice is

$$U = \frac{\log 18 + (1 - 18^{-3})^{-1}}{18\pi^2}. \tag{2.4}$$

For the last bound use $E \leq 1/36$ up to $t_0 = 4\log 18/\pi^2$ and $E \leq Z_2/2$ afterwards, then $n^2 \geq 1 + 3(n-1)$. The first two constants follow from the exact odd-integer series $Z_2 - Z_1$. These bounds will control analytic continuation, sample tails and exact-node conversion.

3 The actual connected aperture and its trace

On $L^2(0,1)$ let $A_D = -\partial_\vartheta^2$ have Dirichlet conditions at both ends, and let A_N have a Dirichlet condition at 0 and Neumann condition at 1. Their normalized real bases are

$$e_n = \sqrt{2}\sin(n\pi\vartheta), \qquad f_m = \sqrt{2}\sin((m - \tfrac{1}{2})\pi\vartheta).$$

Write $Q_D(\lambda) = (A_D + \lambda)^{1/2}$ and $Q_N(\lambda) = (A_N + \lambda)^{1/2}$, with principal roots. Zero extension $J_0 : L^2(0,1) \to L^2(0,2)$ and reflection about 1 give

$$J_0^* F(A_{(0,2),D})J_0 = \tfrac{1}{2}\{F(A_D) + F(A_N)\}. \tag{3.1}$$

It follows by splitting the extension into its even and odd parts; each part has squared norm one half. It holds for the root forms and bounded inverse powers needed below. The aperture operator is therefore

$$\mathfrak{M}(\lambda) = \frac{3Q_D(\lambda) + Q_N(\lambda)}{2}. \tag{3.2}$$

Its closed form has trace domain $H_{00}^{1/2}(0,1)$; this is the zero-extension trace space, including the aperture endpoint conditions. No commutation between the two interval roots is assumed.

Proposition 3.1 (Resolvent trace formula)**.** *For* $\operatorname{Re}\lambda > 0$ *put*

$$q(\lambda) = \int_0^\infty te^{-\lambda t}E(t)\,\mathrm{d}t, \qquad C_\lambda = \mathfrak{M}(\lambda)^{-1}.$$

Then

$$\begin{aligned} q(\lambda) &= \operatorname{Tr}(C_\lambda B_\lambda) + \operatorname{Tr}(C_\lambda A_\lambda C_\lambda A_\lambda) - \tfrac{1}{2}\operatorname{Tr}(A_D + \lambda)^{-2}, \\ A_\lambda &= \tfrac{1}{4}(3Q_D^{-1} + Q_N^{-1}), \qquad B_\lambda = \tfrac{1}{8}(3Q_D^{-3} + Q_N^{-3}). \end{aligned} \tag{3.3}$$

All products defining these traces are trace class.

Proof. For real $\lambda \geq 0$ the Poisson extension on the left half-strip is $K_D u(X) = e^{XQ_D} u$, $X < 0$, and on the right it is $K_2 u(X) = e^{-XQ_2} J_0 u$, $X > 0$. Let K combine them. Spectral integration and (3.1) give

$$K^*K = \tfrac{1}{2}(Q_D^{-1} + J_0^* Q_2^{-1} J_0) = \mathfrak{M}' = A_\lambda.$$

Integration by parts first for traces supported away from the aperture endpoints, followed by closure in $H_{00}^{1/2}$, gives

$$(L + \lambda)^{-1} - (L_{\mathrm{cut}} + \lambda)^{-1} = K \mathfrak{M}^{-1} K^*.$$

The added Poisson solution has common aperture trace and matches the two normal fluxes. The coercive closed aperture form gives uniqueness, including at the reflex endpoint. Moreover

$$\operatorname{Tr}(\mathfrak{M}^{-1/2} \mathfrak{M}' \mathfrak{M}^{-1/2}) \leq \frac{2}{3} \left\{ \frac{3}{4} \operatorname{Tr} Q_D^{-2} + \frac{1}{4} \operatorname{Tr} Q_N^{-2} \right\} < \infty,$$

because $Q_N \leq Q_D$ in form order and $\mathfrak{M} \geq 3Q_D/2$. Thus $K\mathfrak{M}^{-1/2}$ is Hilbert–Schmidt and the resolvent correction is trace class. The nonnegative heat-kernel difference and Tonelli identify its trace with $\int e^{-\lambda t} D(t)\, \mathrm{d}t = \operatorname{Tr}(\mathfrak{M}^{-1}\mathfrak{M}')$.

