Leading bias and positive correction of heat-kernel Sinkhorn metrics

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Abstract

We identify the leading bias of the population heat-kernel Sinkhorn tensor at smooth strictly positive densities on a flat torus. Its second-order quadratic-form coefficient is minus one forty-eighth of the Wasserstein Hessian of Fisher information, with a fourth-order $H^6 \to L^2$ remainder uniform under every fixed order of density-parameter differentiation. An explicit density has bias directions of both signs. A compact same-bandwidth correction removes the bias, remains positive on the full tangent complement, and preserves the original completed form space and square-root range with equivalent norms. Its sharp $H^r \to L^2$ rates are $h^{1+r/2}$ for $0 \le r \le 6$; full L^2 accuracy remains first order. Smooth finite-family metrics and intrinsic curvature inherit fourth-order accuracy. Separate computational appendices give certified metric values and normalization-inclusive approximation for prescribed analytic families close to uniform. The latter has logarithmic per-coordinate resolution under explicit coefficient and tail inputs.

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1 Introduction and statements

The heat-kernel Sinkhorn tensor converges to the inverse weighted Laplacian, with first-order full operator-norm error and second-order error on smooth tangents [1]. This leaves a structural question: what is the density-dependent second-order distortion, and can it be removed while preserving positivity on the complete tangent space? We answer this question in the smooth positive periodic population model. The original tensor is unchanged; the fourth-order results concern new corrected tensors.

The exact identification with Sinkhorn local geometry comes from [2]. Symmetric normalization and its spectral accuracy have important antecedents in diffusion maps [3]. Higher-order potential expansions are also available [4]; the cubic scale calculation used here was developed in the related dimension-three score problem [5]. We re-establish the coefficient and required remainder in every fixed dimension.

Richardson correction of scalar Sinkhorn divergences and Fisher-information bias already appear in [6]. In particular, the extrapolation principle itself is not a contribution of this paper. Scalar asymptotics do not automatically imply their density-Hessian analogues with uniform singular-inverse bounds. Symmetric-product

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coefficient algebra and modified generators are classical [7, 8]; positive fourth-order modified-potential splitting also has established precedents [9]. We use a classical Simpson comparison filter and a rational same-bandwidth two-step Richardson filter. Neither scalar filtering principle is new. Our results concern the explicit variable-density tensor bias, its sign structure, the uniform remainder in specified Sobolev domains, and the additional density-dependent correction needed for compact full-tangent positivity, uniform form comparison and fourth-order smooth-input accuracy. Explicit bias estimates for differently normalized diffusion generators are also known [10]. Fisher information need not be displacement convex, including on the circle [11]; the sign example below is an explicit certificate for the precise Sinkhorn tensor coefficient, not a first discovery of Fisher-information nonconvexity. Nonmonotonicity in the regularization parameter is already illustrated for other Sinkhorn geometric models in [2, Section 4.4]; our example concerns directional leading bias at a fixed smooth periodic density in the present normalization. Related finite-support tensor and curvature convergence is studied in [12]; our geometric consequences concern smooth density families with the bandwidth-dependent periodic heat-kernel cost fixed above. Recent spatial and chain discretizations of the geodesic Sinkhorn distance are studied at fixed regularization in [13]. Our bandwidth asymptotics describe the stated population tensors. Appendix B instead proves a restricted deterministic finite approximation theorem for local tensors of small analytic families.

Saidi, Gunsilius and Mena [14] develop Hadamard differentiability, covariant differentiation and statistical inference for Sinkhorn geometry at fixed regularization. Here the parameter-differentiated estimates are uniform in the vanishing heat bandwidth and lead to an explicit bias correction on the full tangent complement.

1.1 Model and fixed-space conventions

Let $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ have normalized Lebesgue measure, with $d \geq 1$ fixed, and put $H_h = e^{h\Delta/2}$. Its Fourier multiplier is $e^{-2\pi^2 h|n|^2}$. Let $f > 0$ be a smooth probability density and $q = \sqrt{f}$. The positive symmetric scale satisfies

$$b_h H_h(b_h q) = q, \qquad Q_h = M_{b_h} H_h M_{b_h}. \tag{1}$$

Here M_v means multiplication by v. Set

$$V = \frac{\Delta q}{2q}, \quad S = -\Delta/2 + V, \quad \Pi = |q\rangle\langle q|, \quad J = I - \Pi, \quad G = \tfrac{1}{2}S^\dagger, \quad T_h = hQ_h(I - Q_h^2)^{-1}\big|_{q^\perp}. \tag{2}$$

The superscript † denotes inversion on $q^\perp$ and zero on $\operatorname{span}(q)$. All tensors are extended by zero on this span. The scale exists uniquely for small h, Q_h is a positive compact selfadjoint contraction fixing q, and its gap on $q^\perp$ is at least ch; proofs are recalled below.

