Leading bias and positive correction of heat-kernel Sinkhorn metrics

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Luminary Oddities

Abstract

We study how heat-kernel Sinkhorn regularization changes the local transport metric on smooth positive probability densities on a flat torus. The second-order bias is minus one forty-eighth of the Wasserstein Hessian of Fisher information, and has both signs at one explicit density. A same-bandwidth correction removes this bias, remains compact and positive on the full tangent complement, and preserves the original completed form space and square-root range. Its sharp $H^r \to L^2$ error is $h^{1+r/2}$ for $0 \le r \le 6$, including fourth-order accuracy on H^6 inputs. We also show why positivity cannot be assumed for raw two-bandwidth Richardson extrapolation: at explicit bandwidth-dependent densities, even arbitrarily near uniform, no finite quadratic-form bound compares the twice-bandwidth tensor with the original one. The extrapolated tensor therefore has a smooth negative direction. A category argument also gives fixed smooth densities with this failure along any prescribed sequence tending to zero, residually in a specified near-uniform smooth class. In contrast, classical odd hyperpower inverse polynomials admit compact positive realizations with sharp smooth-input orders $6, 10, 14, \ldots$; the first gives sixth-order $H^{10} \to L^2$ accuracy. Full L^2 accuracy remains first order throughout. The operator estimates include every fixed parameter-derivative order and pass to intrinsic geometry on fixed finite-dimensional density families. Analytic appendices give continuum residual certificates and a finite construction with logarithmic per-coordinate resolution for prescribed small analytic families under explicit coefficient and tail inputs.

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1 Introduction and statements

A local transport metric assigns a squared length to an infinitesimal change of a probability density. Heat-kernel Sinkhorn regularization replaces the inverse weighted Laplacian defining the limiting metric by a compact normalized-kernel tensor. The approximation error is not only a scalar rescaling: at a variable density, one direction can be too long and another too short. We identify that distortion and remove its leading term while preserving positivity on the complete tangent complement.

The objects and conventions are fixed in the next subsection. The original tensor is T_h, the limiting tensor is G, and the preferred correction is $\widehat{C}_h$. Theorems 1.1 and 1.2 are population operator statements for smooth strictly positive periodic densities. They do not change the definition of the original Sinkhorn tensor. The bias formula identifies a second-order differential defect, and the correction combines a scalar filter with its density-dependent sandwich.

Two distinctions govern the result. First, fourth-order accuracy is a smooth-input estimate $H^6 \to L^2$, not a fourth-order full operator-norm claim: the sharp full L^2 error remains first order. Second, strict positivity of a compact tensor has no uniform positive L^2 floor. Instead our two-sided

form comparison preserves the original completed form space and, separately, its square-root range. An identity-floor correction would alter that high-frequency structure. The Simpson comparison included below makes this distinction explicit.

Raw extrapolation across separately normalized bandwidths has a structural obstruction. Section 8 gives an explicit smooth density f_h for each sufficiently small h such that $T_{2h}[f_h] \le CT_h[f_h]$ fails for every finite C. Consequently $(4T_h - T_{2h})/3$ has a smooth negative direction, even at densities arbitrarily near uniform. The density is the same within each pair of bandwidths but depends on the matching h. Exact Gaussian operator ranges, rather than a sign calculation of the leading bias, prove this failure. A finite-polynomial extension and Baire category then prove Theorem 8.2: in the stated smooth near-uniform class, a residual set of fixed densities fails every finite cross-band bound at every bandwidth of any prescribed countable sequence. The residual set may depend on that sequence. This rules out eventual full-tangent positivity when the sequence tends to zero.

There is also a systematic higher-order construction. With the local operator S defined below, the first new tensor is

$$\mathcal{C}_{h,3} = 3T_h - 6T_hST_h + 4T_hST_hST_h.$$

It is compact, positive on the full tangent complement and uniformly form-comparable to T_h, while its error is $O(h^6) : H^{10} \to L^2$. Section 9 proves a hierarchy of sharp orders $6, 10, 14, \ldots$ by realizing classical hyperpower inverse polynomials [20] in this infinite-dimensional setting. The inverse-polynomial algebra is inherited; the domain, moving-kernel, positivity and parameter-uniform Sobolev estimates are established here. This does not demonstrate a numerical speed advantage over a limiting weighted-Poisson solve.

The finite-family corollaries are consequences of the operator estimates and their parameter derivatives: they concern intrinsic geometry in a fixed compact parameter region. The computational appendices are secondary and separate. Their residual certificates and small-analytic finite construction have explicit inputs; they are not empirical, runtime, or arbitrary-density discretization guarantees.

The exact identification with Sinkhorn local geometry comes from [1]. Symmetric normalization and its spectral accuracy have important antecedents in diffusion maps [2]. Higher-order potential expansions are also available [3]. We give the cubic scale coefficient and its required remainder in every fixed dimension.

Richardson correction and Fisher-information bias for scalar Sinkhorn divergences already appear in [4]. Symmetric-product and modified-generator algebra are classical [5, 6, 7]; so are the Simpson and Richardson filters used here. Our contribution is the explicit variable-density tensor coefficient, its uniform Sobolev remainder, and the density-dependent sandwich that makes the corrected tensor compact, positive and uniformly form-comparable. Generator bias under other normalizations is studied in [8]. Fisher-information nonconvexity is also known [9]; our sign example certifies the specific coefficient in this heat-kernel normalization.

The exact Sinkhorn geometry in [1] includes regularization nonmonotonicity in other models. Finite-support tensor and curvature limits are studied in [10], and fixed-regularization geodesic discretization in [11]. Here the vanishing-bandwidth estimates apply to smooth periodic population densities and their finite-dimensional pullback geometry. Appendix B gives a separate deterministic approximation theorem for small analytic families.

Saidi, Gunsilius and Mena [12] develop Hadamard differentiability, covariant differentiation and statistical inference for Sinkhorn geometry at fixed regularization. Here the parameter-differentiated estimates are uniform in the vanishing heat bandwidth and lead to an explicit bias correction on the full tangent complement.

An external unrefereed candidate [21] studies a distinct model: ambient squared-distance cost

on a closed embedded manifold, with a first-order extrinsic-curvature bias and a positive correction. Here the cost comes from the intrinsic periodic heat kernel and depends on h; the bias and higher-order conclusions below concern that model. No curved-manifold extension is inferred by changing the kernel notation.

1.1 Model and fixed-space conventions

Let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ have normalized Lebesgue measure, with $d \geq 1$ fixed, and put $H_h = e^{h\Delta/2}$. Its Fourier multiplier is $e^{-2\pi^2 h|n|^2}$. Let $f > 0$ be a smooth probability density and $q = \sqrt{f}$. The positive symmetric scale satisfies

$$b_h H_h(b_h q) = q, \qquad Q_h = M_{b_h} H_h M_{b_h}. \tag{1}$$

Here M_v means multiplication by v. Set

$$V = \frac{\Delta q}{2q}, \quad S = -\Delta/2 + V, \quad \Pi = |q\rangle\langle q|, \quad J = I - \Pi, \quad G = \tfrac{1}{2}S^\dagger, \quad T_h = hQ_h(I - Q_h^2)^{-1}\big|_{q^\perp}. \tag{2}$$

The superscript † denotes inversion on $q^\perp$ and zero on $\operatorname{span}(q)$. All tensors are extended by zero on this span. The scale exists uniquely for small h, Q_h is a positive compact selfadjoint contraction fixing q, and its gap on $q^\perp$ is at least ch; proofs are recalled below.

The unitary map $U : L^2(f\,\mathrm{d}x) \to L^2(\,\mathrm{d}x)$, $Uu = qu$, gives

$$U^{-1}SU = D_f/2, \qquad D_f = -f^{-1}\operatorname{div}(f\nabla), \qquad U^{-1}Q_hU = P_h,$$

where $P_h u = a_h H_h(a_h f u)$ and $a_h = b_h/q$. Thus $U^{-1}GU = D_f^{-1}$ on density-mean-zero functions. With entropic parameter $\varepsilon = 2h$ and the periodic cost $c_h(x,y) = -2h\log(k_h(x-y)/k_h(0))$, the coefficient of t^2 in the debiased divergence between $f\,\mathrm{d}x$ and $(f+tfu)\,\mathrm{d}x$ is $\langle u, hP_h(I-P_h^2)^{-1}u\rangle_{L^2(f)}$ [1]. This fixes the tensor normalization; the model does not use a fixed Euclidean squared cost with boundary.

For parameters, f_θ is jointly smooth on $\mathbb{T}^d \times \Omega$, with $\Omega \subset \mathbb{R}^p$ open, and $K \Subset \Omega$ compact. The densities have mass one and a common positive lower bound on a neighborhood of K. Uniform bounds on the finitely many spatial and parameter derivatives used in each estimate are understood. Sobolev spaces H^r are the ordinary fixed periodic spaces after conjugation. We write $O_{C^k(H^r\to L^2)}(h^a)$ for a bound on all parameter derivatives of order at most k, uniform on K, between these fixed spaces. In particular derivatives include those of the zero extension and its moving projection. Constants can depend on dimension, the lower bound and derivative bounds.

1.2 Bias and correction

For $H = \operatorname{Hess} V$, define

$$R = \frac{1}{12}\left[-\operatorname{div}(H\nabla) + M_{\operatorname{div}(H\nabla q)/q}\right], \qquad L = \tfrac{1}{2}S^\dagger R S^\dagger - S/12. \tag{3}$$

The operator R is symmetric, second order and annihilates q. The bias L is defined on H^2 and is generally unbounded on L^2.

Theorem 1.1 (Leading tensor bias)**.** *Under the preceding hypotheses, for every fixed integer $k \geq 0$,*

$$T_h = G + h^2 L + O_{C^k(H^6\to L^2)}(h^4). \tag{4}$$

For $D_H = -f^{-1}\operatorname{div}(fH\nabla)$, its density-coordinate form is

$$U^{-1}RU = D_H/12, \qquad U^{-1}LU = \tfrac{1}{6}D_f^{-1}D_H D_f^{-1} - D_f/24. \tag{5}$$

In particular the expansion is not a full L^2 operator expansion.

Theorem 1.2 (Fourth-order corrections and sharp scales)**.** *On $q^\perp$ define*

$$T_h^{\mathrm{R}} = (4T_h - T_{2h})/3, \tag{6}$$

$$C_h^0 = T_h + \frac{h}{6}(I - Q_h)(I + Q_h)^{-1} - 2h^2 T_h R T_h, \tag{7}$$

$$\widehat{C}_h = T_h + \frac{h}{3}Q_h(I - Q_h)(I + Q_h)^{-1}(I + Q_h^2)^{-1} - 2h^2 T_h R T_h. \tag{8}$$

The sandwich in (7) *has a bounded selfadjoint extension on L^2. The main correction $\widehat{C}_h$ is compact and, for a uniform constant M, satisfies on $q^\perp$*

$$(1 - 2Mh^2)T_h \;\le\; \widehat{C}_h \;\le\; (4/3 + 2Mh^2)T_h. \tag{9}$$

It is strictly positive on every nonzero tangent for sufficiently small h, but has no positive uniform L^2 lower bound. The comparison operator C_h^0 satisfies $C_h^0 \ge (h/12)I$ there and equals $(h/6)I$ plus a compact operator, so it is noncompact. All three corrected tensors satisfy, for every fixed k,

$$\|T_h^{\mathrm{R}} - G\|_{C^k(H^r\to L^2)} + \|C_h^0 - G\|_{C^k(H^r\to L^2)} + \|\widehat{C}_h - G\|_{C^k(H^r\to L^2)} \le C_{k,r}h^{1+r/2}, \qquad 0 \le r \le 6. \tag{10}$$

Each exponent for each tensor is sharp already for $f = 1$. The tensor T_h^{R} is compact.

The definition of $\widehat{C}_h$ uses exact finite-h operators and local density derivatives, not the limiting inverse G. This is an analytic construction. Positivity refers to its quadratic form on the entire L^2 tangent complement.

For fixed small h, define the tangent form norm on $q^\perp$ by $\|u\|_{T_h} = \langle u, T_h u\rangle^{1/2}$, and similarly for $\widehat{C}_h$. Inequality (9) identifies their completions by the identity map, with equivalent norms uniformly for small h. Separately, Douglas's range theorem [13, Theorem 1] gives

$$\operatorname{ran}\widehat{C}_h^{1/2} = \operatorname{ran}T_h^{1/2},$$

with equivalent inverse-square-root range norms. The completed form space and this range space are distinct constructions; they are not identified with each other.

Proposition 1.3 (Bias has both signs)**.** *Write $\ell = \log f$. For every smooth periodic ϕ,*

$$24\langle D_f\phi, (U^{-1}LU)D_f\phi\rangle_{L^2(f)} = -\int f\left[|\nabla^3\phi|^2 - 3\operatorname{Hess}\ell : (\operatorname{Hess}\phi)^2\right]\mathrm{d}x. \tag{11}$$

At the single density $f(x) = e^{3\cos(2\pi x)}/Z$ on $\mathbb{T}$, the bias has a strictly positive smooth tangent direction and strictly negative smooth tangent directions. Consequently neither universal ordering $T_h \le G$ nor $T_h \ge G$ holds at variable density, even when tested only on smooth tangents for sufficiently small h.

The sign formula also has a variational interpretation. Normalize Fisher information by $\mathcal{I}(f) = \int f|\nabla\log f|^2\,\mathrm{d}x$. If $u = D_f\phi$, the corresponding density tangent is fu and

$$\langle u, (U^{-1}LU)u\rangle_f = -\frac{1}{48}\operatorname{Hess}_W\mathcal{I}(f)[fu, fu]. \tag{12}$$

Here Hess_W means second variation along the local smooth quadratic Wasserstein geodesic. Thus (11) identifies precisely which Fisher second variation is seen by this heat-kernel tensor. The general Fisher nonconvexity principle is classical [9].

Corollary 1.4 (Finite-family intrinsic geometry)**.** *Suppose $\theta \mapsto f_\theta$ is an immersion on a neighborhood of K. Put $\eta_i = (\partial_i f)/q$ and $g_{ij}^A = \langle \eta_i, A\eta_j \rangle$ for $A = G, T_h, T_h^{\mathrm{R}}, C_h^0, \widehat{C}_h$. For every fixed k,*

$$g_{ij}^{T_h} = g_{ij}^G + h^2 \langle \eta_i, L\eta_j \rangle + O_{C^k}(h^4),$$
$$g^{T_h^{\mathrm{R}}} - g^G = O_{C^k}(h^4), \quad g^{C_h^0} - g^G = O_{C^k}(h^4), \quad g^{\widehat{C}_h} - g^G = O_{C^k}(h^4).$$

The corrected matrices are positive definite for small h. For $k \geq 2$, their Christoffel symbols converge in C^{k-1} and intrinsic Riemann curvature components in C^{k-2} at order h^4. These are finite-family statements. Corollary 1.5 gives their restricted distance and initial-value consequences.