Here $\mathfrak{M}'$ is Hilbert–Schmidt and $\mathfrak{M}'' = -B_\lambda$ is trace class, locally uniformly in λ. Difference quotients in these ideals justify differentiation of the trace: the inverse derivative contains two Hilbert–Schmidt factors, and the other term contains $\mathfrak{M}''$. Subtracting the derivative of the narrow-strip correction $Z_1/2$ proves (3.3) on the positive axis. Holomorphic form inversion and the same locally uniform trace ideals extend it to the right half-plane. □

4 Sectorial continuation and Mellin conditioning

The root construction continues to the slit plane $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, as does (3.3). The following bound makes that continuation quantitative.

Lemma 4.1 (Sector bound)**.** *For $\lambda = \rho e^{i\psi}$, $|\psi| \leq \phi < \pi$, put $c_0 = \cos(\phi/2)$. Then*

$$|q(\rho e^{i\psi})| \leq c_0^{-3}\{q(\rho) + D_2(\rho)\}, \qquad D_2(\rho) = \sum_{n \geq 1} (\pi^2 n^2 + \rho)^{-2}, \tag{4.1}$$

$$\int_0^\infty |q(\rho e^{i\psi})|\, \mathrm{d}\rho \leq c_0^{-3}(2U + 1/6). \tag{4.2}$$

Proof. For $a > 0$, elementary scalar algebra gives

$$\operatorname{Re} \sqrt{a + \lambda} \geq c_0 \sqrt{a + \rho}, \qquad |\sqrt{a + \lambda}| \geq \sqrt{c_0}\sqrt{a + \rho}.$$

For the modulus inequality square $|a + \lambda| \geq \cos(\psi/2)(a + \rho)$; the difference of its squares is $(a - \rho)^2 \sin^2(\psi/2)$. The real-part inequality follows from $|a + \lambda| \geq a \cos\psi + \rho$ and the formula for the real part of a principal square root. Apply functional calculus separately in each root. With $M_\rho = \mathfrak{M}(\rho)$,

$$\operatorname{Re} \mathfrak{M}(\lambda) \geq c_0 M_\rho, \qquad \|M_\rho^{1/2} C_\lambda M_\rho^{1/2}\| \leq c_0^{-1}.$$

This is coercivity of a normalized closed form; its inverse is holomorphic locally on the slit plane.

Write $F(\rho) = \operatorname{Tr}(M_\rho^{-1} B_\rho) + \operatorname{Tr}[(M_\rho^{-1/2} A_\rho M_\rho^{-1/2})^2]$. Absolute values of the beta rank-one factors of B_λ are bounded by $c_0^{-3/2} B_\rho$, so the first complex trace is at most $c_0^{-5/2}$ times its real counterpart. If H is the positive operator formed from the absolute alpha factors, then $H \leq c_0^{-1/2} A_\rho$. The second complex trace is bounded by $\|H^{1/2} C_\lambda H^{1/2}\|_{\mathfrak{S}_2}^2$. Schatten Hölder with exponents $4, \infty, 4$ bounds this by

$$c_0^{-2} \operatorname{Tr}[(M_\rho^{-1/2} H M_\rho^{-1/2})^2] \leq c_0^{-3} \operatorname{Tr}[(M_\rho^{-1/2} A_\rho M_\rho^{-1/2})^2].$$

The last step uses positive eigenvalue monotonicity, not commuting roots. Finally the DD subtraction has modulus at most $c_0^{-2}D_2(\rho)/2$, and $F(\rho) = q(\rho) + D_2(\rho)/2$. This proves (4.1). Positive Tonelli gives $\int q = \int E = 2\mathcal{M}_E(1)$ and $\int D_2 = 1/6$, proving (4.2). The estimates also prove local uniform trace convergence for the continuation just used. □

4.1 Explicit exponential propagation into a left sector

Put $F(\lambda) = q(\lambda) - 1/(36\lambda^2)$, $r = |\lambda| \geq 1$. For $|\arg\lambda| \leq \phi < \pi$, write $c_\phi = \cos(\phi/2)$. The sector bound, $q(r) \leq 1/(36r^2)$, and $D_2(r) \leq 1/(4r^{3/2})$ give

$$|\lambda F(\lambda)| \leq c_\phi^{-3}\left(\frac{1}{36r} + \frac{1}{4\sqrt{r}}\right) + \frac{1}{36r} \leq A_\phi := \frac{5}{18}c_\phi^{-3} + \frac{1}{36}. \tag{4.3}$$

The DD sum bound follows by comparison with its integral on $(0,\infty)$. On the positive ray, the flatness estimate of Section 2 gives $r|F(r)| \leq 2e^{-2\sqrt{r}}$ for $r \geq 1$. For example, with $v = e^{-2\sqrt{r}}$ its coefficient is bounded by $(1-v)^{-2} + [2\sqrt{r}(1-v)]^{-1} < 49/36 + 7/12 < 2$, using $e^2 > 7$.