The unitary map $U : L^2(f\,\mathrm{d}x) \to L^2(\,\mathrm{d}x)$, $Uu = qu$, gives

$$U^{-1}SU = D_f/2, \qquad D_f = -f^{-1}\operatorname{div}(f\nabla), \qquad U^{-1}Q_hU = P_h,$$

where $P_h u = a_h H_h(a_h f u)$ and $a_h = b_h/q$. Thus $U^{-1}GU = D_f^{-1}$ on density-mean-zero functions. With entropic parameter $\varepsilon = 2h$ and the periodic cost $c_h(x,y) = -2h\log(k_h(x-y)/k_h(0))$, the coefficient of t^2 in the debiased divergence between $f\,\mathrm{d}x$ and $(f + tfu)\,\mathrm{d}x$ is $\langle u, hP_h(I - P_h^2)^{-1}u\rangle_{L^2(f)}$ [2, 1]. This fixes the tensor normalization; the model does not use a fixed Euclidean squared cost with boundary.

For parameters, f_θ is jointly smooth on $\mathbb{T}^d \times \Omega$, with $\Omega \subset \mathbb{R}^p$ open, and $K \Subset \Omega$ compact. The densities have mass one and a common positive lower bound on a neighborhood of K. Uniform bounds on the finitely many spatial and parameter derivatives used in each estimate are understood. Sobolev spaces H^r are the ordinary fixed periodic spaces after conjugation. We write $O_{C^k(H^r \to L^2)}(h^a)$ for a bound on all parameter derivatives of order at most k, uniform on K, between these fixed spaces. In particular derivatives include those of the zero extension and its moving projection. Constants can depend on dimension, the lower bound and derivative bounds.

1.2 Bias and correction

For $H = \operatorname{Hess} V$, define

$$R = \frac{1}{12}\left[-\operatorname{div}(H\nabla) + M_{\operatorname{div}(H\nabla q)/q}\right], \qquad L = \tfrac{1}{2}S^{\dagger}RS^{\dagger} - S/12. \tag{3}$$

The operator R is symmetric, second order and annihilates q. The bias L is defined on H^2 and is generally unbounded on L^2.

Theorem 1.1 (Leading tensor bias)**.** *Under the preceding hypotheses, for every fixed integer $k \geq 0$,*

$$T_h = G + h^2 L + O_{C^k(H^6 \to L^2)}(h^4). \tag{4}$$

For $D_H = -f^{-1}\operatorname{div}(fH\nabla)$, its density-coordinate form is

$$U^{-1}RU = D_H/12, \qquad U^{-1}LU = \tfrac{1}{6}D_f^{-1}D_H D_f^{-1} - D_f/24. \tag{5}$$

In particular the expansion is not a full L^2 operator expansion.

Theorem 1.2 (Fourth-order corrections and sharp scales)**.** *On $q^{\perp}$ define*

$$T_h^{\mathrm{R}} = (4T_h - T_{2h})/3, \tag{6}$$

$$C_h^0 = T_h + \frac{h}{6}(I - Q_h)(I + Q_h)^{-1} - 2h^2 T_h R T_h, \tag{7}$$

$$\widehat{C}_h = T_h + \frac{h}{3}Q_h(I - Q_h)(I + Q_h)^{-1}(I + Q_h^2)^{-1} - 2h^2 T_h R T_h. \tag{8}$$

The sandwich in (7) *has a bounded selfadjoint extension on L^2. The main correction $\widehat{C}_h$ is compact and, for a uniform constant M, satisfies on $q^{\perp}$*

$$(1 - 2Mh^2)T_h \;\leq\; \widehat{C}_h \;\leq\; (4/3 + 2Mh^2)T_h. \tag{9}$$

It is strictly positive on every nonzero tangent for sufficiently small h, but has no positive uniform L^2 lower bound. The comparison operator C_h^0 satisfies $C_h^0 \geq (h/12)I$ there and equals $(h/6)I$ plus a compact operator, so it is noncompact. All three corrected tensors satisfy, for every fixed k,

$$\|T_h^{\mathrm{R}} - G\|_{C^k(H^r \to L^2)} + \|C_h^0 - G\|_{C^k(H^r \to L^2)} + \|\widehat{C}_h - G\|_{C^k(H^r \to L^2)} \leq C_{k,r}\, h^{1+r/2}, \qquad 0 \leq r \leq 6. \tag{10}$$

Each exponent for each tensor is sharp already for $f = 1$. The tensor T_h^{R} is compact.