Corollary 1.5 (Distances and geodesic initial-value problems)**.** *Under Corollary 1.4, let $g = g^G$ and let g_h be any of the three corrected pullback metrics. On a fixed compact parameter set K in the immersion domain, define d_g^K and $d_{g_h}^K$ by infimizing lengths of piecewise C^1 paths lying in K. For every pair joined by such a path, and all sufficiently small h,*

$$(1 - Ch^4)^{1/2} d_g^K \leq d_{g_h}^K \leq (1 + Ch^4)^{1/2} d_g^K.$$

The same factors $1 \pm Ch^4$ compare infima of path energies on any fixed time interval with the same endpoints and admissible paths.

Suppose also that a compact set of initial positions and velocities has g-geodesic phase trajectories on $[0, \tau]$ lying in a compact subset of a parameter chart and its velocity space. Then the g_h-geodesics with the same initial data exist on this interval for small h, and their positions and velocities differ by $O(h^4)$, uniformly in time and in these initial data. Parallel transports along the paired geodesics, expressed in this chart with the same initial vectors, differ by $O(h^4)$ in operator norm. These statements concern the intrinsic geometry of the density family.

Three correction constructions. The two-bandwidth Richardson tensor cancels the leading bias by combining exact tensors at two scales. The Simpson comparison has an identity-floor contribution. The preferred same-bandwidth rational correction uses an additional density sandwich to remain compact, positive on the full tangent complement, and comparable to the original tensor. Their smooth-input rates are shared, while their compactness and positivity mechanisms differ. Appendix B gives a separate effective theorem for prescribed small analytic families, including normalization and finite resolution; its input conditions remain independent of the general smooth-density bias theorem.

2 Scale, gap and baseline bounds

We include the estimates needed to handle the shrinking inverse gap. They also establish the second-order baseline used below.

Lemma 2.1 (Baseline estimates)**.** *For every fixed k, uniformly on the declared density families,*

$$b_h = e^{-hV/2} + O_{C^k(H^m)}(h^3), \tag{13}$$
$$Q_h - e^{-hS} = O_{C^k(L^2 \to L^2)}(h^2) = O_{C^k(H^2 \to L^2)}(h^3), \tag{14}$$
$$T_h - G = O_{C^k(L^2 \to L^2)}(h) = O_{C^k(H^2 \to L^2)}(h^2). \tag{15}$$

Here $m > d/2$ is any fixed integer, with sufficient density regularity. Moreover, with $B_Q = I - Q_h + h\Pi$,

$$\|B_Q^{-1}\|_{C^k(L^2 \to L^2)} \leq C_k/h. \tag{16}$$

Proof. In the Sobolev algebra H^m, put $F_h(b) = bH_h(bq)/q - 1$. At $b = 1$,

$$DF_h(1) = M_{1/q}(I + H_h)M_q + M_{H_h q/q-1}.$$

The first term has uniformly bounded inverse $M_{1/q}(I+H_h)^{-1}M_q$, and the second is $O(h)$. The second derivative is uniformly bounded bilinearly on a fixed ball. Positive-time heat Taylor expansion of $e^{-hV/2}$ gives residual $O(h^3)$, so quantitative Newton inversion proves (13). The same residual equation differentiated inductively proves its parameter estimates, and positive-order derivatives of b_h are $O(h)$. Sobolev embedding gives positivity. For uniqueness let $w_i = a_i f$, so $w_i H_h w_i = f$. Their difference z satisfies $H_h z = -fz/(w_1 w_2)$. Pairing with z contradicts positivity of H_h unless $z = 0$.

For the full norm comparison reduce to the symmetric product $e^{-hV/2}H_h e^{-hV/2}$. In a lifted Brownian bridge from x to $y = x + z$, the mean difference between $\int_0^h V(X_t)\,\mathrm{d}t$ and $h(V(x) + V(y))/2$ is bounded by $C(h|z|^2+h^2)$. Taylor expansion cancels the linear segment error, and bridge covariance $t(h-t)I/h$ gives Ch^2. Both exponents are $O(h)$, so replacing exponentials by one minus their arguments costs Ch^2. Integrating against the lifted heat kernel uses $\int |z|^2 k_h^{\mathbb{R}^d}(z)\,\mathrm{d}z = dh$. Both absolute row and column errors are $O(h^2)$; summing lifts and Schur's test prove the first norm in (14). Parameter derivatives obey the same bound: their linear terms replace V by a derivative, and nonlinear remainders have at least two time integrals.

For the strong norm let $A = \Delta/2$, $B = -M_V$, $C = [A, B]$. Removing the bounded outer factors from the defect of $F(t) = e^{tB/2}e^{tA}e^{tB/2}$ relative to $A + B$ gives

$$-\int_0^{t/2} e^{-rB}Ce^{rB}\,\mathrm{d}r\, e^{tA} + \frac{1}{2}\int_0^t e^{(t-r)A}Ce^{rA}\,\mathrm{d}r. \tag{17}$$

Subtract Ce^{tA} in each integral; their integrals cancel. The remaining commutators are $[C, B]$, a multiplier, and $[A, C]$, of differential order two. Their integral representations bound the defect by $Ct^2 : H^2 \to L^2$. Variation of constants yields the second norm in (14). Every fixed parameter derivative has the same differential orders; bounded-potential Duhamel controls the differentiated semigroups.

The weighted form $\langle v, Sv\rangle = \frac{1}{2}\int f|\nabla(v/q)|^2$ shows that S has kernel $\mathrm{span}(q)$ and a uniform positive gap on its complement by weighted Poincare. The Markov contraction P_h is unitarily conjugate to Q_h, which is positive and compact. The full norm comparison gives $\|Q_h|_{q^\perp}\| \le 1 - ch$. Thus $B_Q^{-1} = O(h^{-1})$ on L^2. Its positive-order parameter derivatives have the same bound, since $\partial^\alpha B_Q = O(h)$ for $|\alpha| > 0$ and each term of inverse differentiation has one more inverse than differentiated factors.

For $E = e^{-hS}$ the corresponding inverse $(I - E + h\Pi)^{-1}$ is $O_{C^k}(h^{-1})$ on L^2 and H^2, by the gap, graph norms, and bounded-potential differentiation. Use the exact identity

$$hQ(I - Q^2)^{-1} = \frac{h}{2}[(I - Q)^{-1} - (I + Q)^{-1}] \tag{18}$$

on the complement. The resolvent identity and (14) give tensor differences $O(h)$ on L^2 and $O(h^2) : H^2 \to L^2$. For zero extensions the exact fixed-space expression is

$$\frac{h}{2}[B_Q^{-1} - h^{-1}\Pi - J(I + Q)^{-1}J]. \tag{19}$$

The projected terms cancel in the comparison with E. Its scalar tensor multiplier is $h/(2\sinh(h\lambda))$. The difference from $1/(2\lambda)$ is bounded by both Ch and $Ch^2\lambda$ on the positive spectrum.

For completeness these scalar estimates retain parameter derivatives. Put $A_0 = S + \Pi$; it has fixed domain H^2 and positive lower bound. Its positive-order parameter derivatives are bounded multipliers plus smooth finite-rank operators. On fixed sector rays, $\|(z - A_0)^{-1}\| \le C/(1 + |z|)$. The scalar difference $g_h(z) = h/(2\sinh(hz)) - 1/(2z)$ is analytic and bounded by Ch on every strict right sector. Positive-order derivatives of its contour formula contain at least two resolvents, so their integrals are $O(h)$. For the strong estimate use $g_h(A_0) = h^2 j(hA_0)A_0$, where $j(w) = [1/(2\sinh w) - 1/(2w)]/w$ is bounded analytic there. The same two-resolvent argument bounds all derivatives of $j(hA_0)$. Compression by J completes (15) on fixed spaces. □

3 Cubic scale and positive-time local error

Lemma 3.1 (Scale to fifth-order residual)**.** *For every fixed m and k,*

$$b_h = \exp(-hV/2 + h^3 c) + O_{C^k(H^m)}(h^5), \tag{20}$$

where

$$c = \frac{\Delta^3 q}{192q} - \frac{\Delta((\Delta q)^2/q)}{128q} + \frac{(\Delta q)^3}{384q^3}. \tag{21}$$

Proof. First identify finitely many coefficients in the formal ring of smooth functions and a scalar h. The equation $be^{hA}(bq) = q$, $A = \Delta/2$, has a unique formal solution with constant term one: the unknown coefficient b_n appears as $2qb_n$ plus known lower terms. Formal inversion gives $e^{-hA}(q/b) = bq$, so $b(h)^{-1}$ solves the same equation at $-h$. Uniqueness implies $b(-h) = b(h)^{-1}$ and hence $\log b$ is odd as a formal series. This is coefficient algebra, not an application of an inverse heat operator to a function.

Let $u = -V/2$. Substitution of $b = 1 + hu + h^2u^2/2 + h^3(c + u^3/6)$ gives cubic residual coefficient

$$2qc + qu^3/3 - A^3 q/12 + A(qu^2)/2.$$

Its cancellation gives (21). Formal oddness removes the fourth logarithmic coefficient, so $p_h = \exp(-hV/2 + h^3 c)$ has zero residual coefficients through degree four. Positive-time heat Taylor's integral remainder through degree four, with sufficiently many derivatives of the fixed smooth coefficients, gives $F_h(p_h) = O_{C^k(H^m)}(h^5)$. For example ten additional derivatives suffice for the heat remainder of a fixed coefficient. Every actual heat time lies in $[0, h]$. The uniform Newton inverse from Lemma 2.1 converts this residual into (20). Differentiating the exact residual equation inductively proves the parameter remainder. This does not differentiate an uncontrolled pointwise asymptotic. □

Lemma 3.2 (Local comparison on H^6)**.** *For $Z = -S = A + B$ and R from* (3)*,*

$$Q_h - e^{hZ} = h^3 R + \tfrac{h^4}{2}(ZR + RZ) + O_{C^k(H^6 \to L^2)}(h^5). \tag{22}$$

Proof. Put $C = [A, B]$, $D = [A, C]$ and $K_0 = [C, B]$. The multiplier K_0 commutes with B, so $e^{-rB} C e^{rB} - C = rK_0$ exactly. Changing variables in the heat commutator part of (17), its remaining expression is

$$-\frac{t^2}{8} K_0 e^{tA} + \frac{1}{2}\int_0^t (t - u) e^{uA} D e^{(t-u)A}\, \mathrm{d}u. \tag{23}$$

Expanding the first heat factor once, and each heat factor in the integral once, gives

$$t^2 D_2 + t^3 D_3 + O(t^4) : H^6 \to L^2, \quad D_2 = D/4 - K_0/8, \quad D_3 = AD/12 + DA/6 - K_0 A/8.$$

The omitted products A^2D, ADA, DA^2 have order at most six; integral Taylor remainders use only nonnegative times and bounded heat operators on the corresponding Sobolev spaces. Restoring outer multiplications adds $(BD_2 + D_2B)/2$ to the cubic defect coefficient. Variation of constants, with its left semigroup expanded once, now proves an actual $O(h^5) : H^6 \to L^2$ remainder and cubic coefficient $R_0 = D/12 - K_0/24$. The fourth coefficient is $(ZR_0 + R_0Z)/2$. One can obtain it by inserting D_2, D_3; equivalently the finite formal identity $F(-h) = F(h)^{-1}$ forces $\log F(h) = hZ + h^3R_0 + O(h^5)$ and gives the same coefficient. The analytic remainder has already been established by (23), independently of this formal identity.

The scale expansion inserts outer multiplications e^{h^3c}. Their contribution is $2h^3M_c + h^4(M_cZ + ZM_c)$, with the same remainder domain. Thus $R = R_0 + 2M_c$ in (22). Direct differentiation gives

$$C = -\nabla V \cdot \nabla - \Delta V/2, \quad [A, C] = -\operatorname{div}(\operatorname{Hess} V \nabla) - \Delta^2 V/4, \quad [C, B] = |\nabla V|^2.$$

Since both exact operators in (22) fix the smooth q, its cubic coefficient annihilates q. Its principal part therefore determines its scalar term, giving exactly (3). Every parameter derivative preserves the displayed differential orders and the positive-time remainder bounds. □

4 Modified generator and singular-inverse expansion

The modified operator

$$S_h = S - h^2R \tag{24}$$

is elliptic for small h. Indeed direct conjugation gives

$$U^{-1}RU = D_H/12, \qquad U^{-1}S_hU = \tfrac{1}{2}D_{f,M_h}, \quad M_h = I - \tfrac{h^2}{6}\operatorname{Hess} V, \tag{25}$$

where $D_{f,M} = -f^{-1}\operatorname{div}(fM\nabla)$. If $h^2\|\operatorname{Hess} V\|_\infty \le 3$, then $M_h \ge I/2$, $S_h \ge S/2$ as quadratic forms, and its exact kernel is $\operatorname{span}(q)$. Elliptic graph norms of $S_h + \Pi$ are uniformly equivalent to the corresponding ordinary even Sobolev norms.

Lemma 4.1 (Modified comparison and parameter bounds)**.** *Let $E_h^* = e^{-hS_h}$ and $B_* = I - E_h^* + h\Pi$. Then*

$$Q_h - E_h^* = O_{C^k(H^6 \to L^2)}(h^5), \qquad B_*^{-1} = O_{C^k(H^6 \to H^6)}(h^{-1}). \tag{26}$$

The inverse bound holds on L^2 as well.

Proof. One Duhamel insertion about e^{hZ} is $h^2\int_0^h e^{(h-t)Z}Re^{tZ}\,\mathrm{d}t$. Expanding its outer factors through first order yields $h^3R + h^4(ZR + RZ)/2$, with $O(h^5) : H^6 \to L^2$ remainder: the highest omitted compositions are Z^2R, ZRZ, RZ^2. The exact twice-iterated remainder has two R insertions, coefficient h^4 and time-simplex volume $O(h^2)$, hence is $O(h^6) : H^4 \to L^2$. Uniform semigroup bounds on the required Sobolev spaces suffice; their parameter justification is given next. Lemma 3.2 then proves the first assertion.