Proposition 4.2 (Exterior-sector flatness)**.** *For $0 \leq \theta < \phi < \pi$ and $r \geq 1$,*

$$|F(re^{-i\theta})| \leq \frac{C_\phi}{r}e^{-\kappa\sqrt{r}}, \qquad \kappa = \frac{2\sin((\phi-\theta)/2)}{\sin(\phi/2)}, \qquad C_\phi = \max(2, A_\phi e^{2/\sin(\phi/2)}). \tag{4.4}$$

In particular, $\phi = 5\pi/6$, $\theta = 3\pi/4$ permit $C_\phi < 150$ and $\kappa > 1/4$.

Proof. Work first in the upper exterior sector. Set

$$K = \frac{2e^{i(\pi/2-\phi/2)}}{\sin(\phi/2)}, \qquad G(\lambda) = \lambda F(\lambda)e^{K\sqrt{\lambda}}.$$

On its two bounding rays, $|G| \leq 2$ and $|G| \leq A_\phi$, respectively; on its unit connecting arc, $|G| \leq A_\phi e^{|K|}$. Everywhere $|G| \leq A_\phi e^{|K|\sqrt{r}}$ by (4.3). For $\delta > 0$ apply the maximum principle on a truncated exterior sector to $G(\lambda)e^{-\delta e^{-i\phi/2}\lambda}$. The last multiplier has modulus at most one, and on a large outer arc its decay at least $e^{-\delta\cos(\phi/2)r}$ dominates the displayed square-root growth. Let the outer radius tend to infinity, then $\delta \downarrow 0$. This gives $|G| \leq C_\phi$. Evaluating the real part of $Ke^{i\theta/2}$ gives the sine quotient; complex conjugation gives the lower sector.

For the stated constants, $\cos(5\pi/12) > 1/4$ and $\sin(5\pi/12) > 24/25$, hence $A_\phi < 641/36$, $|K| < 25/12$ and $e^{25/12} < 33/4$, so $C_\phi < 150$. Also $\kappa = 2\sin(\pi/24)/\sin(5\pi/12) > 1/4$. The elementary rational/Taylor comparisons are checked in the driver. □

The angular interpolation formula and its multiplier are prior art: Rogers [8, Corollary 8.2 and its proof] gives this precise sine-weighted propagation with exponent 1/2 and positive-ray rate2. We have written the exterior-sector argument to retain its unit-arc constant explicitly. Related analytic propagation appears in Coulhon–Sikora [9]. Here the essential input is the actual connected trace and its computable flat remainder.

4.2 The full Mellin contour

Put $b = 5\pi/(2\log 2)$, $s = 1 + ib$, $z_5 = 2s - 1$, $c = -1 - i$. Absolute Fubini initially on the positive axis gives

$$\mathcal{M}_E(z_5) = \frac{1}{2\Gamma(2-s)}\int_0^\infty \lambda^{1-s}q(\lambda)\,\mathrm{d}\lambda.$$

The sector bound is $O(1)$ at zero and $O(r^{-3/2})$ at infinity, uniformly on each closed sector strictly inside the slit plane. Thus the small and large circular arcs vanish with orders $O(r)$

and $O(r^{-1/2})$, including the bounded angular frequency factors. Contour deformation through the lower sector proves

$$\mathcal{M}_E(z_5) = PI, \qquad I = \int_0^\infty r^{1-s} q(cr)\,\mathrm{d}r, \qquad P = \frac{c^{2-s}}{2\Gamma(2-s)}. \tag{4.5}$$

All powers use principal logarithms. This is continuation of the resolvent trace. Gamma reflection gives

$$|P| = \sqrt{2}\left(\frac{e^{\pi b} - e^{-\pi b}}{8\pi b}\right)^{1/2} e^{-3\pi b/4} < 1.144 \cdot 10^{-5}. \tag{4.6}$$

The small prefactor is the decisive conditioning improvement.

5 A finite certificate for the infinite aperture

We give the finite bounds evaluated at $\lambda = (-1-i)r$. Let $\rho = \sqrt{2}r$, $D = (A_D + \rho)^{1/2}$, $d_n = \sqrt{\pi^2 n^2 + \rho}$, and $n_m = \sqrt{\pi^2(m - 1/2)^2 + \rho}$. With $c_0 = \cos(3\pi/8) > 3/8$, the scalar inequalities above give

$$\operatorname{Re}\mathfrak{M}(\lambda) \geq \frac{3c_0}{2} D, \quad \|D^{1/2} C_\lambda D^{1/2}\| \leq \chi := \frac{16}{9}, \quad |Q_D(\lambda)^{-1}| \leq \eta D^{-1}, \quad \eta := \frac{5}{3}. \tag{5.1}$$

The analogous inverse-root inequality holds in the DN basis. In code a downward approximation to ρ only weakens these bounds.