The definition of $\widehat{C}_h$ uses exact finite-h operators and local density derivatives, not the limiting inverse G. This is an analytic construction. Positivity refers to its quadratic form on the entire L^2 tangent complement.

For fixed small h, define the tangent form norm on $q^{\perp}$ by $\|u\|_{T_h} = \langle u, T_h u\rangle^{1/2}$, and similarly for $\widehat{C}_h$. Inequality (9) identifies their completions by the identity map, with equivalent norms uniformly for small h. Separately, Douglas's range theorem [15, Theorem 1] gives

$$\operatorname{ran} \widehat{C}_h^{1/2} = \operatorname{ran} T_h^{1/2},$$

with equivalent inverse-square-root range norms. The completed form space and this range space are distinct constructions; they are not identified with each other.

Proposition 1.3 (Bias has both signs)**.** *Write $\ell = \log f$. For every smooth periodic ϕ,*

$$24\langle D_f\phi, (U^{-1}LU)D_f\phi\rangle_{L^2(f)} = -\int f\left[|\nabla^3\phi|^2 - 3\,\mathrm{Hess}\,\ell : (\mathrm{Hess}\,\phi)^2\right]\,\mathrm{d}x. \tag{11}$$

At the single density $f(x) = e^{3\cos(2\pi x)}/Z$ on $\mathbb{T}$, the bias has a strictly positive smooth tangent direction and strictly negative smooth tangent directions. Consequently neither universal ordering $T_h \le G$ nor $T_h \ge G$ holds at variable density, even when tested only on smooth tangents for sufficiently small h.

The sign formula also has a variational interpretation. Normalize Fisher information by $\mathcal{I}(f) = \int f|\nabla \log f|^2\,\mathrm{d}x$. If $u = D_f\phi$, the corresponding density tangent is fu and

$$\langle u, (U^{-1}LU)u\rangle_f = -\frac{1}{48}\,\mathrm{Hess}_W\,\mathcal{I}(f)[fu, fu]. \tag{12}$$

Here Hess_W means second variation along the local smooth quadratic Wasserstein geodesic. Thus (11) identifies precisely which Fisher second variation is seen by this heat-kernel tensor. The general Fisher nonconvexity principle is classical [11].

Corollary 1.4 (Finite-family intrinsic geometry)**.** *Suppose $\theta \mapsto f_\theta$ is an immersion on a neighborhood of K. Put $\eta_i = (\partial_i f)/q$ and $g^A_{ij} = \langle \eta_i, A\eta_j\rangle$ for $A = G, T_h, T_h^{\mathrm{R}}, C_h^0, \widehat{C}_h$. For every fixed k,*

$$g^{T_h}_{ij} = g^G_{ij} + h^2\langle \eta_i, L\eta_j\rangle + O_{C^k}(h^4),$$
$$g^{T_h^{\mathrm{R}}} - g^G = O_{C^k}(h^4), \quad g^{C_h^0} - g^G = O_{C^k}(h^4), \quad g^{\widehat{C}_h} - g^G = O_{C^k}(h^4).$$

The corrected matrices are positive definite for small h. For $k \ge 2$, their Christoffel symbols converge in C^{k-1} and intrinsic Riemann curvature components in C^{k-2} at order h^4. These are finite-family statements, not ambient infinite-dimensional curvature or global distance convergence.

Three correction constructions. The two-bandwidth Richardson tensor cancels the leading bias by combining exact tensors at two scales. The Simpson comparison has an identity-floor contribution. The preferred same-bandwidth rational correction uses an additional density sandwich to remain compact, positive on the full tangent complement, and comparable to the original tensor. Their smooth-input rates are shared, while their compactness and positivity mechanisms differ. Appendix B gives a separate effective theorem for prescribed small analytic families, including normalization and finite resolution; its input conditions remain independent of the general smooth-density bias theorem.

2 Scale, gap and baseline bounds

We include the estimates needed to handle the shrinking inverse gap. They also fix precisely which second-order results are used from [1].