Parameter derivatives of S_h include second-order terms, so a bounded-potential argument alone would be insufficient. Put $A_h = S_h + \Pi$ and $A_0 = S + \Pi$. On $X = H^{2j}$ for each fixed needed integer j, elliptic graph norms and selfadjoint spectral bounds give, on sectorial rays away from the positive axis,

$$\|(z - A_h)^{-1}\|_{X\to X} \le \frac{C}{1 + |z|}, \quad \|(z - A_h)^{-1}\|_{X\to H^{2j+2}} \le C. \tag{27}$$

Consequently every positive-order parameter derivative satisfies

$$\|(\partial^\alpha A_h)(z - A_h)^{-1}\|_{X\to X} \le C_\alpha[(1 + |z|)^{-1} + h^2]. \tag{28}$$

The order-zero part comes from derivatives of A_0; the principal part comes with $h^2\partial^\alpha R$.

For $0 < t \le h_0$, use the semigroup contour with rays starting at radius $1/t$ and an arc through the left half-plane at that radius. It avoids the positive spectrum. The exponential e^{-tz} is bounded on the arc and decays on the rays. In the resolvent difference there is a factor h^2R; (27) bounds it by $Ch^2/|z|$. Arc length and ray integration therefore give

$$e^{-tA_h} - e^{-tA_0} = O_{C^k(X\to X)}(h^2). \tag{29}$$

After differentiation every term retains a factor h^2R or its derivative; all other insertions are bounded by (28). On this contour $(1+|z|)^{-1} \le Ct$, so the same integration proves every fixed derivative bound. The common finite-rank kernel adjustment cancels in the semigroup difference. Together with bounded-potential Duhamel for S, this proves uniform differentiated semigroup bounds on the Sobolev spaces used in the twice-iterated remainder. At $t = h$, positive-order derivatives of E_h^* are $O(h)$ on X.

The uniform gap and graph norms give $B_*^{-1} = O(h^{-1})$ on X. For $|\alpha| > 0$, $\partial^\alpha B_* = O(h)$. Repeated inverse differentiation therefore gives $\partial^\alpha B_*^{-1} = O(h^{-1})$ for every fixed derivative order. Take $j = 0, 3$ to obtain the stated cases. □

Proof of Theorem 1.1. Apply (19) to Q_h and E_h^*. The minus-inverse difference is $B_Q^{-1}(Q_h - E_h^*)B_*^{-1}$. Using Lemmas 2.1 and 4.1 gives $h \cdot h^{-1} \cdot h^5 \cdot h^{-1} = h^4$ from H^6 to L^2, with all fixed parameter derivatives. The plus-inverse difference is of higher order. Thus $T_h - T_h^* = O_{C^k(H^6\to L^2)}(h^4)$, where T_h^* is the tensor of E_h^*.

The scalar function

$$m(w) = \frac{1/(2\sinh w) - 1/(2w) + w/12}{w^3}$$

is bounded analytic on every fixed strict right sector, with $m(0) = 7/720$ and $m(w) = O(|w|^{-2})$ at sectorial infinity. Functional calculus on $q^\perp$ yields

$$T_h^* = (2S_h)^\dagger - h^2 S_h/12 + h^4 m(hS_h) S_h^3. \tag{30}$$

Here the final term is $O_{C^k(H^6\to L^2)}(h^4)$, including parameters of the principal part. To verify this explicitly, use $A_h = S_h + \Pi$ and compress by J. Its zeroth derivative is bounded by the spectral theorem. Differentiated contour terms contain one resolvent and relative insertions (28). Terms with only order-zero insertions have at least two decaying resolvents and bounded integrals. A term with $j \ge 1$ second-order insertions has bound Ch^{2j} times

$$\int_0^\infty \frac{\min(1, [h(1+r)]^{-2})}{1+r}\,\mathrm{d}r \le C(1 + |\log h|).$$

Its product with h^{2j} is uniformly bounded. These estimates also justify differentiation on truncated contours and passage to the limit for each $h > 0$. Finally A_h^3 and all fixed derivatives map H^6 to L^2 uniformly. Compression removes its artificial eigenvalue on q and proves the remainder assertion in (30).

On $q^\perp$ the exact inverse expansion is

$$S_h^{-1} = S^{-1} + h^2 S^{-1} R S^{-1} + h^4 S^{-1} R S_h^{-1} R S^{-1}.$$

The last coefficient and all its fixed parameter derivatives are bounded on L^2 by elliptic regularity; equivalently regularize by Π and then compress, using $R\Pi = \Pi R = 0$. Also $-h^2 S_h/12 = -h^2 S/12 + h^4 R/12$. Substitution proves (4). Conjugation in (25) gives (5), including its normalization factors. □

5 Sign structure and Fisher information

Proof of Proposition 1.3. From (5), with $u = D_f\phi$,

$$24\langle u, (U^{-1}LU)u\rangle_f = 4\int f \operatorname{Hess} V(\nabla\phi, \nabla\phi) - \int f|\nabla D_f\phi|^2. \tag{31}$$

Here and below integrals are over the torus. We give the weighted identity needed to simplify the second term. Write $\mathcal{L} = \Delta + \nabla\ell\cdot\nabla$, $w = \nabla\phi$, $A = \operatorname{Hess}\phi$, $B = \operatorname{Hess}\ell$. Then $\partial_i\mathcal{L}\phi = \mathcal{L}w_i + B_{ij}w_j$. Weighted Bochner, integrated componentwise, gives

$$\int f|\mathcal{L}w|^2 = \int f|\nabla^3\phi|^2 - \int fB : A^2.$$

Integration by parts in the cross term gives

$$2\int f(\mathcal{L}w_i)B_{ij}w_j = -2\int fB : A^2 - 2\int fA_{ik}\ell_{ijk}w_j.$$

Symmetry of the third derivatives turns the last term into $\int f[\operatorname{Hess}\Delta\ell + \nabla\ell\cdot\nabla\operatorname{Hess}\ell][w,w]$. Adding $\int f|Bw|^2$ supplies the B^2 term. Since $4V = \Delta\ell + |\nabla\ell|^2/2$, their sum is $4\int f \operatorname{Hess} V[w,w]$. Substituting into (31) proves (11).

For an explicit positive direction in dimension one, set $\theta = 2\pi x$, $f = e^{3\cos\theta}/Z$, and prescribe

$$\phi''(x) = e^{-3\cos\theta}\sin\theta, \qquad \phi'(x) = e^{-3\cos\theta}/(6\pi) - C,$$

where C makes ϕ' have zero unweighted mean. This gives a smooth periodic ϕ and a nonzero density-mean-zero $u = D_f\phi$. Let $w = e^{-3\cos\theta}$, $c = \cos\theta$, $s = \sin\theta$ and write av for angular average. The integral inside the leading minus sign of (11) is

$$\frac{(2\pi)^2}{Z}\operatorname{av} w[(c+3s^2)^2 + 9cs^2].$$

Integrating the derivatives of $w\sin^3\theta$, $w\sin\theta\cos\theta$ and $w\sin\theta$ gives respectively

$$3\operatorname{av}(ws^4) = -3\operatorname{av}(wcs^2), \quad 3\operatorname{av}(wcs^2) = \operatorname{av}(w(s^2-c^2)), \quad 3\operatorname{av}(ws^2) = -\operatorname{av}(wc).$$

Expanding and applying these identities reduces the average to $\operatorname{av}(w(2-3c^2)) = -\operatorname{av}(w(1+c)) < 0$. Consequently

$$\langle u, (U^{-1}LU)u\rangle_f = \frac{(2\pi)^2}{24Z}\operatorname{av}(e^{-3\cos\theta}(1+\cos\theta)) > 0.$$

For negative directions take $\phi_n = \cos(2\pi nx)$. The bracket integral in (11) is at least

$$\frac{\min f}{2}(2\pi n)^6 - \frac{3\|\ell''\|_\infty \max f}{2}(2\pi n)^4 > 0$$

for large n. Thus the corresponding bias is negative at the same fixed density. Applying (4) to either fixed smooth direction transfers its strict bias sign to $T_h - G$ for small h. This ordering obstruction says nothing by itself about positivity of the two-bandwidth Richardson tensor. □

Proof of the Fisher-Hessian identity (12). For a smooth mass-zero density perturbation g, direct variation of $\mathcal{I}(f) = \int|\nabla f|^2/f$ gives

$$\mathcal{I}'[f](g) = \int(-2\Delta\ell - |\nabla\ell|^2)g = -8\int Vg, \qquad \mathcal{I}''[f](g,g) = 2\int f|\nabla(g/f)|^2.$$

Take $f_t = (\mathrm{Id} + t\nabla\phi)_{\#} f$. For sufficiently small $|t|$, the displacement stays within the torus injectivity range and $I + t\,\mathrm{Hess}\,\phi$ is positive definite; this is the local smooth quadratic Wasserstein geodesic with initial velocity $\nabla\phi$. Differentiating its action on smooth test functions gives

$$\dot{f}_0 = -\operatorname{div}(f\nabla\phi) = fD_f\phi, \qquad \ddot{f}_0 = \partial_i\partial_j(f\phi_i\phi_j).$$

The ordinary second derivative chain rule and periodic integration by parts therefore give

$$\mathrm{Hess}_W\,\mathcal{I}(f)[fD_f\phi, fD_f\phi] = 2\int f|\nabla D_f\phi|^2 - 8\int f\,\mathrm{Hess}\,V(\nabla\phi,\nabla\phi).$$

Comparison with (31) proves (12). In particular (11) is equivalent to the corresponding weighted third-derivative formula for this Fisher Hessian; its one-dimensional antecedent is contained in [9, Lemma 8]. □

6 Uniform smoothing and positive correction

The differential sandwich cannot be justified by formal cancellation alone. The following uniform smoothing estimate supplies its domain.

Lemma 6.1 (Two-derivative smoothing)**.** *For every fixed nonnegative integer s and k,*

$$\|T_h\|_{C^k(H^s\to H^{s+2})} \le C_{k,s}. \tag{32}$$

Proof. Put

$$K_h = (I - H_h)/h, \quad W_h = (b_h^{-2} - 1)/h, \quad v_h = q/b_h, \quad \widehat{A}_h = K_h + M_{W_h} + |v_h\rangle\langle v_h|.$$

There is an exact rank-one factorization

$$I - Q_h + h\Pi = hM_{b_h}\widehat{A}_h M_{b_h}. \tag{33}$$

The gap proves that $\widehat{A}_h$ has a uniform positive lower bound on L^2, hence a uniformly bounded inverse. The functions W_h, v_h and all fixed spatial and parameter derivatives are uniformly bounded: (13) gives $W_h = V + O(h)$.

The inverse is uniformly bounded on each fixed H^s. Indeed K_h commutes with spatial derivatives. Differentiating $\widehat{A}_h w = u$ j times gives the same operator on $\partial^j w$, with commutators involving only lower derivatives of w and smooth derivatives of W_h. The finite-rank commutators are bounded on L^2 with smooth ranges: move derivatives off w in scalar pairings by integration by parts. Induction using the L^2 inverse bound proves the H^j bounds. For smooth u, existence of these derivatives can first be justified at fixed h from

$$w = (1/h + W_h)^{-1}[u + H_h w/h - |v_h\rangle\langle v_h|w].$$

Its right side is smooth, and its denominator is positive for small h. The uniform estimates come from the derivative induction, not from this last formula. Density extends them to H^j inputs. Parameter derivatives of $\widehat{A}_h$ are uniformly bounded on each H^j, so inverse differentiation proves all fixed parameter versions.

Let $R_0 = (K_h + I)^{-1}$. On a Fourier mode with $\mu = 4\pi^2|n|^2$,

$$\frac{(1+\mu)e^{-h\mu/2}}{1 + (1 - e^{-h\mu/2})/h} \le C.$$

For $h\mu/2 \le 1$ use $1 - e^{-t} \ge ct$; otherwise use $te^{-t} \le C$. Hence $H_h R_0 : H^s \to H^{s+2}$ is uniformly bounded. The inverse identity

$$H_h \widehat{A}_h^{-1} = H_h R_0 + H_h R_0 (I - M_{W_h} - |v_h\rangle\langle v_h|)\widehat{A}_h^{-1}$$

gives the same smoothing, including parameters, for its left side. From (33), exactly on the full space,

$$hQ_h(I - Q_h)^\dagger = M_{b_h} H_h \widehat{A}_h^{-1} M_{1/b_h} - \Pi. \tag{34}$$

The subtraction is necessary: $hQ_h(I - Q_h + h\Pi)^{-1}$ acts as the identity on q. Finally $(I + Q_h)^{-1}$ is uniformly bounded on every fixed H^s, with all fixed parameter derivatives. This follows from $Q_h - H_h = O_{C^k(H^s \to H^s)}(h)$ and a Neumann perturbation of the Fourier multiplier inverse $(I + H_h)^{-1}$. Multiply (34) by this plus inverse to obtain (32). □

Lemma 6.2 (Relative form control)**.** *Uniformly on the declared density classes, for every $u \in L^2$,*

$$\big|\langle u, T_h R T_h u\rangle\big| \le M\langle u, T_h u\rangle. \tag{35}$$

Proof. Lemma 6.1 and selfadjoint duality give compatible uniform maps $T_h : L^2 \to H^2$ and $T_h : H^{-2} \to L^2$. Interpolation gives $T_h : H^{-1} \to H^1$. For smooth v, positivity therefore implies

$$\|T_h^{1/2} v\|_2^2 = \langle v, T_h v\rangle \le C\|v\|_{H^{-1}}^2.$$

Density extends the square root to $H^{-1} \to L^2$; duality gives $T_h^{1/2} : L^2 \to H^1$ uniformly. Since the symmetric differential operator R is a bounded form $H^1 \to H^{-1}$,

$$|\langle T_h u, R T_h u\rangle| \le C\|T_h u\|_{H^1}^2 \le M\|T_h^{1/2} u\|_2^2.$$

This proves (35). The argument uses no parameter derivatives of a square root; it applies to the full-space zero extension and its moving kernel pointwise in the parameter. □

Proof of Theorem 1.2. The smoothing lemma makes RT_h uniformly bounded on L^2, and therefore $T_h R T_h$ is uniformly bounded. Its form $\langle T_h u, RT_h v\rangle$ is symmetric, proving that its extension is selfadjoint. It is compact since its left T_h is compact and RT_h is bounded.

For every $0 \le p < 1$,

$$\frac{hp}{1-p^2} + \frac{h}{6}\frac{1-p}{1+p} = \frac{h(p^2+4p+1)}{6(1-p^2)} \ge h/6. \tag{36}$$

Functional calculus and a uniform bound $\|T_h R T_h\| \le M$ give $C_h^0 \ge hI/6 - 2Mh^2 I \ge hI/12$ once $h \le 1/(24M)$, in addition to the previous small-h restrictions. The scalar expression in (36) equals

$$\frac{h}{6} + \frac{2h}{3}\sum_{j\ge 0} p^{2j+1} + \frac{h}{3}\sum_{j\ge 1} p^{2j}.$$

At $p = e^{-h\lambda}$ this is half the classical composite Simpson approximation to $\int_0^\infty e^{-t\lambda}\,\mathrm{d}t$. It also shows the $h/6$ identity term plus a compact remainder; the differential sandwich is compact as already proved.