5.1 Root entries with a leading infinite-tail correction

Direct integration gives the exact overlap

$$O_{nm} = \langle e_n, f_m \rangle = \frac{(-1)^{n+m} 2n}{\pi(n^2 - u_m^2)}, \qquad u_m = m - \tfrac{1}{2}. \tag{5.2}$$

For $n, l \leq N$, the DN root entry is $\sum_{m \geq 1} O_{nm} O_{lm} \sqrt{\pi^2 u_m^2 + \lambda}$. Choose $M + 1/2 \geq 2N$ and put $\xi = N^2/(M + 1/2)^2$. The omitted entry after M terms has the explicit center

$$L_{nl} = (-1)^{n+l} \frac{2nl}{\pi M^2} \tag{5.3}$$

and an absolute remainder no greater than

$$\frac{4nl}{\pi}\left\{\frac{1}{8M^4} + \frac{1}{2M^5} + \frac{n^2 + l^2 + |\lambda|/\pi^2}{4M^4(1-\xi)^2}\right\}. \tag{5.4}$$

To see this, the leading summand is $(-1)^{n+l}(4nl/\pi)u_m^{-3}$. Convex midpoint quadrature gives

$$0 \leq \frac{1}{2M^2} - \sum_{m>M} u_m^{-3} \leq \frac{1}{8M^4} + \frac{1}{2M^5}, \qquad \sum_{m>M} u_m^{-5} \leq \frac{1}{4M^4}.$$

For example bound each midpoint error of x^{-3} on a unit cell by the maximum of its second derivative divided by 24, and bound the resulting decreasing sum by its first term plus its integral. The normalized root differs from 1 by at most $|\lambda|/(\pi^2 u_m^2)$, while the two inverse overlap denominators differ from 1 by at most $(n^2 + l^2)/(u_m^2(1-\xi)^2)$. Combining them proves (5.4). This is a bound on the actual omitted entry, not an extrapolation from successive finite sections.

5.2 Arbitrary trial columns and their full residuals

Use the input list $g_i = (e_1, \dots, e_K, f_1, \dots, f_K)$. For arbitrary finite DD trial columns v_i, define the bilinear matrices

$$\begin{aligned} X_{ij} &= g_i^T C_\lambda g_j, & B_{ij} &= g_i^T v_j + v_i^T g_j - v_i^T \mathfrak{M} v_j, \\ r_i &= g_i - \mathfrak{M} v_i, & |X_{ij} - B_{ij}| &\le \chi \sqrt{w_i w_j}, \qquad w_i = \langle r_i, D^{-1} r_i \rangle. \end{aligned} \tag{5.5}$$

Both roots, hence the inverse, are complex symmetric under transpose in these real bases. Expansion gives $X_{ij} - B_{ij} = r_i^T C_\lambda r_j$; coercivity and Cauchy–Schwarz prove (5.5). The disks are two-sided even on the diagonal. A numerical linear solve merely proposes v_i.

All residual quantities can be enclosed using finite root entries. For $v = \sum_{n \le N} v_n e_n$ and $\rho_i = \|r_i\|^2$,

$$w_i \le \sum_{n \le N} \frac{|\langle e_n, r_i \rangle|^2}{d_n} + \frac{\rho_i - \sum_{n \le N} |\langle e_n, r_i \rangle|^2}{d_{N+1}}. \tag{5.6}$$

When evaluated by intervals, the subtracted sum is bounded below and ρ_i above. To obtain ρ_i, use

$$\|\mathfrak{M} v\|^2 = \tfrac{1}{4}\{9\|Q_D v\|^2 + \|Q_N v\|^2 + 6\,\mathrm{Re}\langle Q_D v, Q_N v \rangle\}. \tag{5.7}$$

Here $\|Q_D v\|^2 = \sum |q_n|^2 |v_n|^2$, with $q_n = \sqrt{\pi^2 n^2 + \lambda}$. The distinct DN norm is

$$\begin{aligned} \|Q_N v\|^2 &= \int_0^1 |v'|^2 \, \mathrm{d}\vartheta - r\|v\|^2 + \sum_{m \ge 1} J(\pi^2 u_m^2) |f_m^T v|^2, \\ J(x) &= \sqrt{(x-r)^2 + r^2} - (x - r). \end{aligned} \tag{5.8}$$