Lemma 2.1 (Baseline estimates)**.** *For every fixed k, uniformly on the declared density families,*

$$b_h = e^{-hV/2} + O_{C^k(H^m)}(h^3), \tag{13}$$
$$Q_h - e^{-hS} = O_{C^k(L^2\to L^2)}(h^2) = O_{C^k(H^2\to L^2)}(h^3), \tag{14}$$
$$T_h - G = O_{C^k(L^2\to L^2)}(h) = O_{C^k(H^2\to L^2)}(h^2). \tag{15}$$

Here $m > d/2$ is any fixed integer, with sufficient density regularity. Moreover, with $B_Q = I - Q_h + h\Pi$,

$$\|B_Q^{-1}\|_{C^k(L^2\to L^2)} \le C_k/h. \tag{16}$$

Proof. In the Sobolev algebra H^m, put $F_h(b) = bH_h(bq)/q - 1$. At $b = 1$,

$$DF_h(1) = M_{1/q}(I + H_h)M_q + M_{H_hq/q-1}.$$

The first term has uniformly bounded inverse $M_{1/q}(I + H_h)^{-1}M_q$, and the second is $O(h)$. The second derivative is uniformly bounded bilinearly on a fixed ball. Positive-time heat Taylor expansion of $e^{-hV/2}$ gives residual $O(h^3)$, so quantitative Newton inversion proves (13). The same residual equation differentiated inductively proves its parameter estimates, and positive-order derivatives of b_h are $O(h)$. Sobolev embedding gives positivity. For uniqueness let $w_i = a_i f$, so $w_i H_h w_i = f$. Their difference z satisfies $H_h z = -fz/(w_1 w_2)$. Pairing with z contradicts positivity of H_h unless $z = 0$.

For the full norm comparison reduce to the symmetric product $e^{-hV/2}H_h e^{-hV/2}$. In a lifted Brownian bridge from x to $y = x + z$, the mean difference between $\int_0^h V(X_t)\,\mathrm{d}t$ and $h(V(x) + V(y))/2$ is bounded by $C(h|z|^2 + h^2)$. Taylor expansion cancels the linear segment error, and bridge covariance $t(h-t)I/h$ gives Ch^2. Both exponents are $O(h)$, so replacing exponentials by one minus their arguments costs Ch^2. Integrating against the lifted heat kernel uses $\int |z|^2 k_h^{\mathbb{R}^d}(z)\,\mathrm{d}z = dh$. Both absolute row and column errors are $O(h^2)$; summing lifts and Schur's test prove the first norm in (14). Parameter derivatives obey the same bound: their linear terms replace V by a derivative, and nonlinear remainders have at least two time integrals.

For the strong norm let $A = \Delta/2$, $B = -M_V$, $C = [A, B]$. Removing the bounded outer factors from the defect of $F(t) = e^{tB/2}e^{tA}e^{tB/2}$ relative to $A + B$ gives

$$-\int_0^{t/2} e^{-rB}Ce^{rB}\,\mathrm{d}r\, e^{tA} + \frac{1}{2}\int_0^t e^{(t-r)A}Ce^{rA}\,\mathrm{d}r. \tag{17}$$

Subtract Ce^{tA} in each integral; their integrals cancel. The remaining commutators are $[C, B]$, a multiplier, and $[A, C]$, of differential order two. Their integral representations bound the defect by $Ct^2 : H^2 \to L^2$. Variation of constants yields the second norm in (14). Every fixed parameter derivative has the same differential orders; bounded-potential Duhamel controls the differentiated semigroups.

The weighted form $\langle v, Sv\rangle = \frac{1}{2}\int f|\nabla(v/q)|^2$ shows that S has kernel $\mathrm{span}(q)$ and a uniform positive gap on its complement by weighted Poincare. The Markov contraction P_h is unitarily conjugate to Q_h, which is positive and compact. The full norm comparison gives $\|Q_h|_{q^\perp}\| \le 1 - ch$. Thus $B_Q^{-1} = O(h^{-1})$ on L^2. Its positive-order parameter derivatives have the same bound, since $\partial^\alpha B_Q = O(h)$ for $|\alpha| > 0$ and each term of inverse differentiation has one more inverse than differentiated factors.