For the compact correction, its additional scalar term divided by the multiplier of T_h is exactly

$$\frac{(h/3)p(1-p)/[(1+p)(1+p^2)]}{hp/(1-p^2)} = \frac{(1-p)^2}{3(1+p^2)} \in [0, 1/3], \qquad 0 < p < 1. \tag{37}$$

Spectral calculus and Lemma 6.2 prove (9). The heat operator is injective and b_h is positive, so Q_h and T_h have no null vector on $q^\perp$. Choosing $2Mh^2 < 1$ gives strict positivity of $\widehat{C}_h$ there. The scalar added filter is continuous on the spectrum of Q_h and vanishes at zero. It is compact, as are T_h and T_hRT_h. This proves compactness and explains why no positive uniform L^2 floor can hold in infinite dimension.

The scalar base has the exact identity on $q^\perp$

$$T_h + \frac{h}{3}Q_h(I-Q_h)(I+Q_h)^{-1}(I+Q_h^2)^{-1} = \frac{4T_h - 2hQ_h^2(I-Q_h^4)^{-1}}{3}. \tag{38}$$

Both sides divided by T_h have multiplier $1 + (1-p)^2/[3(1+p^2)]$. This is a rational two-step Richardson filter constructed from one normalized operator. In general $Q_h^2 \neq Q_{2h}$: separately balancing the doubled heat time changes the scale. Thus (38) is not a positivity theorem for the raw two-bandwidth extrapolation T_h^{R}.

The two form completions are equivalent by (9). Applying Douglas's theorem to the bounded operators $T_h^{1/2}$ and $\widehat{C}_h^{1/2}$ in both directions gives their equal ranges. If $aT_h \le \widehat{C}_h \le bT_h$, then on that range the inverse-square-root norms satisfy $b^{-1/2}\|T_h^{-1/2}v\| \le \|\widehat{C}_h^{-1/2}v\| \le a^{-1/2}\|T_h^{-1/2}v\|$. Taking $a = 1 - 2Mh^2$ and $b = 4/3 + 2Mh^2$ gives uniform constants for small h.

We next prove accuracy without applying R to a rough tangent. Theorem 1.1 gives

$$T_h = G + h^2(2GRG - S/12) + O_{C^k(H^6\to L^2)}(h^4). \tag{39}$$

For the Cayley term, the exact identity between two Cayley filters uses only their plus inverses. Lemma 4.1 therefore gives

$$\frac{h}{6}\left[(I-Q_h)(I+Q_h)^{-1} - (I-E_h^*)(I+E_h^*)^{-1}\right] = O_{C^k(H^6\to L^2)}(h^6).$$

The reference scalar filter is $(h/6)\tanh(hS_h/2)$, and

$$(h/6)\tanh(hz/2) = h^2 z/12 + h^4 z^3 e(hz),$$

where e is bounded analytic on each strict right sector, removable at zero and decaying quadratically at infinity. The relative-insertion contour proof used for $m(hA_h)$ applies verbatim to $e(hA_h)$: bounded insertions give two resolvents, and $j \ge 1$ principal insertions cost $h^{2j}(1 + |\log h|)$. Compressing by J gives

$$\frac{h}{6}(I-Q_h)(I+Q_h)^{-1} = h^2 S/12 + O_{C^k(H^6\to L^2)}(h^4). \tag{40}$$

Replacing S_h by S costs only $O(h^4) : H^2 \to L^2$; the Cayley filter vanishes on the common kernel.

The exact sandwich identity is

$$T_hRT_h - GRG = (T_h - G)RT_h + GR(T_h - G).$$

Its right side is $O_{C^k(H^2\to L^2)}(h^2)$. For the first term, smoothing with $s = 2$ gives $RT_h : H^2 \to H^2$ uniformly and (15) applies. For the second, RG is bounded on L^2 by elliptic regularity, so its adjoint GR extends boundedly there. The product rule handles all parameters between the same fixed domains. Thus the Cayley term cancels $-h^2S/12$ and the sandwich cancels $2h^2GRG$ in (39). This proves the H^6 endpoint for C_h^0. Substitution of (4) at h and $2h$ proves that endpoint for T_h^{R}.

To pass to the compact correction, on the full fixed space exactly

$$\widehat{C}_h - C_h^0 = -\frac{h}{6} J(I - Q_h)^3 (I + Q_h)^{-1} (I + Q_h^2)^{-1} J. \tag{41}$$

Indeed its Cayley addition is that of C_h^0 multiplied by $2Q_h(I + Q_h^2)^{-1}$, whose difference from I is $-(I - Q_h)^2(I + Q_h^2)^{-1}$. For each required integer $s \geq 0$,

$$I - Q_h = O_{C^k(H^{s+2} \to H^s)}(h).$$

This follows from the heat multiplier estimate for $I - H_h$ and $b_h - 1 = O_{C^k}(h)$ in the needed multiplier norms. Parameter derivatives of positive order preserve the factor h. Also $(I + Q_h^2)^{-1}$ is uniformly C^k bounded on every required H^s: the identity $Q_h^2 - H_h^2 = (Q_h - H_h)Q_h + H_h(Q_h - H_h)$ is $O_{C^k}(h)$ there, and a Neumann perturbation of the bounded Fourier multiplier $(I + H_h^2)^{-1}$ proves the assertion. Compose three increments through $H^6 \to H^4 \to H^2 \to L^2$ in (41). The plus inverses and smooth projections are uniformly bounded with every fixed parameter derivative, so the product rule gives $\widehat{C}_h - C_h^0 = O_{C^k(H^6 \to L^2)}(h^4)$. No square-root differentiation is used for this estimate.

At the L^2 endpoint, (15) and the triangle inequality give $O(h)$ for Richardson. For C_h^0, its Cayley term is $O(h)$ and its bounded sandwich is $O(h^2)$, so the same endpoint holds. The right side of (41) is $O_{C^k(L^2 \to L^2)}(h)$, which proves the L^2 endpoint for $\widehat{C}_h$ as well. Interpolation between the fixed-space endpoints gives (10) with all fixed parameter derivatives.

To prove sharpness put $f = 1$. Then $R = 0$, $S = -\Delta/2$ and $Q_h = e^{-hS}$. On an eigenmode of S with eigenvalue λ, the errors of Richardson and the positive correction respectively are $hF_{\mathrm{R}}(h\lambda)$ and $hF_+(h\lambda)$, where

$$F_{\mathrm{R}}(t) = \frac{2}{3\sinh t} - \frac{1}{3\sinh(2t)} - \frac{1}{2t} = -7t^3/180 + O(t^5),$$
$$F_+(t) = \frac{1}{2\sinh t} + \frac{1}{6}\tanh(t/2) - \frac{1}{2t} = t^3/360 + O(t^5).$$

Choose a fixed positive t with both functions nonzero and lattice frequencies

$$n_h = (\mathrm{round}\,\sqrt{t/(2\pi^2 h)}, 0, \ldots, 0).$$

Then $h\lambda_{n_h} \to t$ and their H^r norm is of order $h^{-r/2}$. Each operator error is at least $c_r h^{1+r/2}$ for all small h. A fixed nonzero mode additionally gives the nonzero fourth-order coefficients $-7\lambda^3/180$ and $\lambda^3/360$. This proves sharpness for these two tensors. For $\widehat{C}_h$, (38) agrees with ordinary Richardson at constant density because $Q_h^2 = Q_{2h}$ and $R = 0$. Its error function is therefore F_{R} and its nonzero fourth-order coefficient is also $-7\lambda^3/180$. The same lattice modes establish sharpness for every $r \in [0, 6]$. □

7 Family geometry and limits

Proof of Corollary 1.4. Since $\langle \eta_i, q \rangle = \int \partial_i f = 0$, all tangent contractions are on the stated complement. Joint smoothness provides the finitely many spatial and parameter bounds on η_i required by the H^6 estimates. The product rule and Theorems 1.1– 1.2 give the stated matrix expansions in C^k. The inverse weighted-Laplacian form is strictly positive on nonzero tangents. Immersion and compactness give a uniform positive lower bound for g^G. Uniform convergence then makes the corrected matrices positive and uniformly invertible. Matrix inversion is smooth on this set; the coordinate Christoffel formula uses one metric derivative and the curvature formula uses two. The rates therefore persist in C^{k-1} and C^{k-2}, respectively. □

Proof of Corollary 1.5. The uniform positive Gram margin and the C^0 metric estimate give

$$(1 - Ch^4)g(v, v) \le g_h(v, v) \le (1 + Ch^4)g(v, v)$$

for every parameter tangent v on K, with a constant independent of the path. Taking square roots, integrating and then taking infima proves the distance comparison, without assuming a minimizing path exists. Integrating the quadratic forms proves the energy comparison.

For the initial-value assertion, take a compact phase neighborhood of the reference trajectories whose base points remain inside the chart. The C^2 metric estimate and uniform inverse Gram bound imply $\Gamma_h - \Gamma = O_{C^1}(h^4)$ there. Thus the geodesic vector fields

$$F_h(\theta, v) = (v, -\Gamma_h(\theta)[v, v])$$

are uniformly Lipschitz and differ by $O(h^4)$. Until a perturbed trajectory leaves this neighborhood, its difference $e(t)$ from the reference trajectory satisfies

$$|e(t)| \le C \int_0^t |e(s)| \, \mathrm{d}s + C\tau h^4.$$

Gronwall's inequality gives $\sup_{[0,\tau]} |e| \le C_\tau h^4$. The reference trajectories have a positive margin inside the chosen neighborhood, so this estimate prevents a first exit for small h and provides existence throughout the interval. Along the paired trajectories, the parallel-transport coefficient matrices $\Gamma_h(\theta_h)[v_h, \cdot\,]$ and $\Gamma(\theta)[v, \cdot\,]$ are bounded and differ by $O(h^4)$. Subtracting their linear matrix ODEs and applying the same integral estimate proves the operator-norm transport bound. These are standard finite-dimensional metric and ODE consequences of the uniform bias correction; they do not identify family distances or geodesics with ambient optimal-transport ones. □

In density coordinates the compact positive construction is explicitly

$$\widehat{C}_{h,f} = T_{h,f} + \frac{h}{3} P_h (I - P_h)(I + P_h)^{-1}(I + P_h^2)^{-1} - \frac{h^2}{6} T_{h,f} D_H T_{h,f}$$

on $L^2_0(f)$, with the bounded sandwich interpretation proved above. The local Hessian of V is the density-specific mobility defect; the scalar extrapolation filter alone does not remove it.

The topology qualifications are essential. The original tensor has an order-two bias, and fourth-order corrected accuracy requires H^6 inputs. No fourth-order full L^2 estimate is possible for these constructions. The compact correction preserves the original tensor's completed tangent form space and square-root range up to equivalent norms, while the Simpson comparison has a noncompact identity floor. Raw two-bandwidth Richardson is positive at constant density; Section 8 proves failure at explicit variable densities. Beyond the restricted analytic construction of Appendix B, effective discretized or sample-based guarantees, ambient vanishing-bandwidth/geodesic approximation, and extension to vanishing densities or boundary-domain fixed costs require separate arguments. The fixed-regularization discretization theory of [11] concerns its stated fixed-regularization regime.

8 Raw two-bandwidth Richardson can be indefinite

The same-bandwidth corrections above should be distinguished from $T_h^{\mathrm{R}} = (4T_h - T_{2h})/3$, where each bandwidth is separately normalized at the same density. The latter operator need not be positive, even for arbitrarily small nonuniformity. The obstruction concerns exact operator ranges rather than the sign of the leading smooth-input bias. Kernel-order and reproducing-kernel Hilbert space inclusion criteria are classical [22]; here the bandwidth-dependent nonlinear normalizers are essential, and ordinary Gaussian bandwidth nesting does not settle the comparison.

Theorem 8.1 (Explicit cross-bandwidth non-domination)**.** *On the circle fix $0 < a < 1$ and put*

$$u(x) = 1 + a\cos(2\pi x), \qquad v_h(x) = H_h u = 1 + ae^{-2\pi^2 h}\cos(2\pi x),$$
$$Z_h = 1 + \tfrac{1}{2}a^2 e^{-2\pi^2 h}, \qquad f_h(x) = u(x)v_h(x)/Z_h.$$

For every sufficiently small $h > 0$, no finite constant C satisfies

$$T_{2h}[f_h] \le CT_h[f_h]$$

on their common ground-state complement. In particular $(4T_h[f_h]-T_{2h}[f_h])/3$ has a strictly negative quadratic-form direction, which may be chosen smooth. These examples approach uniform density in every fixed C^m norm as $a \downarrow 0$. This explicit density depends on its matching bandwidth h. Theorem 8.2 below supplies a separate nonconstructive fixed-density conclusion.

Proof. The family f_h, including $h = 0$, has mass one, is smooth and has uniform positive lower and derivative bounds on compact amplitude ranges $0 \le a \le a_0 < 1$. The baseline therefore supplies both exact scales at h and $2h$ for all sufficiently small h. Both scales in what follows use this SAME density f_h, so $q = \sqrt{f_h}$ and $\Pi = |q\rangle\langle q|$ are common to them.

First record the exact range consequence of a proposed order. For a fixed bandwidth s let $A_s = T_s + \Pi$. Since Q_s fixes q and has spectrum in $[0, \rho_s]$, $\rho_s < 1$, on $q^\perp$, the scalar ratio of A_s to Q_s is one on q and

$$\frac{sp/(1-p^2)}{p} = \frac{s}{1-p^2}$$

on the complement. It is bounded above and below by positive fixed-s constants. Douglas's theorem gives

$$\operatorname{ran} A_s^{1/2} = \operatorname{ran} Q_s^{1/2} = b_s \operatorname{ran} H_{s/2}. \tag{42}$$

For the second equality write $Q_s = B_s B_s^*$ with $B_s = M_{b_s} H_{s/2}$ and use $\operatorname{ran}(B_s B_s^*)^{1/2} = \operatorname{ran} B_s$. These are exact, possibly nonclosed ranges. Adding Π is essential: T_s annihilates q, whereas Q_s fixes it.