The signed baseline is retained with an absolute rounding bound. Compute $J(x)$ as $r^2/(\sqrt{(x-r)^2 + r^2} + x - r)$ if $x \ge r$; otherwise the displayed difference adds positive terms. For $x \ge 2r$, $J(x) \le r^2/x$. If $S_v = \sum_{n \le N} n|v_n|$ and $a = M - 1/2$, the omitted correction is

$$\frac{64 r^2 S_v^2}{45 \pi^4 a^5}. \tag{5.9}$$

Indeed $|O_{nm}| \le 8n/(3\pi u_m^2)$ and $\sum_{m > M} u_m^{-6} \le 1/(5a^5)$. The cutoff guard ensures $\pi^2 (M + 1/2)^2 > 2r$. The cross term in (5.7) uses only the finite DN root block. For a DN input, $f_m^T \mathfrak{M} v = (3/2) f_m^T Q_D v + (\sqrt{\pi^2 u_m^2 + \lambda}/2) f_m^T v$. These identities enclose the full residual, including its part outside the DD trial span.

5.3 Absolute input-factor tails of cubic order

Assign the combined input coefficients

$$\alpha(e_n) = \frac{3}{4 q_n}, \quad \alpha(f_m) = \frac{1}{4 r_m}, \qquad \beta(e_n) = \frac{3}{8 q_n^3}, \quad \beta(f_m) = \frac{1}{8 r_m^3}, \quad r_m = \sqrt{\pi^2 u_m^2 + \lambda}.$$

The truncated trace is

$$F_K = \sum_{i \le 2K} \beta_i X_{ii} + \sum_{i,j \le 2K} \alpha_i \alpha_j X_{ij} X_{ji}. \tag{5.10}$$

These are complex products, not absolute squares. Absolute values enter only the following error bounds. Let $H = \sum_i |\alpha_i| g_i g_i^*$ and $H_>$ its omitted input factors. Set

$$t_K = \frac{3}{4 d_{K+1}^2} + \frac{1}{4 n_{K+1}^2}, \qquad H_> \le \eta t_K D. \tag{5.11}$$

The omitted beta trace is at most

$$B_> = \frac{\chi\eta^3}{8\pi^4}\left\{\frac{1}{K^3} + \frac{1}{3(K-1/2)^3}\right\}. \tag{5.12}$$

This follows from C_λ's normalized norm χ and the separate DD/DN fourth-power spectral sums.

For the high–high squared trace, Schatten Hölder and (5.11) give

$$HH \le \frac{\chi^2\eta^2 t_K}{\pi^2}\left\{\frac{3}{4K} + \frac{1}{4(K-1/2)}\right\}. \tag{5.13}$$

Indeed the positive normalized $H_>$ has norm at most ηt_K; its squared trace is bounded by norm times trace. No interval-root commutation is used. For each low input, let $T_i = \langle v_i, H_> v_i\rangle$. Then

$$LH \le \sum_{i\le 2K} |\alpha_i|\left\{\sqrt{T_i} + \chi\sqrt{\eta t_K w_i}\right\}^2. \tag{5.14}$$

This follows by applying $H_>^{1/2}$ to $C_\lambda g_i = v_i + C_\lambda r_i$ and using normalized coercivity. The complete squared-trace tail is at most $2LH + HH$. The finite sums defining T_i use DD coefficients above K and DN coefficients from $K+1$ through M. Their remaining DN sum is at most

$$\frac{4\eta S_{v_i}^2}{9\pi^3(M-1/2)^4}. \tag{5.15}$$

For (5.15), multiply the squared overlap bound by $\eta/(4\pi u_m)$ and sum u_m^{-5}. Thus $B_> + 2LH + HH$ is a fully finite-data bound on the omitted trace. The final DD subtraction in (3.3) is summed through L, with remaining modulus at most $2/(3\pi^4 L^3)$ after its factor $1/2$, since $|\pi^2 j^2 + \lambda| \ge \pi^2 j^2/2$ in this guarded tail.

6 Arithmetic enclosure and a runnable certificate

The computation uses binary64 round-to-nearest basic arithmetic and correctly rounded real square roots. It assumes those properties for the pinned NumPy implementation; it is not a formal verification of hardware or the Python runtime. All reported runs must satisfy the finite/range checks in the supplied source. Scalar integer powers of floating values are explicit repeated multiplications. Complex reciprocal centers use real divisions by $x^2 + y^2$.

Put $u = 2^{-53}$ and $\gamma_k = ku/(1-ku)$. A complex entrywise ball $(\widehat{A}, R_A)$ encloses A when $|A - \widehat{A}| \le R_A$. The source uses the following rules, with every positive upper expression additionally inflated by $1 + \gamma_{64}$ and rounded with `nextafter` toward $+\infty$. An absolute 10^{-250} floor covers negligible underflow terms in the checked computation.