For $E = e^{-hS}$ the corresponding inverse $(I - E + h\Pi)^{-1}$ is $O_{C^k}(h^{-1})$ on L^2 and H^2, by the gap, graph norms, and bounded-potential differentiation. Use the exact identity

$$hQ(I - Q^2)^{-1} = \frac{h}{2}[(I - Q)^{-1} - (I + Q)^{-1}] \tag{18}$$

on the complement. The resolvent identity and (14) give tensor differences $O(h)$ on L^2 and $O(h^2) : H^2 \to L^2$. For zero extensions the exact fixed-space expression is

$$\frac{h}{2}[B_Q^{-1} - h^{-1}\Pi - J(I + Q)^{-1}J]. \tag{19}$$

The projected terms cancel in the comparison with E. Its scalar tensor multiplier is $h/(2\sinh(h\lambda))$. The difference from $1/(2\lambda)$ is bounded by both Ch and $Ch^2\lambda$ on the positive spectrum.

For completeness these scalar estimates retain parameter derivatives. Put $A_0 = S + \Pi$; it has fixed domain H^2 and positive lower bound. Its positive-order parameter derivatives are bounded multipliers plus smooth finite-rank operators. On fixed sector rays, $\|(z - A_0)^{-1}\| \le C/(1 + |z|)$. The scalar difference $g_h(z) = h/(2\sinh(hz)) - 1/(2z)$ is analytic and bounded by Ch on every strict right sector. Positive-order

derivatives of its contour formula contain at least two resolvents, so their integrals are $O(h)$. For the strong estimate use $g_h(A_0) = h^2 j(hA_0)A_0$, where $j(w) = [1/(2\sinh w) - 1/(2w)]/w$ is bounded analytic there. The same two-resolvent argument bounds all derivatives of $j(hA_0)$. Compression by J completes (15) on fixed spaces. □

3 Cubic scale and positive-time local error

Lemma 3.1 (Scale to fifth-order residual)**.** *For every fixed m and k,*

$$b_h = \exp(-hV/2 + h^3 c) + O_{C^k(H^m)}(h^5), \tag{20}$$

where

$$c = \frac{\Delta^3 q}{192q} - \frac{\Delta((\Delta q)^2/q)}{128q} + \frac{(\Delta q)^3}{384q^3}. \tag{21}$$

Proof. First identify finitely many coefficients in the formal ring of smooth functions and a scalar h. The equation $be^{hA}(bq) = q$, $A = \Delta/2$, has a unique formal solution with constant term one: the unknown coefficient b_n appears as $2qb_n$ plus known lower terms. Formal inversion gives $e^{-hA}(q/b) = bq$, so $b(h)^{-1}$ solves the same equation at $-h$. Uniqueness implies $b(-h) = b(h)^{-1}$ and hence $\log b$ is odd as a formal series. This is coefficient algebra, not an application of an inverse heat operator to a function.

Let $u = -V/2$. Substitution of $b = 1 + hu + h^2u^2/2 + h^3(c + u^3/6)$ gives cubic residual coefficient

$$2qc + qu^3/3 - A^3q/12 + A(qu^2)/2.$$

Its cancellation gives (21). Formal oddness removes the fourth logarithmic coefficient, so $p_h = \exp(-hV/2 + h^3c)$ has zero residual coefficients through degree four. Positive-time heat Taylor's integral remainder through degree four, with sufficiently many derivatives of the fixed smooth coefficients, gives $F_h(p_h) = O_{C^k(H^m)}(h^5)$. For example ten additional derivatives suffice for the heat remainder of a fixed coefficient. Every actual heat time lies in $[0, h]$. The uniform Newton inverse from Lemma 2.1 converts this residual into (20). Differentiating the exact residual equation inductively proves the parameter remainder. This does not differentiate an uncontrolled pointwise asymptotic. □

Lemma 3.2 (Local comparison on H^6)**.** *For $Z = -S = A + B$ and R from* (3)*,*

$$Q_h - e^{hZ} = h^3 R + \tfrac{h^4}{2}(ZR + RZ) + O_{C^k(H^6 \to L^2)}(h^5). \tag{22}$$

Proof. Put $C = [A, B]$, $D = [A, C]$ and $K_0 = [C, B]$. The multiplier K_0 commutes with B, so $e^{-rB}Ce^{rB} - C = rK_0$ exactly. Changing variables in the heat commutator part of (17), its remaining expression is

$$-\frac{t^2}{8}K_0 e^{tA} + \frac{1}{2}\int_0^t (t-u)e^{uA}De^{(t-u)A}\,\mathrm{d}u. \tag{23}$$