If $T_{2h} \le CT_h$, then $A_{2h} \le \max(C, 1)A_h$. A further application of Douglas's theorem and (42) gives

$$b_{2h} \operatorname{ran} H_h \subset b_h \operatorname{ran} H_{h/2}.$$

Since $H_h 1 = 1$, this forces

$$r := b_{2h}/b_h \in \operatorname{ran} H_{h/2}. \tag{43}$$

The first scale has an exact expression. Indeed $w_h = b_h q = u/\sqrt{Z_h}$ satisfies $w_h H_h w_h = f_h$, so uniqueness gives $b_h = \sqrt{u/v_h}$. Write $w_{2h} = b_{2h} q = r w_h$ and set $F = H_{2h}(ru)$. The second scaling equation then gives, on the real circle, the exact identity

$$v_h = rF. \tag{44}$$

All normalizing factors Z_h have canceled.

We give the complex-analytic facts used to contradict this identity. For $s > 0$ and $g \in L^2(\mathbb{T})$, the Fourier series of $H_s g$ has coefficients $\widehat{g}(n)e^{-2\pi^2 s n^2}$ and defines an entire one-periodic function. Cauchy–Schwarz and completing the square in n give

$$|H_s g(z)| \le C_s \|g\|_2 \exp(C_s |z|^2), \qquad z \in \mathbb{C}. \tag{45}$$

The series is locally uniformly convergent, and real-valued boundary data give conjugation symmetry.

We also need the following elementary rigidity fact: a zero-free entire one-periodic function satisfying (45) and positive on the real axis is a positive constant. To prove it, write $g = e^H$ with H entire; this logarithm exists since $\mathbb{C}$ is simply connected. Then $\operatorname{Re} H(z) \le C(1 + |z|^2)$. On $|z| \le 2R$, choose $A_R > 0$ of order $1 + R^2$ such that $\operatorname{Re}(H - H(0)) \le A_R$. The analytic function

$$\frac{H - H(0)}{2A_R - (H - H(0))}$$

has modulus at most one and vanishes at zero. Schwarz's lemma bounds its modulus by $1/2$ on $|z| \le R$, hence $|H - H(0)| \le 2A_R$ there. Cauchy's estimates as $R \to \infty$ make all derivatives of H of order at least three vanish. Thus H is quadratic. Periodicity makes $H(z+1) - H(z)$ a constant in $2\pi i\mathbb{Z}$, so $H(z) = 2\pi ikz + c$ for an integer k. Positivity on the entire real axis forces $k = 0$, proving the fact.

Under (43), the positive real function r has an entire extension with the Gaussian growth above. The same is true of $F = H_{2h}(ru)$, since ru is smooth on the real circle. The identity theorem extends (44) to $\mathbb{C}$. Put $c_h = ae^{-2\pi^2 h} \in (0, 1)$. The zeros of $v_h(z) = 1 + c_h \cos(2\pi z)$ are exactly the simple zeros

$$z = \tfrac{1}{2} + k \pm i\beta, \qquad k \in \mathbb{Z}, \qquad \beta = \frac{\operatorname{arcosh}(1/c_h)}{2\pi} > 0.$$

Every zero of r belongs to this set. Periodicity propagates it along its integer orbit, and conjugation symmetry propagates it to the other orbit. Therefore either r has no zeros or it contains both orbits. In the latter case simplicity and $rF = v_h$ make F zero-free. One of r, F is consequently a zero-free positive real entire periodic function of Gaussian growth, and is constant by the rigidity fact.

If $r = c_0 > 0$ is constant, then $H_h u = c_0^2 H_{2h} u$. Its constant Fourier coefficient gives $c_0^2 = 1$, while its cosine coefficient would require $e^{-2\pi^2 h} = e^{-4\pi^2 h}$, impossible for $h > 0$. If $F = c_0 > 0$ is constant, heat injectivity gives $ru = c_0$. Then $r = v_h/c_0$ and $uv_h = c_0^2$ would be constant, contrary to its nonzero $\cos(4\pi x)$ coefficient $a^2 e^{-2\pi^2 h}/2$. Both alternatives contradict (44). Thus no finite order constant C exists.

For $C = 4$, failure of the order gives a strict negative quadratic-form value of $4T_h - T_{2h}$ on $q^\perp$. Both operators are bounded selfadjoint and preserve real functions. Splitting real and imaginary parts gives a real negative vector. Smooth real functions are dense in L^2, and their projections by J remain smooth because q is smooth. Approximating a strict negative vector by these projected functions preserves its sign. Finally the explicit formula for f_h gives convergence to one in every fixed C^m norm as $a \downarrow 0$; the small-h existence bounds can be chosen uniformly on $0 \le a \le a_0 < 1$. □

The ratio $\langle v, T_{2h} v\rangle / \langle v, T_h v\rangle$ is in fact unbounded on nonzero smooth vectors of the common complement, since every finite upper order fails and T_h is injective there. The negative directions may depend on h and need not remain in a fixed bounded smoothness class; uniform estimates on fixed smooth family tangents are not contradicted. The circle example also embeds in any fixed torus dimension by taking the density independent of the other coordinates and restricting to functions constant in them.

For small amplitude these are small analytic densities as well. With $b = e^{-2\pi^2 h}$ their only nonconstant Fourier modes are

$$f_h = 1 + \theta_1 \cos(2\pi x) + \theta_2 \cos(4\pi x), \quad \theta_1 = \frac{a(1+b)}{1 + a^2 b/2}, \quad \theta_2 = \frac{a^2 b/2}{1 + a^2 b/2}.$$

For example $a = 1/100$ and $\rho = 1/5$ give $\|f_h - 1\|_{B_\rho} \le 6ae^{1/5} + (5a^2/2)e^{2/5} < 1/8$; a small complex parameter thickening retains the strict inequality. This is a full-tangent ordering obstruction, not a

new numerical certificate. This explicit example leaves the positive same-bandwidth constructions intact. The next theorem addresses fixed densities along prescribed sequences, using an additional category argument.

8.1 Fixed densities on prescribed bandwidth sequences

The preceding explicit formula depends on its matching bandwidth. We next obtain a fixed-density conclusion by finite-Fourier approximation and Baire category. Its fixed density is obtained nonconstructively, and its genericity is relative to the following smooth topology. Define the circle Fourier algebra

$$\|g\|_{A_0} = \sum_{m\in\mathbb{Z}} |\widehat{g}(m)|, \qquad \mathcal{U} = \{f \in C^\infty(\mathbb{T};\mathbb{R}) : \textstyle\int f = 1,\ \|f-1\|_{A_0} < 1/8\}.$$

Every member is positive, since $f > 7/8$. We give $\mathcal{U}$ the inherited real C^∞ Fréchet topology.

Theorem 8.2 (Fixed-density cross-band obstruction)**.** *For any prescribed countable sequence $h_j > 0$, a dense residual subset of $\mathcal{U}$ consists of densities f such that, for every j, no finite C satisfies $T_{2h_j}[f] \le CT_{h_j}[f]$ on the common $q_f^\perp$. In particular one fixed such density has a smooth strictly negative raw Richardson direction at every listed bandwidth. If $h_j \to 0$, these fixed densities have no eventual full-tangent positivity of raw Richardson at all sufficiently small bandwidths. The residual set and the chosen density may depend on the whole prescribed sequence; the chosen density does not vary with its index.*

Proof. We first establish exact scaling and continuity throughout $\mathcal{U}$ at every fixed positive bandwidth s. Both H_s and $L_s = (I+H_s)^{-1}$ are contractions on the Banach algebra A_0. Write $f = 1+g$ and

$$r_* = (1-1/\sqrt{2})/2, \qquad z = L_s(g - zH_sz).$$

Because $1/8 + r_*^2 = r_*$, the right side maps the closed radius-r_* ball into itself and has Lipschitz constant $2r_* < 1$. Banach contraction gives a real fixed point and $w = 1+z > 0$, with

$$wH_sw = f. \tag{46}$$

It is smooth: H_sw is positive and smooth for fixed $s > 0$, and $w = f/(H_sw)$. This gives the exact scale $b_s = w/\sqrt{f}$. Uniqueness among positive scales follows directly: if $w_iH_sw_i = f$, their difference v satisfies $H_sv = -fv/(w_1w_2)$; its L^2 pairing contradicts positivity of heat unless $v = 0$.

The contraction also gives

$$\|w(f) - w(\widetilde{f})\|_{A_0} \le (1-2r_*)^{-1}\|f-\widetilde{f}\|_{A_0}.$$

The A_0 norm is continuous in the smooth topology, for example by Cauchy–Schwarz and a fixed H^1 norm. Fixed positive-time heat maps A_0 continuously into every C^m. Equation (46) then upgrades the scale's continuity to the Fréchet topology. The same follows for b_s, q_f and Π_f.

For completeness there is a strict gap at this fixed s without a small-s restriction. The density-conjugate Markov operator is $P_sv = H_s(wv)/(H_sw)$, reversible with respect to $f\,dx$. It is a contraction, and $\langle v, P_sv\rangle_f = \langle wv, H_s(wv)\rangle \ge 0$. Let $\kappa_s = \min_{x\in\mathbb{T}} k_s(x) > 0$. Its Dirichlet form satisfies

$$\langle v, (I-P_s)v\rangle_f = \frac{1}{2}\iint w(x)w(y)k_s(x-y)(v(x)-v(y))^2\,dx\,dy \ge (1-r_*)^2\kappa_s \operatorname{Var}_{dx}(v).$$

For f-centered v, $\|v\|_f^2 = \operatorname{Var}_f(v) \le \|f\|_\infty \operatorname{Var}_{dx}(v) \le (9/8)\operatorname{Var}_{dx}(v)$. Thus the complement gap is at least $\delta_s = (1-r_*)^2\kappa_s/(9/8) > 0$. Compactness and heat injectivity give the other spectral properties used earlier. In particular the full-space identity

$$T_s = sQ_s(I - Q_s^2 + \Pi_f)^{-1}J_f$$

is well-defined and continuous in f in L^2 operator norm, by norm continuity of Q_s, Π_f and bounded inverses. Constants may depend on this fixed s; no bandwidth-uniform inverse bound is needed here.

We now extend the range contradiction to any real strictly positive nonconstant trigonometric polynomial u, of degree $d \geq 1$. Put $P = H_h u$, $Z = \int uP$ and $f = uP/Z$, and suppose $f \in \mathcal{U}$. If any finite order $T_{2h} \leq CT_h$ held, the exact $T_s + \Pi_f$/Douglas range argument from the preceding proof would give

$$r = b_{2h}/b_h \in \operatorname{ran} H_{h/2}, \qquad F = H_{2h}(ru), \qquad rF = P.$$

Both factors are positive on the real axis, entire, one-periodic and real-symmetric, with at most quadratic-exponential growth.

Here is the required finite-factor argument with multiplicities. Write $P(z) = W^{-d}p(W)$, $W = e^{2\pi i z}$, where p has degree $2d$ and nonzero constant and leading coefficients. Positivity on the real axis excludes unit-circle zeros. Its zeros therefore form finitely many conjugate pairs of integer orbits $\alpha_i \pm i\beta_i + \mathbb{Z}$, $\beta_i > 0$, with equal paired multiplicities. A factor r can have only a subcollection of these orbits, also conjugate-paired. If its selected multiplicities are m_i, form

$$R(z) = \prod_i [\cosh(2\pi\beta_i) - \cos(2\pi(z - \alpha_i))]^{m_i}.$$

This is positive on the real axis and has exactly the zeros of r, with their multiplicities. The quotient r/R is entire, zero-free, positive-real and periodic. It retains quadratic-exponential growth: outside a fixed horizontal strip, each displayed factor has a lower bound $ce^{2\pi|\operatorname{Im} z|}$; inside the strip the quotient, with removable zeros filled, is bounded on a compact period cell. The elementary zero-free rigidity argument in the preceding proof therefore makes r/R constant. Hence r is a finite trigonometric polynomial, say of degree D.

Real nonzero trigonometric polynomials have symmetric extreme degrees, and products add degrees because their highest coefficients multiply to a nonzero coefficient. Heat preserves every nonzero Fourier mode. Consequently

$$\deg P = d, \qquad \deg F = \deg H_{2h}(ru) = D + d, \qquad \deg(rF) = 2D + d.$$

Thus $D = 0$. The resulting $r = c > 0$ would imply $H_h u = c^2 H_{2h} u$; its positive constant Fourier coefficient gives $c^2 = 1$, while any nonzero nonconstant coefficient requires unequal positive-time heat multipliers to agree. This is impossible. Every such manufactured finite-Fourier density therefore fails every finite cross-band order.

These manufactured densities are dense in $\mathcal{U}$ at each fixed h. Indeed, for any $f \in \mathcal{U}$, its exact w_h is positive and smooth. Real Fourier truncations give positive nonconstant trigonometric polynomials $u_m \to w_h$ in C^∞, adding a vanishing nonzero cosine if the scale is constant. Then

$$f_m = \frac{u_m H_h u_m}{\int u_m H_h u_m} \longrightarrow f \quad \text{in } C^\infty,$$

and $f_m \in \mathcal{U}$ eventually. Each f_m has the non-domination property just proved.

For a positive integer N, let $\mathcal{O}_{h,N}$ consist of those $f \in \mathcal{U}$ for which a real smooth test v satisfies

$$\langle v, (T_{2h}[f] - NT_h[f])v \rangle > 0.$$

It is dense by the manufactured examples and smooth approximation of a strict bounded-form witness. It is open by the fixed-bandwidth operator-norm continuity: keep the same test near f.

The moving complement causes no loss, since replacing v by $J_f v$ preserves its quadratic value and smoothness.

The real smooth integral-one affine space is Fréchet and complete; $\mathcal{U}$ is a nonempty open subset because $\|\cdot\|_{A_0}$ is continuous there. It is therefore Baire. For the prescribed sequence,

$$\bigcap_{j\geq 1}\bigcap_{N\geq 1}\mathcal{O}_{h_j,N}$$

is dense residual in $\mathcal{U}$. A density in this intersection fails every integer upper bound at each h_j, hence every finite upper bound by choosing an integer $N \geq C$. Taking $N = 4$ gives the strict negative raw Richardson direction, with smooth projection as above. If $h_j \to 0$, any proposed eventual-positivity threshold is contradicted by some listed bandwidth. □

This fixed-density conclusion is relative to $\mathcal{U}$ with its smooth Fréchet topology. It yields such densities in every open neighborhood of the uniform density contained in $\mathcal{U}$, but not an explicit coefficient formula for one fixed density, an analytic-category statement, a claim for every real bandwidth, or a claim about every positive density. The negative directions can vary with bandwidth and need not have bounded smooth norms. The positive same-bandwidth constructions are unaffected.

9 A positive higher-order hierarchy

The original finite-bandwidth tensor also admits a positive correction of unbounded smooth-input order without computing further modified-generator coefficients. The inverse polynomial used here is classical hyperpower iteration; its cubic instance and general residual-power mechanism occur in Altman [20]. The assertions below concern its realization on the compact heat-kernel tensor, including its domains, moving kernel, parameter derivatives and sharp Sobolev rates. No norm-small residual assumption is needed for the positivity argument.