Operation	Propagated bound before final outward inflation
Complex sum	$R_A + R_B + \gamma_8(\lvert\widehat{A}\rvert + \lvert\widehat{B}\rvert)$
Entry product	$\lvert\widehat{A}\rvert R_B + R_A\lvert\widehat{B}\rvert + R_A R_B + \gamma_{16}\lvert\widehat{A}\rvert\lvert\widehat{B}\rvert$
Length-k matrix product	$\lvert\widehat{A}\rvert R_B + R_A\lvert\widehat{B}\rvert + R_A R_B + \gamma_{8k+32}\lvert\widehat{A}\rvert\lvert\widehat{B}\rvert$
Positive length-k dot	Its computed value divided by $1 - \gamma_{2k+8}$
Reciprocal of z	$R/(\lvert\widehat{z}\rvert(\lvert\widehat{z}\rvert - R))$ plus a γ_{64} center allowance, using lower moduli
Squared modulus	$2\lvert\widehat{z}\rvert R + R^2$ plus a γ_{16} center allowance

All absolute center factors in this table are themselves bounded upwards. The matrix-product bound is componentwise. The familiar product-of-$(1+\delta)$ estimate proves the γ rules; regrouping a conventional sum of k real products has at most $2k$ elementary factors on any path. A complex

product and its two real sums fit the conservative $8k + 32$ allowance. Fused operations do not enlarge these bounds.

Proof-relevant products call NumPy 2.3.5 `einsum('ik,kj->ij',...,optimize=False)`. Its pinned source dispatches to conventional C sum-products [5]. A generic BLAS interface would not justify that componentwise model [6]; BLAS is used only to propose trial columns. Their accuracy is immaterial by (5.5).

The overlap denominators are exactly representable quarter-integers in the checked index range. The stored binary64 π value is bounded using rational Machin series. Principal roots are evaluated using the larger component:

$$h_0 = \sqrt{a^2 + b_0^2}, \qquad \ell = \sqrt{(h_0 + |a|)/2}.$$

For $a \geq 0$ take $(\operatorname{Re}\sqrt{z}, \operatorname{Im}\sqrt{z}) = (\ell, b_0/(2\ell))$; for $a < 0$ take $(|b_0|/(2\ell), \operatorname{sgn}(b_0)\ell)$. For $z = x - r - ir$, $(x + r)/|z| < 3$ uniformly for $x \geq 0$. This bounds the conditioning of the signed input subtraction. Every used root formula then has only positive sums, square roots and division by the larger component. The implementation allows γ_{2048} for root inputs, γ_{8192} for squared root moduli and the stable DN norm correction, and $\gamma_{8M+32768}$ for finite DN root entries. These margins exceed the conditioned elementary-operation budgets of these formulas. Unused vectorized branches are finite on the checked domain. Insertion of (5.3) and evaluation of (5.4) have separate outward allowances. The number reserves exceed the finite-operation counts in these positive formulas.

For the final scalar assembly the source uses outward decimal intervals, rational bounds for π and logarithms, and sine/cosine Taylor polynomials with explicit remainder. Floating node centers and radii are converted by exact hexadecimal or exact decimal expansion; a nearest-decimal display is never treated as an outward radius. The public driver can regenerate its inputs, checks every acceptance inequality, and returns the actual bound; it does not accept a saved success label in place of recomputation.

For this certificate $N = 512$, $K = 256$, $M = 4096$, and $L = 4096$. Source guards require $10^{-12} \leq r \leq 10^4$, $N \leq 2048$, $K \leq 256$, $M < 2^{20}$, $M + 1/2 \geq 2N$, finite ball bounds and trial real/imaginary components of modulus at most1. Exceptions on overflow, invalid operations or zero division stop the run. The finite operation count is below 10^{20} per node. True root and overlap denominators used in proof quantities stay bounded away from zero on this domain; underflow-sized rounding perturbations remain far below the 10^{-250} floor. This is a stated machine-arithmetic model, not verification of the hardware implementation.

7 Complete analytic quadrature and target separation

Use $h = 1/5$, $d = 7/10$, and $f(x) = e^{(2-s)x} q(ce^x)$. It is analytic on $|\operatorname{Im} x| \leq d$, because $3\pi/4 + d < \pi$. Its two boundary integrals obey

$$M_y \leq \frac{e^{by}}{\sqrt{2}\cos^3((y - 3\pi/4)/2)}(2U + 1/6), \qquad y = \pm d, \qquad 2U + 1/6 < .211.$$

Here $\rho = \sqrt{2}e^x$ in (4.2); the extra factor e^{by} comes from $e^{(2-s)(x+iy)}$. Classical analytic-strip trapezoid error [4, Theorem5.1 and the extension following (5.15)] therefore gives

$$\left| I - h \sum_{j \in \mathbb{Z}} f(jh) \right| \leq \frac{M_d + M_{-d}}{e^{2\pi d/h} - 1}. \tag{7.1}$$

One may prove it directly by shifting the Fourier integral to each boundary, bounding its transforms by $M_{\pm d} e^{-d|\xi|}$, and summing the nonzero Poisson frequencies. The decay established above justifies the limits.