Expanding the first heat factor once, and each heat factor in the integral once, gives

$$t^2 D_2 + t^3 D_3 + O(t^4) : H^6 \to L^2, \quad D_2 = D/4 - K_0/8, \quad D_3 = AD/12 + DA/6 - K_0A/8.$$

The omitted products A^2D, ADA, DA^2 have order at most six; integral Taylor remainders use only nonnegative times and bounded heat operators on the corresponding Sobolev spaces. Restoring outer multiplications adds $(BD_2 + D_2B)/2$ to the cubic defect coefficient. Variation of constants, with its left semigroup expanded once, now proves an actual $O(h^5) : H^6 \to L^2$ remainder and cubic coefficient $R_0 = D/12 - K_0/24$. The fourth coefficient is $(ZR_0 + R_0Z)/2$. One can obtain it by inserting D_2, D_3; equivalently the finite formal

identity $F(-h) = F(h)^{-1}$ forces $\log F(h) = hZ + h^3R_0 + O(h^5)$ and gives the same coefficient. The analytic remainder has already been established by (23), independently of this formal identity.

The scale expansion inserts outer multiplications $e^{h^3 c}$. Their contribution is $2h^3M_c + h^4(M_cZ + ZM_c)$, with the same remainder domain. Thus $R = R_0 + 2M_c$ in (22). Direct differentiation gives

$$C = -\nabla V \cdot \nabla - \Delta V/2, \quad [A, C] = -\operatorname{div}(\operatorname{Hess} V \nabla) - \Delta^2 V/4, \quad [C, B] = |\nabla V|^2.$$

Since both exact operators in (22) fix the smooth q, its cubic coefficient annihilates q. Its principal part therefore determines its scalar term, giving exactly (3). Every parameter derivative preserves the displayed differential orders and the positive-time remainder bounds. □

4 Modified generator and singular-inverse expansion

The modified operator

$$S_h = S - h^2 R \tag{24}$$

is elliptic for small h. Indeed direct conjugation gives

$$U^{-1}RU = D_H/12, \qquad U^{-1}S_hU = \tfrac{1}{2}D_{f,M_h}, \quad M_h = I - \tfrac{h^2}{6}\operatorname{Hess} V, \tag{25}$$

where $D_{f,M} = -f^{-1}\operatorname{div}(fM\nabla)$. If $h^2\|\operatorname{Hess} V\|_\infty \le 3$, then $M_h \ge I/2$, $S_h \ge S/2$ as quadratic forms, and its exact kernel is $\operatorname{span}(q)$. Elliptic graph norms of $S_h + \Pi$ are uniformly equivalent to the corresponding ordinary even Sobolev norms.

Lemma 4.1 (Modified comparison and parameter bounds)**.** *Let $E_h^* = e^{-hS_h}$ and $B_* = I - E_h^* + h\Pi$. Then*

$$Q_h - E_h^* = O_{C^k(H^6\to L^2)}(h^5), \qquad B_*^{-1} = O_{C^k(H^6\to H^6)}(h^{-1}). \tag{26}$$

The inverse bound holds on L^2 as well.

Proof. One Duhamel insertion about e^{hZ} is $h^2\int_0^h e^{(h-t)Z}Re^{tZ}\,\mathrm{d}t$. Expanding its outer factors through first order yields $h^3R + h^4(ZR + RZ)/2$, with $O(h^5) : H^6 \to L^2$ remainder: the highest omitted compositions are Z^2R, ZRZ, RZ^2. The exact twice-iterated remainder has two R insertions, coefficient h^4 and time-simplex volume $O(h^2)$, hence is $O(h^6) : H^4 \to L^2$. Uniform semigroup bounds on the required Sobolev spaces suffice; their parameter justification is given next. Lemma 3.2 then proves the first assertion.

Parameter derivatives of S_h include second-order terms, so a bounded-potential argument alone would be insufficient. Put $A_h = S_h + \Pi$ and $A_0 = S + \Pi$. On $X = H^{2j}$ for each fixed needed integer j, elliptic graph norms and selfadjoint spectral bounds give, on sectorial rays away from the positive axis,

$$\|(z - A_h)^{-1}\|_{X\to X} \le \frac{C}{1 + |z|}, \quad \|(z - A_h)^{-1}\|_{X\to H^{2j+2}} \le C. \tag{27}$$

Consequently every positive-order parameter derivative satisfies

$$\|(\partial^\alpha A_h)(z - A_h)^{-1}\|_{X\to X} \le C_\alpha[(1 + |z|)^{-1} + h^2]. \tag{28}$$

The order-zero part comes from derivatives of A_0; the principal part comes with $h^2\partial^\alpha R$.