Use the standing fixed-dimensional smooth positive density-family model, and keep S, G, T_h, Π, J and their full-space zero extensions. Let $\ell \geq 1$ be a fixed odd integer. Define

$$\mathcal{C}_{h,\ell} = \sum_{j=0}^{\ell-1} T_h (I - 2ST_h)^j. \tag{47}$$

Every application of the local differential operator S follows an application of T_h; the smoothing lemma makes each ST_h bounded. Thus this definition uses finite compositions of the original finite-h tensor and a known second-order differential operator. It does not require the limiting inverse G, a square-root computation, or a new density-bias coefficient.

Theorem 9.1 (Positive odd hyperpowers)**.** *For each fixed odd ℓ, the operator $\mathcal{C}_{h,\ell}$ is compact and selfadjoint, vanishes on q, and is strictly positive on every nonzero vector in $q^\perp$. Uniformly for sufficiently small h there is $B_\ell < \infty$ such that*

$$\tfrac{1}{2} T_h \leq \mathcal{C}_{h,\ell} \leq B_\ell T_h. \tag{48}$$

It has the same completed tangent form space and square-root range as T_h, with equivalent respective norms. It has no positive uniform L^2 lower bound. For every fixed parameter order k and $0 \leq r \leq 4\ell - 2$,

$$\|\mathcal{C}_{h,\ell} - G\|_{C^k(H^r \to L^2)} \leq C_{k,r,\ell} h^{1+r/2}. \tag{49}$$

In particular the endpoint is $O_{C^k}(h^{2\ell}) : H^{4\ell-2} \to L^2$. Every exponent is sharp already at constant density. The operator $\mathcal{C}_{h,\ell} : H^s \to H^{s+2}$ is uniformly bounded, including all fixed parameter derivatives, for each nonnegative integer s.

The case $\ell = 1$ is the original tensor. The first new member is

$$\mathcal{C}_{h,3} = 3T_h - 6T_hST_h + 4T_hST_hST_h, \tag{50}$$

with sixth-order accuracy on H^{10} inputs. Odd lengths $3, 5, 7, \ldots$ give orders $6, 10, 14, \ldots$ with the stated increasing regularity loss. This is an unbounded hierarchy of orders, not a claim that this formula has every even order or a better full L^2 rate. All constants and regularity requirements may depend on the fixed length ℓ.

Proof. We first lift the existing strong error estimate to higher output Sobolev norms. Put $\mathcal{E}_h = T_h - G$. For every nonnegative integer s, the baseline estimates imply

$$\mathcal{E}_h = O_{C^k}(h^2) : H^{s+2} \to H^s. \tag{51}$$

Here is a fixed-space justification. Adjoin small spatial-translation parameters a to the density family, $f_{\theta,a}(x) = f_\theta(x + a)$, and write $U_a u(x) = u(x + a)$. Heat translation invariance and uniqueness of the positive symmetric scale show that Q_h, T_h, S, G, Π, J are all conjugated by U_a. Thus a translation derivative of $\mathcal{E}_h$ is $[\partial_{x_i}, \mathcal{E}_h]$, and repeated translation derivatives give iterated commutators. Apply the established $O_{C^k}(h^2) : H^2 \to L^2$ estimate to this augmented family. Commuting ∂_x^α, $|\alpha| \le s$, through $\mathcal{E}_h$ gives

$$\partial_x^\alpha \mathcal{E}_h u = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \mathrm{ad}^\beta_{\partial_x}(\mathcal{E}_h) \partial_x^{\alpha-\beta} u.$$

Each term is bounded by $Ch^2 \|u\|_{H^{s+2}}$. Density proves (51), with mixed original-parameter jets as well. Translation conjugacy includes the moving projection; no parameter-dependent Sobolev space or uncontrolled differentiated asymptotic is used.

The smoothing lemma and elliptic regularity for $S + \Pi$ also give uniform $T_h, G, \mathcal{E}_h : H^s \to H^{s+2}$ bounds for nonnegative integer s, with fixed parameter jets. Consequently ST_h and $S\mathcal{E}_h$ are uniformly bounded on every such H^s.

For positivity, define the form operator

$$\mathcal{M}_h = 2T_h^{1/2} S T_h^{1/2}.$$

It is a bounded nonnegative selfadjoint operator on L^2, with a uniform norm bound K. Indeed the same half-power argument as in the relative-form lemma gives $T_h^{1/2} : L^2 \to H^1$: selfadjoint duality of $T_h : L^2 \to H^2$ gives the compatible $H^{-2} \to L^2$ extension, interpolation gives $H^{-1} \to H^1$, and positivity yields $\|T_h^{1/2} v\|_2^2 \le C \|v\|_{H^{-1}}^2$ for smooth v. Density and duality prove the stated half-power map. The nonnegative bounded form $S : H^1 \to H^{-1}$ then defines $\mathcal{M}_h$ with its uniform bound. These two negative-index spaces have been explicitly derived; no arbitrary negative-order smoothing is presumed.

Let $P_\ell(x) = \sum_{j=0}^{\ell-1} (1 - x)^j$. Intertwining the bounded form identity $\mathcal{M}_h T_h^{1/2} = 2T_h^{1/2} S T_h$ gives

$$\mathcal{C}_{h,\ell} = T_h^{1/2} P_\ell(\mathcal{M}_h) T_h^{1/2}. \tag{52}$$

For $x > 0$,

$$P_\ell(x) = \frac{1-(1-x)^\ell}{x}, \qquad P_\ell(0) = \ell.$$

Since ℓ is odd, this polynomial is at least one on $[0,1]$, at least $1/2$ on $[1,2]$, and at least one on $[2,\infty)$. For the last interval use $(x-1)^\ell \geq x-1$; for the middle interval use $1+(x-1)^\ell \geq 1$. On $[0,K]$ its upper bound is $\ell(1+K)^{\ell-1}$. Functional calculus proves (48) with that choice of B_ℓ. For $\ell = 3$, $P_3(x) = (x-3/2)^2 + 3/4$ improves the lower constant to $3/4$.

Formula (47) makes compactness explicit: its left T_h is compact and each remaining factor is bounded. Equation (52) proves selfadjointness and positivity, and injectivity of T_h on $q^\perp$ makes that positivity strict there. The form-completion consequence follows by the identity map, and the square-root-range assertion follows from Douglas's theorem in both directions. These two constructions are not identified with each other. Compactness excludes a positive infinite-dimensional L^2 floor. All fixed parameter bounds and the claimed $H^s \to H^{s+2}$ smoothing follow directly by product rules in the finite-polynomial formula, not by differentiating a square root.

The exact error identity accounts for the common kernel. Put $\mathcal{D}_h = I - 2ST_h$. Since $2SG = 2GS = J$ and $\mathcal{E}_h q = 0$, on the full space

$$\mathcal{D}_h = \Pi - 2S\mathcal{E}_h, \quad \Pi S\mathcal{E}_h = S\mathcal{E}_h\Pi = 0, \quad \mathcal{D}_h^\ell = \Pi + (-2S\mathcal{E}_h)^\ell.$$

Geometric telescoping and multiplication by G give the bounded fixed-space identity

$$\mathcal{C}_{h,\ell} - G = 2^{\ell-1}\mathcal{E}_h(S\mathcal{E}_h)^{\ell-1}. \tag{53}$$

These identities may first be checked on smooth vectors and extended using the stated smoothing. There is no commutation assumption on S and T_h, nor any residual contraction assumption. All artificial ground-state terms vanish after multiplication by G.

Apply (51) successively from the right in (53). Each of the ℓ error factors contributes h^2 and loses two derivatives, while each of the $\ell-1$ differential factors loses two more. The total loss is $4\ell-2$, proving the $h^{2\ell}$ endpoint. Every parameter-derivative product has the same domain chain; derivatives of S have no higher order. At the L^2 endpoint use the leftmost $\mathcal{E}_h = O_{C^k}(h)$ and the uniform bounds for $S\mathcal{E}_h$. Interpolation of each fixed parameter jet proves (49).

At constant density, set $t = h\lambda$ for a nonzero eigenvalue λ of $S = -\Delta/2$. The scalar multipliers of $\mathcal{M}_h$ and of the error are respectively

$$\frac{t}{\sinh t}, \qquad \frac{1}{2\lambda}\left(\frac{t}{\sinh t} - 1\right)^\ell = hF_\ell(t), \quad F_\ell(t) = \frac{(t/\sinh t - 1)^\ell}{2t} < 0.$$

For a fixed nonzero mode the leading term is

$$-\frac{h^{2\ell}\lambda^{2\ell-1}}{2\,6^\ell} + O(h^{2\ell+2}\lambda^{2\ell+1});$$

for $\ell = 3$ its coefficient is $-\lambda^5/432$. For every r in the asserted range, choose nearest lattice frequencies $n_h = (\text{round}\,\sqrt{t_0/(2\pi^2 h)}, 0, \ldots, 0)$ with $t_0 > 0$. Then $h\lambda_{n_h} \to t_0$ and the H^r norm of the mode is comparable to $h^{-r/2}$. The nonzero $F_\ell(t_0)$ gives the matching lower bound $ch^{1+r/2}$. Zeroth parameter derivatives already establish sharpness of the C^k estimate. □

Corollary 9.2 (Finite-family higher-order geometry)**.** *Under the immersion and compactness hypotheses of Corollary 1.4, the pullback tensor of $\mathcal{C}_{h,\ell}$ is a positive metric and differs from g^G by $O_{C^k}(h^{2\ell})$. For $k \geq 2$, its Christoffel symbols and intrinsic curvature components differ at that order in C^{k-1} and C^{k-2}. The restricted-distance, energy, compact-chart geodesic initial-value and parallel-transport conclusions of Corollary 1.5 hold with h^4 replaced by $h^{2\ell}$ under the same respective hypotheses.*

Proof. The family tangents have every fixed required spatial and parameter regularity, so contraction of (49) at its smooth endpoint gives the metric estimate. The limiting Gram margin and immersion give uniform invertibility for small h. Matrix inversion, the coordinate formulas and the compact-chart Gronwall/first-exit argument used earlier prove the stated consequences with their original derivative losses and path/trajectory restrictions. □

This hierarchy concerns exact population operators at each fixed length. It does not provide a growing-ℓ algorithm, numerical precision or runtime bound, an empirical theorem, an ambient metric or curvature limit, or positivity of the separately normalized raw two-bandwidth Richardson tensor. It preserves the sharp first-order full L^2 scale.

A Continuum residual certificates for a nonconstant family

We give a finite continuum certificate for metric values, rather than using differences between numerical grids as an error estimate. General normalized-diffusion refinement has established precedents [14]; the calculation below concerns the corrected inverse tensor and its differential sandwich. The certificate concerns metric values.

A.1 An exact scale and rational residuals

On the circle put

$$w_\theta = 1 + \theta_1 \cos(2\pi x) + \theta_2 \cos(4\pi x), \qquad v = H_h w_\theta,$$
$$Z = \int w_\theta v, \qquad f_{h,\theta} = w_\theta v / Z, \qquad b = \sqrt{w_\theta / v}.$$

For $|\theta_1| + |\theta_2| < 1$, these are positive and $bH_h(b\sqrt{f}) = \sqrt{f}$ exactly. This is the classical symmetric Schrödinger endpoint product construction [15]. Its Markov operator and actual parameter scores are

$$Pu = H_h(wu)/v, \qquad Y_i = \partial_i \log f = w_i/w + v_i/v - Z_i/Z.$$

Here and below $w = w_\theta$. The density varies with h: comparisons are always with $G[f_{h,\theta}]$, not $G[f_{0,\theta}]$. Joint smoothness and positivity on compact parameter sets allow the uniform analytic estimates above to apply along this family.

Let $\mathcal{R}_z = (I - zP)^{-1}$ for $z = -1, i$, and let $\mathcal{R}_1$ be the inverse on $L^2_0(f)$. Scalar functional calculus gives

$$T = \frac{h}{2}(\mathcal{R}_1 - \mathcal{R}_{-1}), \qquad C_{\mathrm{base}} = \frac{h}{6}(3\mathcal{R}_1 - 5\mathcal{R}_{-1} + 2\,\mathrm{Re}\,\mathcal{R}_i), \tag{54}$$

where C_{base} is the scalar part of $\widehat{C}_{h,f}$. In particular

$$g^{\widehat{C}}_{ij} = \langle Y_i, C_{\mathrm{base}} Y_j \rangle_f - \frac{h^2}{6} \int f V''(TY_i)'(TY_j)'. \tag{55}$$

The polynomial $F_i = wvY_i = w_i v + w v_i - (Z_i/Z)wv$ has degree at most four and zero mean. For any trigonometric polynomial s, the trial $x = s/w$ has the exact continuum residual

$$Y_i - (I - zP)x = E/(wv), \qquad E = F_i - vs + zwH_h s. \tag{56}$$

Thus every residual Fourier coefficient is obtained by finite, unaliasing polynomial operations. All generated modes are retained. Project a trial to $L^2_0(f)$ by replacing s with $s - cw$, $c = \int vs/Z$. At $z = 1$ this leaves the residual unchanged; at the other values it is recomputed after projection.

A.2 A computable continuum error bound

Write

$$w_0 = 1 - |\theta_1| - |\theta_2|, \quad v_0 = 1 - |\theta_1|e^{-2\pi^2 h} - |\theta_2|e^{-8\pi^2 h}, \quad w_1 = 1 + |\theta_1| + |\theta_2|.$$

Reversibility and heat Poincaré comparison give the whole-space gap

$$I - P \geq \delta I \text{ on } L_0^2(f), \qquad \delta = (w_0/w_1)^2(1 - e^{-2\pi^2 h}). \tag{57}$$

Indeed the Dirichlet form is $(2Z)^{-1}\iint w(x)w(y)k_h(x-y)(u(x)-u(y))^2$. It bounds w_0^2/Z times the unweighted heat form, whereas $\int fu^2 \leq (w_1^2/Z)\operatorname{Var}_{dx}(u)$ for $\int fu = 0$. This proves (57), including the unresolved Fourier complement.

For (56), Parseval gives

$$\|r\|_f \leq \frac{\|E\|_2}{\sqrt{Z w_0 v_0}}, \qquad \|r'\|_f \leq \frac{\|E'\|_2 + L\|E\|_2}{\sqrt{Z w_0 v_0}}, \quad L = \frac{\|w'\|_\infty}{w_0} + \frac{\|v'\|_\infty}{v_0}. \tag{58}$$

Use the sum of squared real and imaginary polynomial norms for complex E. No discrete residual is substituted for these continuum norms. The following estimate pays for the differential sandwich.