Only $j = -24, \dots, 42$ are aperture calculations. For $j \le -25$, replace $q(ce^{jh})$ by its real endpoint q_0. The spectral gap $\pi^2/4$ and $E \le (Z_2 - Z_1)/2$ give

$$|q'(\lambda)| \le \int_0^\infty t^2 e^{.01t} E(t)\,\mathrm{d}t \le \frac{1}{15}(1 - 4(.01)/\pi^2)^{-3} < \frac{1}{14}, \qquad |\lambda| \le .01.$$

The unweighted moment $1/15$ follows from $(64-1)\zeta(6)/\pi^6$. Since $\sqrt{2}e^{-5} < .01$, the low sample center and its error are

$$L = q_0 \frac{he^{-25(2-s)h}}{1 - e^{-(2-s)h}}, \qquad e_L \le \frac{\sqrt{2}he^{-10}}{14(1 - e^{-2h})}. \tag{7.2}$$

To reproduce q_0 without another package, calculate the same left solver at the exact binary input $r_0 = 2^{-39}$. Its disk's real projection expanded by $3r_0/28$ encloses q_0, because $\sqrt{2} < 3/2$. It lies inside the retained conservative box

$$.00550048488672214 \le q_0 \le .00585907094669114.$$

The full box is retained as a rectangle in L, not counted twice as an extra node error.

For $j \ge 43$ use $q(cr) = 1/(36c^2r^2) + F(cr)$. The exact high sample center and the complete remainder bound are

$$H = \frac{he^{-43sh}}{36c^2(1 - e^{-sh})}, \qquad e_H \le 150e^{-\sqrt{R}/4}\left(h + \frac{8}{\sqrt{R}}\right), \quad R = e^{43h}. \tag{7.3}$$

Indeed $|c| \ge 1$ permits weakening (4.4) to $|rF(cr)| \le 150e^{-\sqrt{r}/4}$; compare the decreasing logarithmic sample tail to its integral, which is at most $8e^{-\sqrt{R}/4}/\sqrt{R}$.

Each actual solver input $\widehat{r}_j$ is exact binary, while $r_j = e^{jh}$ and its weight are directed intervals. For input conversion use a radius-$r/8$ Cauchy disk about cr. It stays in $|\arg\lambda| < 2.5$, $|\lambda| > r$. The sector bound and $q(r) \le q_0 < .006$, $D_2(r) \le 1/90$ bound $|q| < 1$ there, because $\cos(1.25) > .3$. Thus $|q'(cr)| < 8/r$, and radial displacement is bounded safely by

$$20|r_j - \widehat{r}_j| / \min(r_j, \widehat{r}_j).$$

Every center and radius is serialized exactly. Directed assembly combines (7.1)–(7.3), all67 node radii and these input displacements. The scalar/analytic error is below $4.310 \cdot 10^{-11}$ in $\mathcal{M}_E$ units, including the width of the retained q_0 rectangle.

For a stronger comparison without a complex Gamma phase, reflection $\Gamma(s)\Gamma(2-s) = \pi b/\sinh(\pi b)$ yields

$$\frac{\mathcal{T}}{P} = -\frac{Ab\pi\zeta(2+2ib)}{3\sinh(\pi b)\pi^{2+2ib}c^{1-ib}}. \tag{7.4}$$

The first1024 zeta terms are evaluated with directed elementary phases; the omitted complex modulus is at most $1/1024$. After the prefactors this adds a disk radius below $1.545 \cdot 10^{-6}$ in I units. Separately the Euler product gives $|\zeta(2+2ib)| \ge \zeta(4)/\zeta(2) = \pi^2/15$. Gamma reflection and rational elementary bounds then prove

$$|\mathcal{T}| \ge A|\Gamma(1+ib)|/90 > 1.18 \cdot 10^{-8}. \tag{7.5}$$

8 The certificate and the spectral conclusion

The supplied directed calculation gives the following outward bounds.