For $0 < t \le h_0$, use the semigroup contour with rays starting at radius $1/t$ and an arc through the left half-plane at that radius. It avoids the positive spectrum. The exponential e^{-tz} is bounded on the arc and decays on the rays. In the resolvent difference there is a factor h^2R; (27) bounds it by $Ch^2/|z|$. Arc length and ray integration therefore give

$$e^{-tA_h} - e^{-tA_0} = O_{C^k(X\to X)}(h^2). \tag{29}$$

After differentiation every term retains a factor h^2R or its derivative; all other insertions are bounded by (28). On this contour $(1+|z|)^{-1} \le Ct$, so the same integration proves every fixed derivative bound. The common finite-rank kernel adjustment cancels in the semigroup difference. Together with bounded-potential Duhamel for S, this proves uniform differentiated semigroup bounds on the Sobolev spaces used in the twice-iterated remainder. At $t = h$, positive-order derivatives of E_h^* are $O(h)$ on X.

The uniform gap and graph norms give $B_*^{-1} = O(h^{-1})$ on X. For $|\alpha| > 0$, $\partial^\alpha B_* = O(h)$. Repeated inverse differentiation therefore gives $\partial^\alpha B_*^{-1} = O(h^{-1})$ for every fixed derivative order. Take $j = 0, 3$ to obtain the stated cases. □

Proof of Theorem 1.1. Apply (19) to Q_h and E_h^*. The minus-inverse difference is $B_Q^{-1}(Q_h - E_h^*)B_*^{-1}$. Using Lemmas 2.1 and 4.1 gives $h \cdot h^{-1} \cdot h^5 \cdot h^{-1} = h^4$ from H^6 to L^2, with all fixed parameter derivatives. The plus-inverse difference is of higher order. Thus $T_h - T_h^* = O_{C^k(H^6 \to L^2)}(h^4)$, where T_h^* is the tensor of E_h^*.

The scalar function

$$m(w) = \frac{1/(2\sinh w) - 1/(2w) + w/12}{w^3}$$

is bounded analytic on every fixed strict right sector, with $m(0) = 7/720$ and $m(w) = O(|w|^{-2})$ at sectorial infinity. Functional calculus on $q^\perp$ yields

$$T_h^* = (2S_h)^\dagger - h^2 S_h/12 + h^4 m(hS_h) S_h^3. \tag{30}$$

Here the final term is $O_{C^k(H^6 \to L^2)}(h^4)$, including parameters of the principal part. To verify this explicitly, use $A_h = S_h + \Pi$ and compress by J. Its zeroth derivative is bounded by the spectral theorem. Differentiated contour terms contain one resolvent and relative insertions (28). Terms with only order-zero insertions have at least two decaying resolvents and bounded integrals. A term with $j \ge 1$ second-order insertions has bound Ch^{2j} times

$$\int_0^\infty \frac{\min(1, [h(1+r)]^{-2})}{1+r}\, \mathrm{d}r \le C(1 + |\log h|).$$

Its product with h^{2j} is uniformly bounded. These estimates also justify differentiation on truncated contours and passage to the limit for each $h > 0$. Finally A_h^3 and all fixed derivatives map H^6 to L^2 uniformly. Compression removes its artificial eigenvalue on q and proves the remainder assertion in (30).

On $q^\perp$ the exact inverse expansion is

$$S_h^{-1} = S^{-1} + h^2 S^{-1} R S^{-1} + h^4 S^{-1} R S_h^{-1} R S^{-1}.$$

The last coefficient and all its fixed parameter derivatives are bounded on L^2 by elliptic regularity; equivalently regularize by Π and then compress, using $R\Pi = \Pi R = 0$. Also $-h^2 S_h/12 = -h^2 S/12 + h^4 R/12$. Substitution proves (4). Conjugation in (25) gives (5), including its normalization factors. □

5 Sign structure and Fisher information

Proof of Proposition 1.3. From (5), with $u = D_f \phi$,

$$24\langle u, (U^{-1}LU)u\rangle_f = 4\int f \operatorname{Hess} V(\nabla\phi, \nabla\phi) - \int f |\nabla D_f \phi|^2. \tag{31}$$