Lemma A.1 (Explicit residual smoothing)**.** *Set*

$$\mathcal{F} = 2(\|v'\|_\infty/v_0)^2 + \frac{2w_1}{v_0 h}, \quad K_-^2 = \frac{2\mathcal{F}h^2}{\delta} + \frac{h^2}{\delta^2}, \quad K_+ = h\sqrt{1 + 2\mathcal{F}}.$$

For $r \in L_0^2(f)$,

$$\|hP\mathcal{R}_1 r\|_{H^1(f)} \leq K_-\|r\|_f, \qquad \|hP\mathcal{R}_{-1} r\|_{H^1(f)} \leq K_+\|r\|_f.$$

The constant K_- remains bounded as $h \downarrow 0$ on compact positive parameter sets.

Proof. The Markov kernel is $p(x,y) = k_h(x-y)w(y)/v(x)$. The wrapped Gaussian score satisfies $\int (k_h')^2/k_h \leq 1/h$: Cauchy–Schwarz on its Gaussian translates and unfolding a period reduce this to Gaussian Fisher information. Consequently $\int p|\partial_x \log p|^2 \leq \mathcal{F}$. Subtracting $u(x)$ inside the differentiated Markov integral and using reversibility yields

$$\|(Pu)'\|_f^2 \leq 2\mathcal{F}\langle u, (I-P)u\rangle_f.$$

Apply this to $u = \mathcal{R}_1 r$, using $\langle u, (I-P)u\rangle_f \leq \delta^{-1}\|r\|_f^2$ and $\|u\|_f \leq \delta^{-1}\|r\|_f$. For $u = \mathcal{R}_{-1} r$, both $\|u\|_f \leq \|r\|_f$ and $\langle u, (I-P)u\rangle_f \leq \|r\|_f^2$ follow from $0 \leq P \leq I$. These prove the two estimates. Since $\delta \asymp h$ and $\mathcal{F} = O(h^{-1})$, the final assertion follows. □

For each column let $r_{1,j}, r_{-1,j}, r_{i,j}$ be the residuals of its three trial inverses, and put $U_j = (h/2)(x_{1,j} - x_{-1,j})$. Define the computable bounds

$$e_j = \tfrac{1}{2}[h(\|r_{1,j}\|_{H^1(f)} + \|r_{-1,j}\|_{H^1(f)}) + K_-\|r_{1,j}\|_f + K_+\|r_{-1,j}\|_f],$$
$$d_j = \tfrac{h}{6}(3\|r_{1,j}\|_f/\delta + 5\|r_{-1,j}\|_f + 2\|r_{i,j}\|_f).$$

Then $\|TY_j - U_j\|_{H^1(f)} \leq e_j$ and the error in a symmetrized trial value of (55) is at most

$$\frac{\|Y_i\|_f d_j + \|Y_j\|_f d_i}{2} + \frac{h^2}{6}\|V''\|_\infty(e_i\|U_j'\|_f + e_j\|U_i'\|_f + e_i e_j). \tag{59}$$

For the first claim use $\mathcal{R}_1 = I + P\mathcal{R}_1$ and $\mathcal{R}_{-1} = I - P\mathcal{R}_{-1}$; retaining the identity terms accounts for the residual derivatives. The base error uses inverse norms $\delta^{-1}, 1, 1$ in (54). Expanding the last integral in (55) proves the remaining terms of (59). This is a direct value certificate; no additional adjoint solve is needed.

A.3 A finite analytic certificate protocol

The preceding formulas specify a finite continuum certificate for any chosen trigonometric inverse trials. Here are the inputs and all the remaining integration steps, without assuming an accompanying trial file. Fix $h > 0$ and real θ with $|\theta_1| + |\theta_2| < 1$. Choose finite even trigonometric polynomials $s_{z,j}$ for $z = 1, -1, i$ and $j = 1, 2$, giving rational real and imaginary coefficients. Form w, v, Z, F_j from their displayed exact formulas, project every trial to invariant mean zero as above, and form $E = F_j - vs + zwH_h s$ by full finite Fourier convolution. Retain every generated frequency: no collocation or aliased residual replaces E. Parseval and the explicit residual-smoothing lemma give all e_j, d_j and the continuum budget (59). The trial choice affects the budget but is not presumed accurate.

For example the concrete input $(\theta_1, \theta_2) = (1/5, 1/10)$ admits the fixed integration strip $|\operatorname{Im} x| \le a = 1/16$, since

$$|w| \ge 1 - \tfrac{1}{5}\cosh(2\pi a) - \tfrac{1}{10}\cosh(4\pi a) > 0;$$

the analogous lower bound for v is larger. For any finite polynomial s, the weighted sum $\sum_n |s_n| e^{2\pi a|n|}$ bounds its strip supremum; inserting $(2\pi|n|)^j$ gives the corresponding derivative bound. Together with these strict reciprocal margins, finite products and quotients bound each metric integrand by an explicit strip constant M. The periodic trapezoidal rule with J equally spaced nodes then has error at most $2M/(e^{2\pi aJ} - 1)$, by contour-shift Fourier bounds and summing nonzero alias frequencies. Add this integration radius to (59). When evaluating V'' set $l = (w'/w + v'/v)/2$ and use $V'' = l'''/2 + (l')^2 + ll''$.

For a separate limiting comparison, the odd periodic primitive B_i of $\partial_i f$ gives $f\phi_i' = -B_i$: parity makes the constant flux vanish. Hence $g_{ij}^G = \int B_i B_j / f$, with its own reciprocal margin and trapezoidal enclosure. Heat multipliers, reciprocal factors, all trial coefficients and quadrature values must be evaluated exactly or enclosed with directed rounding; finite arithmetic uncertainty is added to the two analytic budgets. Interval endpoints can be rendered outward using rational bounds and integer division. Symmetrize the matrix and apply any desired explicit Gram-margin test to the resulting intervals. This is a complete a posteriori recipe conditional on its chosen finite trial inputs, not a claim of particular numerical metric values or of a computed whole-parameter-box certificate. The universal convergence and positivity proofs above do not depend on executing it.

B Prescribed analytic densities and logarithmic resolution

The preceding example has an h-dependent density and an exact scale. We now include normalization for a prescribed density which stays fixed as $h \downarrow 0$. The smallness and analyticity assumptions below are substantial. Exponential spectral approximation and uniform nonlocal limits are established principles [16]; bandwidth-independent normalization iteration also predates this work [2]. The result here concerns their specific integration with the corrected inverse tensor and its differential term. At fixed regularization, Sinkhorn information-matrix approximation already has logarithmic tolerance dependence [17]; band-limited heat-kernel transport with complexity estimates is also established [18]. We control the joint vanishing-bandwidth resolution of a particular corrected local tensor, rather than uniformly approximating the entire narrow heat kernel with logarithmic bandwidth.

For $n \in \mathbb{Z}^d$ write $|n|_1 = \sum_j |n_j|$, and define the Banach Fourier algebras

$$\|u\|_{A_\rho} = \sum_n |\widehat{u}(n)| e^{\rho|n|_1}, \qquad \|u\|_{B_\rho} = \sum_n (1 + 2|n|_1)|\widehat{u}(n)| e^{\rho|n|_1}.$$

Products are contractive in these norms: both weights are submultiplicative. Fourier projections and H_h are contractions. If $0 < \rho' < \rho$ and $|\alpha| = j \geq 1$, then

$$\|\partial_x^\alpha u\|_{A_{\rho'}} \leq (2\pi)^j \left(\frac{j}{e(\rho - \rho')}\right)^j \|u\|_{A_\rho}. \tag{60}$$

It follows by maximizing $t^j e^{-(\rho-\rho')t}$.

Theorem B.1 (Fixed prescribed densities)**.** *Fix $d \geq 1$, $h_0 = 1/100$, $\rho, \eta > 0$ and a compact real parameter box $K \subset \mathbb{R}^p$. Suppose $f_\theta = 1 + g_\theta$ has integral one, is independent of h, and extends holomorphically to the complex polydisc neighborhood K_η with*

$$\sup_{\theta \in K_\eta} \|g_\theta\|_{B_\rho} \leq 1/8. \tag{61}$$

For real parameters assume real Fourier symmetry. For every fixed integer $k \geq 0$, there is a symmetric finite matrix construction M_h with

$$\|M_h - g^{\widehat{C}_h[f]}\|_{C^k(K)} \leq h^4, \qquad \|M_h - g^{G[f]}\|_{C^k(K)} \leq C_k h^4$$

for $0 < h \leq h_0$, in the entrywise matrix norm. The first bound refers to exact evaluation of the finite construction; a separately enclosed evaluation error at most h^4 changes it to $2h^4$. The Fourier numerator degrees in each coordinate, normalization and root iteration counts, and quadrature resolution in each coordinate are $O_k(1+\log(1/h))$. Thus total retained modes and quadrature nodes are $O_k((1 + \log(1/h))^d)$, with constants depending on d.

Finite execution requires exact or certified input Fourier coefficients on K_η, with the norm/tail information in (61)*; nonpolynomial inputs are not assumed to arrive with a free coefficient oracle.*

We prove the theorem and give sufficient integer choices. The proof separates normalization, the shrinking inverse, density/score changes, root truncation and integration. In particular it does not replace normalization at f silently by normalization at a different density.

B.1 Normalization and its finite approximation

Set

$$r = \frac{1 - 1/\sqrt{2}}{2}, \quad \lambda = 2r < 1, \quad L_h = (I + H_h)^{-1}.$$

The Fourier multiplier of L_h has modulus at most one, so it is a contraction on both algebras. The map

$$\Phi_h(u) = L_h(g - uH_h u) \tag{62}$$

preserves the closed B_ρ ball of radius r, since $1/8 + r^2 = r$, and is λ-Lipschitz there. Its unique fixed point is holomorphic in parameters by uniform holomorphic iteration. Put

$$w = 1 + u_h, \quad v = H_h w, \quad wv = f, \quad Z = \int wv = 1.$$

For real parameters $w, v > 0$, and $b = \sqrt{w/v}$ is the exact scale. Thus this construction normalizes the prescribed fixed density.

Let P_N retain $|n|_\infty \leq N$, and form

$$t_0 = 0, \qquad t_{j+1} = \mathsf{P}_N \Phi_h(t_j). \tag{63}$$

Every iterate has degree at most N in each coordinate and norm at most r. Comparing the exact and projected fixed points and then iterating the contraction gives, for $\rho_1 = 3\rho/4$,

$$\|u_h - t_L\|_{B_{\rho_1}} \le \varepsilon_N := \frac{re^{-(\rho-\rho_1)N}}{1-\lambda} + r\lambda^L. \tag{64}$$

Indeed the exact fixed-point tail is at most $re^{-(\rho-\rho_1)N}$; the remaining difference is absorbed by $1-\lambda$. The iteration error from zero is at most $r\lambda^L$. Only the retained input coefficients enter (63); certified coefficient uncertainty adds its separately enclosed budget.

Write $w_t = 1 + t_L$, $v_t = H_h w_t$, $Z_t = \int w_t v_t$ and $f_t = w_t v_t / Z_t$. Generally $f_t \ne f$ and $Z_t \ne 1$. Their changes will be paid for below. On $K_{3\eta/4}$, parameter Cauchy estimates give

$$\|w_i\|_{B_{\rho_1}}, \|(w_t)_i\|_{B_{\rho_1}} \le B_0 := 4r/\eta, \qquad \|w_i - (w_t)_i\|_{B_{\rho_1}} \le 4\varepsilon_N/\eta. \tag{65}$$

B.2 A uniform scaled inverse on a fixed complex neighborhood

For this step let $w = 1 + u$ be any complex element with $\|u\|_{B_{\rho_1}} \le R = 1/5$. Set $v = H_h w$ and $Z = \int wv$. With $U = 1 + R$, $z_0 = 1 - 2R - R^2$, we have

$$\|1/w\|_{A_{\rho_1}}, \|1/v\|_{A_{\rho_1}} \le (1-R)^{-1}, \qquad |Z| \ge z_0 > 0.$$

These follow from geometric series and $|Z - 1| \le 2R + R^2$.

The transformed root equation is again

$$\mathcal{L}_z(w)s = vs - zwH_h s = F, \qquad x = s/w, \quad F = wvY, \quad z \in \{1, -1, i, -i\}.$$

Put $e_n = e^{-2\pi^2 h|n|_2^2}$ and $\mathcal{D}_z = \mathrm{diag}(1 - ze_n)$. For $z = 1$ omit the zero Fourier row and column and choose $s_0 = 0$. The full zero row vanishes identically, and $F_0 = 0$ is exactly compatibility. Since $|w_0| \ge 1 - R$, this gauge is transverse to the true kernel $\mathrm{span}(w)$. Write $\mathcal{L}_z = \mathcal{D}_z(I + \mathcal{E}_z(w))$. The constant perturbation u_0 is retained in $\mathcal{E}_z$. For $n = m + a \ne 0$,

$$\frac{|e_a - e_m|}{1 - e_n} \le \max\left(1, \frac{\big||a|_2^2 - |m|_2^2\big|}{|n|_2^2}\right) \le 1 + 2|a|_1. \tag{66}$$

For the first inequality use $|e^{-A} - e^{-B}| \le 1 - e^{-|A-B|}$ and $(1 - e^{-at})/(1 - e^{-as}) \le \max(1, t/s)$. For the second use $m = n - a$ and $|n|_2 \ge 1$. Weighted column summation therefore gives

$$\|\mathcal{E}_1\|_{A_{\rho_1} \to A_{\rho_1}} \le R, \qquad \|\mathcal{E}_z\|_{A_{\rho_1} \to A_{\rho_1}} \le 2R \quad (z = -1, i, -i). \tag{67}$$

The latter uses $|1 - ze_n| \ge 1$ and $|e_a - ze_m| \le 2$. The B norm controls coefficients; the inverse bounds are on A, not on B with an unaccounted shift weight.

Set $c_0 = (1 - e^{-2\pi^2 h_0})/h_0$. Then

$$\|h\mathcal{D}_1^{-1}\| \le c_0^{-1}, \quad \|h\mathcal{L}_1^{-1}\| \le [c_0(1-R)]^{-1}, \quad \|\mathcal{L}_z^{-1}\| \le (1-2R)^{-1} \ (z = -1, i, -i).$$

To select invariant mean zero, replace s/w by $s/w - \int vs/Z$. This map from s has norm at most

$$C_P = (1-R)^{-1} + U/z_0. \tag{68}$$

All estimates hold on the same complex parameter neighborhood. The map $w \mapsto \mathcal{E}_z(w)$ is linear, with difference norm at most $\|w - \widetilde{w}\|_{B_{\rho_1}}$ for $z = 1$, and twice that bound for the stable roots. The

resolvent identity for $I+\mathcal{E}$ therefore gives uniform Lipschitz bounds for $h\mathcal{L}_1^{-1}$. For example, at fixed compatible F its difference is at most

$$\frac{\|w-\widetilde{w}\|_{B_{\rho_1}}\|F\|_{A_{\rho_1}}}{c_0(1-R)^2}.$$

This cancellation avoids an unnecessary h^{-1} normalization-error loss. It also supplies the complex bounds needed for parameter differentiation; a real spectral gap alone would not suffice.