Quantity	Certified bound
Unscaled node-radius contribution	$< 7.297728 \cdot 10^{-5}$
Unscaled input-conversion error	$< 3.008 \cdot 10^{-11}$
Unscaled analytic error	$< 3.621127 \cdot 10^{-6}$
Complete additional I disk radius	$< 7.659843 \cdot 10^{-5}$
$\lvert\mathcal{M}_E(z_5)\rvert$	$< 3.198 \cdot 10^{-9}$
$\operatorname{dist}(I, \mathcal{T}/P)$	$> .00118496509035011988$
$\lvert C_5\rvert$	$> 2.16 \cdot 10^{-7}$

The center rectangle has real part between $-.000102118991$ and $-.000101837801$, and imaginary part between .000175226088 and .000175365786. These displayed bounds are weakened outwards from the full endpoints in `certificate.json`. Both the modulus comparison (7.5) and the complete complex separation prove $\mathcal{M}_E(z_5) \neq \mathcal{T}$. The latter implies

$$|C_5| = \frac{3|z_5|}{2\sqrt{\pi}}|P|\,|I - \mathcal{T}/P| > 16|P|\,|I - \mathcal{T}/P| > 2.1686290376151944 \cdot 10^{-7}.$$

The simple factor16 is valid since $b > 11$ and $\pi < 4$.

Finally the inherited actual-tube theorem gives (1.3). Subtract the leading term, divide by T^3, put $u = \log T$, average against $e^{-5i\tau u}$ over a period and let the starting point tend to infinity. The bounded-core constant vanishes in that Fourier coefficient, and the remainder tends to zero. The resulting nonzero multiplier recovers ϵ. For two specified identical ends both profiles double, leaving the same determination. This proves Theorem 1.1 in precisely its stated family.

9 Replay, prior work and limits

Build `main.tex` twice with a standard LaTeX installation; the bibliography is inline and there are no external figures. With Python3.12, run `python -B certify.py` for fresh directed assembly of the supplied disks. To regenerate every disk, install NumPy2.3.5 and run `python -B certify.py -recompute -output reproduced.json`. Expect minutes. Do not disable assertions with optimization flags. The second command performs all68 actual aperture calculations. No private files, review tokens, sibling package, network access or SciPy are needed. A trial solve's harmless last-bit changes may change centers, but every accepted result must pass the enclosure and strict separation checks.

The heat bridge and fixed geometry are inherited from [1]; the noncommuting residual architecture builds on [7]. Classical gluing, sector propagation and analytic-strip quadrature are credited at their points of use. The contribution here is the explicit exterior-sector flatness bound for this trace, the negative- real-part aperture certificate, and the original fifth-mode conclusion. Independent internal calculations reproduced the full node collection, scalar assembly and a separately authored192-bit Arb/Acb comparison using direct Gamma/Zeta. Such checks are internal AI-assisted analytic and code verification.

The statement concerns ideal complete-spectrum recovery of the original fifth mode in the fixed one-end or specified identical-two-end model. The result continues the same spectral-observability family.

References

[1] S. Schubr, *Heat-Trace Recovery of a Weyl-Blind Twisting Mode*, first edition, Luminary Oddities, 2026. `https://luminaryoddities.com/researcharchive/heat-trace-recovery-weyl-blind-twisting-mode/v/1`.

[2] M. van den Berg and S. Srisatkunarajah, Heat equation for a region in $\mathbb{R}^2$ with a polygonal boundary, *J. London Math. Soc.* (2) **37** (1988), 119–127. doi:10.1112/jlms/s2-37.121.119.

[3] J. Behrndt, F. Gesztesy and S. Nakamura, *Spectral shift functions and Dirichlet-to-Neumann maps*, arXiv:1609.08292. `https://arxiv.org/abs/1609.08292`.

[4] L. N. Trefethen and J. A. C. Weideman, The exponentially convergent trapezoidal rule, *SIAM Review* **56** (2014), 385–458. doi:10.1137/130932132.

[5] NumPy developers, NumPy 2.3.5 source: `einsumfunc.py`, `einsum_sumprod.c.src`, and `loops_umath_fp.dispatch.c.src`. `https://github.com/numpy/numpy/tree/v2.3.5/numpy/_core`.

[6] E. Anderson et al., *LAPACK Users' Guide*, third edition, SIAM, 1999, "Error Bounds for Fast Level 3 BLAS." `https://netlib.org/lapack/lug/node108.html`.

[7] S. Schubr, *Certified Resolvent Detection of a Third Weyl-Blind Twisting Mode*, prepared companion manuscript, Luminary Oddities, 2026.

[8] L. G. Rogers, *Estimates for the resolvent kernel of the Laplacian on p.c.f. self similar fractals and blowups*, arXiv:1009.5334, Section8, Corollary8.2 and its proof. `https://arxiv.org/abs/1009.5334`.

[9] T. Coulhon and A. Sikora, Gaussian heat kernel upper bounds via the Phragmén–Lindelöf theorem, *Proc. London Math. Soc.* **96** (2008), 507–544. doi:10.1112/plms/pdm050.