B.3 An explicit stability and integration ledger

We give conservative constants to make every cutoff below definite. For w_i of norm one, the actual score forcing is

$$Y_i = w_i/w + H_h w_i/v - Z_i/Z, \qquad F_i = w_i v + w H_h w_i - (Z_i/Z)wv, \quad |Z_i| \le 2U\|w_i\|_{A_{\rho_1}}.$$

Define

$$K_f = U^2/z_0, \qquad K_Y = 2/(1-R) + 2U/z_0,$$
$$K_F = 2U(1+U^2/z_0), \qquad K_X = C_P K_F \max\{[c_0(1-R)]^{-1}, h_0/(1-2R)\}.$$

Thus every mean-zero scaled root $h\mathcal{R}_z Y_i$ has A norm at most $K_X\|w_i\|_{B_{\rho_1}}$; the same bound holds for the finite projected iterates below. Put $\sigma=\rho/4$, $\rho_2=\rho/2$, $\rho_3=\rho/4$, and

$$D_j = (2\pi)^j[j/(e\sigma)]^j \quad (1\le j\le 4), \qquad L_0 = -\log(1-R),$$
$$K_H = d^2\left[\tfrac{1}{2}D_4 L_0 + (D_2^2 + D_1 D_3)L_0^2\right].$$

To verify K_H, let $\ell = (\log w + \log v)/2$ up to its spatially constant normalization. Then $V = (\Delta\ell + \nabla\ell\cdot\nabla\ell)/2$ and (60) bounds each Hessian entry by K_H/d. Summing products with two gradients bounds the full integrand by $dK_H D_1^2$ times their undifferentiated norms.

The holomorphic extension of the scalar base uses all four roots:

$$T = \tfrac{h}{2}(\mathcal{R}_1 - \mathcal{R}_{-1}), \qquad C_{\text{base}} = \tfrac{h}{6}(3\mathcal{R}_1 - 5\mathcal{R}_{-1} + \mathcal{R}_i + \mathcal{R}_{-i}).$$

On the real slice this agrees with (54). Taking a real part on complex parameters would not be holomorphic. The complete metric expression is

$$g_{ij}(w) = \int f_w Y_i C_{\text{base}} Y_j - \frac{h^2}{6}\int f_w \nabla(TY_i)^{\mathsf{T}}\,\mathrm{Hess}\,V\,\nabla(TY_j), \quad f_w = wv/Z. \tag{69}$$

Define

$$B_* = K_f\left[\tfrac{5}{3}K_Y K_X + \tfrac{h_0^2}{6}dK_H D_1^2 K_X^2\right],$$
$$B_Q = B_* B_0^2, \qquad C_R = 2B_Q, \qquad C_S = B_*\left[B_0^2/(R-r) + 8B_0/\eta\right],$$
$$A_k = k!\max(1, 2/\eta)^k.$$

Here B_* bounds the A_{ρ_2} metric integrand for unit parameter directions, uniformly on the complex B_{ρ_1} ball of radius R. Cauchy's estimate in that Banach ball bounds its derivative in w, at norm at most r, by $B_*/(R-r)$. The expression is bilinear in w_i, w_j. Applying (65) therefore gives the explicit transfer bound

$$\sup_{K_{3\eta/4}} |g_{ij}(w) - g_{ij}(w_t)| \le C_S\varepsilon_N. \tag{70}$$

This includes the changes in density, mass, scores, reciprocal factors and invariant mean. Parameter Cauchy estimates on radius $\eta/2$ multiply this bound by at most A_k for derivatives of order at most k. The ledger gives a conservative bound.

B.4 Finite root solves and the total error budget

For w_t use one fixed Fourier cube and iterate on the scaled unknown $a = hs$:

$$a_0 = 0, \qquad a_{j+1} = \mathsf{P}_M[h\mathcal{D}_z^{-1}F_i - \mathcal{E}_z(w_t)a_j]. \tag{71}$$

Omit $n = 0$ for $z = 1$. Products are formed at their full degree before projection. The contraction is at most $2R$. The exact scaled solution is uniformly bounded on A_{ρ_1}. Comparing projected and exact fixed points on A_{ρ_2} and iterating gives a root error at most

$$K_X B_0\, \xi, \qquad \xi = \frac{e^{-\sigma M}}{1-2R} + (2R)^J,$$

after division and invariant-mean projection. For the scaled polynomial this projection is explicitly $a \leftarrow a - w_t \int v_t a / Z_t$. The numerator degree is at most $\max(M, N)$ in each coordinate. An unprojected iteration could increase degree by N at every step; (71) avoids that logarithm-squared degree.

Use the direct root combinations in (69), and symmetrize the finite matrix. The error in T is at most $K_X B_0 \xi$, and in the scalar base at most $5K_X B_0 \xi/3$. Paying the additional strip loss from ρ_2 to ρ_3 for gradient errors, the sandwich difference is bounded by the sum of one-error/one-field products. Both exact and trial root norms are at most $K_X B_0$, so the entrywise error is at most $C_R \xi$. This proves the root part of the ledger without an adjoint solve or nonpolynomial forcing oracle.

All finite metric integrands have A_{ρ_3} norm at most B_Q on $K_{3\eta/4}$. For a product trapezoidal rule with Q nodes in each coordinate, the error is at most $B_Q e^{-\rho_3 Q}$. Indeed only Fourier coefficients in $Q\mathbb{Z}^d \setminus \{0\}$ alias into the integral, and their $|n|_1$ is at least Q. This weighted-norm version of exponential periodic quadrature [19] avoids an additional lattice-sum constant. Consequently, with exact finite evaluation,

$$\|M_h - g^{\widehat{C}_h[f]}\|_{C^k(K)} \le A_k \left[C_S \varepsilon_N + C_R \xi + B_Q e^{-\rho_3 Q}\right]. \tag{72}$$

For example choose positive integers satisfying

$$e^{-\sigma N} \le \frac{(1-\lambda)h^4}{6A_k C_S r}, \qquad \lambda^L \le \frac{h^4}{6A_k C_S r},$$
$$e^{-\sigma M} \le \frac{(1-2R)h^4}{6A_k C_R}, \qquad (2R)^J \le \frac{h^4}{6A_k C_R},$$
$$e^{-\rho_3 Q} \le \frac{h^4}{3A_k B_Q}.$$

Taking ceilings of the corresponding logarithms, with a lower bound of one, makes every choice $O_k(1 + \log(1/h))$ and proves the first bound in Theorem B.1. Theorem 1.2 gives the second. Parameter estimates are applied to holomorphic functions with cutoffs fixed at each h, not by differentiating the integer cutoffs or a scalar norm bound. Initially the second bound holds for sufficiently small h. It extends, with a larger C_k, to $0 < h \le h_0$: on any remaining compact positive h interval the normalization, root bounds and smooth coefficients above uniformly bound both exact metrics and their fixed parameter derivatives.

For polynomial input, the construction uses only finite convolutions, projections, heat multipliers, nonzero scalar divisions and rational point evaluations. Input-enclosure error and arithmetic

error must be added separately to (72). At a fixed h the finite operations and strict reciprocal/inverse margins permit validated precision to be increased until the chosen finite-evaluation budget is met. Numerical parameter-jet validation likewise requires enclosures of those jets, not just a real-parameter scalar interval. No uniform bit-complexity statement follows. For nonpolynomial input the coefficient/tail information stated in the theorem remains an explicit input requirement.

B.5 A fixed-density certificate and finite-family geometry

On the circle take

$$f_\theta = 1 + \theta_1 \cos(2\pi x) + \theta_2 \cos(4\pi x), \qquad |\theta_1| \le .02, \quad |\theta_2| \le .005.$$

With $\rho = .2$ and complex thickening $\eta = .0005$, $3(.0205)e^{\cdot 2} + 5(.0055)e^{\cdot 4} < 1/8$, so the theorem applies to these fixed prescribed densities. For any finite polynomial trial t in the normalization radius-r ball, the continuum residual gives

$$\|u_h - t\|_{B_\rho} \le \frac{\|\Phi_h(t) - t\|_{B_\rho}}{1 - \lambda}.$$

Generate the entire product tH_ht before taking this norm, including all modes up to twice its cutoff. This supplies an a posteriori normalization certificate directly from chosen finite coefficients, without a stored residual or trial-data premise. Together with the root, transfer and quadrature ledger above it specifies a finite construction and its errors for the stated input class. It does not assert any previously computed pointwise error or full parameter-box metric/curvature enclosure.

For this fixed family an explicit limiting Gram margin is

$$g^G_\theta(c, c) \ge \frac{|c|^2}{32\pi^2(1.025)}.$$

The density perturbation is $\zeta = c_1 \cos(2\pi x) + c_2 \cos(4\pi x)$. The variational inverse weighted-Laplacian form is at least $\|\zeta\|^2_{\dot{H}^{-1}}/1.025$, and its two modes give this bound. For general immersed finite families compactness supplies a positive limiting Gram margin. Sufficiently accurate symmetric M_h are therefore positive for small h; inversion and the coordinate formulas give fourth-order connection and intrinsic-curvature approximation in C^{k-1} and C^{k-2} for $k \ge 2$. These are finite-family analytic consequences, not ambient curvature, ambient distance, numerical curvature certificates or a general-density normalization theorem without (61).

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References

[1] H. Lavenant, J. Luckhardt, G. Mordant, B. Schmitzer and L. Tamanini, *The Riemannian geometry of Sinkhorn divergences*, Ann. Inst. H. Poincaré Anal. Non Linéaire 43 (2026), no. 5, 1079–1144. doi:10.4171/AIHPC/165; arXiv:2405.04987v2.

[2] C. L. Wormell and S. Reich, *Spectral convergence of diffusion maps: improved error bounds and an alternative normalisation*, arXiv:2006.02037v3, 2021.

[3] R. Yao, Y.-H. Kim and G. Schiebinger, *Quantitative Stability of Many-Marginal Schrödinger Bridge*, arXiv:2604.15191v1, 2026.

[4] L. Chizat, P. Roussillon, F. Léger, F.-X. Vialard and G. Peyré, *Faster Wasserstein Distance Estimation with the Sinkhorn Divergence*, Advances in Neural Information Processing Systems 33 (2020). arXiv:2006.08172v2.

[5] S. Blanes, F. Casas and A. Murua, *Splitting methods for differential equations*, arXiv:2401.01722v3, 2024.

[6] T. Ichinose and H. Tamura, *Results on Convergence in Norm of Exponential Product Formulas and Pointwise of the Corresponding Integral Kernels*, arXiv:0905.3458, 2009.

[7] S. Blanes, F. Casas, C. González and M. Thalhammer, *Efficient Splitting Methods Based on Modified Potentials*, Communications in Computational Physics 33 (2023), 937–961. doi:10.4208/cicp.OA-2022-0247.

[8] S. Sule, L. Evans and M. Cameron, *Sharp error estimates for target measure diffusion maps with applications to the committor problem*, Applied and Computational Harmonic Analysis 79 (2025), 101803. doi:10.1016/j.acha.2025.101803; arXiv:2312.14418v1.

[9] J. A. Carrillo and D. Slepčev, *Example of a displacement convex functional of first order*, Calculus of Variations and Partial Differential Equations 36 (2009), no. 4, 547–564. doi:10.1007/s00526-009-0243-4.

[10] G. Mordant and L. Xu, *Convergence of the Sinkhorn Riemannian metric for finitely supported measures*, arXiv:2609.14651v1, 2026.

[11] H. Lavenant, J. Luckhardt and B. Schmitzer, *Approximation and computation of the geodesic Sinkhorn distance*, arXiv:2610.02007v1, 2026.

[12] T. L. Saidi, F. Gunsilius and G. Mena, Second-order Sinkhorn Geometry and $\sqrt{n}$-Inference, arXiv:2610.36488v1, 2026. `https://arxiv.org/abs/2610.36488v1`.

[13] R. G. Douglas, *On majorization, factorization, and range inclusion of operators on Hilbert space*, Proceedings of the American Mathematical Society 17 (1966), 413–415. doi:10.1090/S0002-9939-1966-0203464-1.

[14] N. Kessler, R. Magnet and J. Feydy, *Sinkhorn Normalization of Diffusion Kernels*, Proceedings of Machine Learning Research 306 (2026), 56764–56795. ICML 2026.

[15] Y. Chen, T. T. Georgiou and M. Pavon, *Stochastic control liaisons: Richard Sinkhorn meets Gaspard Monge on a Schrödinger bridge*, arXiv:2005.10963v2.

[16] Q. Du and J. Yang, *Asymptotically Compatible Fourier Spectral Approximations of Nonlocal Allen–Cahn Equations*, SIAM Journal on Numerical Analysis 54 (2016), no. 3, 1899–1919. doi:10.1137/15M1039857.

[17] Z. Shen, Z. Wang, A. Ribeiro and H. Hassani, *Sinkhorn Natural Gradient for Generative Models*, Advances in Neural Information Processing Systems 33 (2020). NeurIPS 2020.

[18] R. J. Berman, *The Sinkhorn algorithm, parabolic optimal transport and geometric Monge–Ampère equations*, Numerische Mathematik 145 (2020), 771–836. doi:10.1007/s00211-020-01127-x.

[19] L. N. Trefethen and J. A. C. Weideman, *The Exponentially Convergent Trapezoidal Rule*, SIAM Review 56 (2014), no. 3, 385–458. doi:10.1137/130932132.

[20] M. Altman, *An optimum cubically convergent iterative method of inverting a linear bounded operator in Hilbert space*, Pacific Journal of Mathematics 10 (1960), no. 4, 1107–1113. doi:10.2140/pjm.1960.10.1107.

[21] Anonymous, *Extrinsic curvature bias in the small-regularisation limit of Sinkhorn metrics*, unrefereed candidate, version 0.2.0, 2026. `https://www.evidencepress.org/releases/sinkhorn-metric-curvature-bias/`.

[22] H. Zhang and L. Zhao, *On the Inclusion Relation of Reproducing Kernel Hilbert Spaces*, arXiv:1106.4075v1, 2011.