Here and below integrals are over the torus. We give the weighted identity needed to simplify the second term. Write $\mathcal{L} = \Delta + \nabla\ell \cdot \nabla$, $w = \nabla\phi$, $A = \operatorname{Hess}\phi$, $B = \operatorname{Hess}\ell$. Then $\partial_i \mathcal{L}\phi = \mathcal{L} w_i + B_{ij} w_j$. Weighted Bochner, integrated componentwise, gives

$$\int f |\mathcal{L} w|^2 = \int f |\nabla^3 \phi|^2 - \int f B : A^2.$$

Integration by parts in the cross term gives

$$2\int f(\mathcal{L}w_i)B_{ij}w_j = -2\int fB : A^2 - 2\int fA_{ik}\ell_{ijk}w_j.$$

Symmetry of the third derivatives turns the last term into $\int f[\operatorname{Hess}\Delta\ell + \nabla\ell\cdot\nabla\operatorname{Hess}\ell][w,w]$. Adding $\int f|Bw|^2$ supplies the B^2 term. Since $4V = \Delta\ell + |\nabla\ell|^2/2$, their sum is $4\int f\operatorname{Hess}V[w,w]$. Substituting into (31) proves (11).

For an explicit positive direction in dimension one, set $\theta = 2\pi x$, $f = e^{3\cos\theta}/Z$, and prescribe

$$\phi''(x) = e^{-3\cos\theta}\sin\theta, \qquad \phi'(x) = e^{-3\cos\theta}/(6\pi) - C,$$

where C makes ϕ' have zero unweighted mean. This gives a smooth periodic ϕ and a nonzero density-mean-zero $u = D_f\phi$. Let $w = e^{-3\cos\theta}$, $c = \cos\theta$, $s = \sin\theta$ and write av for angular average. The integral inside the leading minus sign of (11) is

$$\frac{(2\pi)^2}{Z}\operatorname{av} w[(c+3s^2)^2 + 9cs^2].$$

Integrating the derivatives of $w\sin^3\theta$, $w\sin\theta\cos\theta$ and $w\sin\theta$ gives respectively

$$3\operatorname{av}(ws^4) = -3\operatorname{av}(wcs^2), \quad 3\operatorname{av}(wcs^2) = \operatorname{av}(w(s^2-c^2)), \quad 3\operatorname{av}(ws^2) = -\operatorname{av}(wc).$$

Expanding and applying these identities reduces the average to $\operatorname{av}(w(2-3c^2)) = -\operatorname{av}(w(1+c)) < 0$. Consequently

$$\langle u, (U^{-1}LU)u\rangle_f = \frac{(2\pi)^2}{24Z}\operatorname{av}(e^{-3\cos\theta}(1+\cos\theta)) > 0.$$

For negative directions take $\phi_n = \cos(2\pi nx)$. The bracket integral in (11) is at least

$$\frac{\min f}{2}(2\pi n)^6 - \frac{3\|\ell''\|_\infty \max f}{2}(2\pi n)^4 > 0$$

for large n. Thus the corresponding bias is negative at the same fixed density. Applying (4) to either fixed smooth direction transfers its strict bias sign to $T_h - G$ for small h. This ordering obstruction says nothing by itself about positivity of the two-bandwidth Richardson tensor. □

Proof of the Fisher-Hessian identity (12). For a smooth mass-zero density perturbation g, direct variation of $\mathcal{I}(f) = \int|\nabla f|^2/f$ gives

$$\mathcal{I}'[f](g) = \int(-2\Delta\ell - |\nabla\ell|^2)g = -8\int Vg, \qquad \mathcal{I}''[f](g,g) = 2\int f|\nabla(g/f)|^2.$$

Take $f_t = (\mathrm{Id} + t\nabla\phi)_\# f$. For sufficiently small $|t|$, the displacement stays within the torus injectivity range and $I + t\operatorname{Hess}\phi$ is positive definite; this is the local smooth quadratic Wasserstein geodesic with initial velocity $\nabla\phi$. Differentiating its action on smooth test functions gives

$$\dot f_0 = -\operatorname{div}(f\nabla\phi) = fD_f\phi, \qquad \ddot f_0 = \partial_i\partial_j(f\phi_i\phi_j).$$

The ordinary second derivative chain rule and periodic integration by parts therefore give

$$\operatorname{Hess}_W\mathcal{I}(f)[fD_f\phi, fD_f\phi] = 2\int f|\nabla D_f\phi|^2 - 8\int f\operatorname{Hess}V(\nabla\phi,\nabla\phi).$$

Comparison with (31) proves (12). In particular (11) is equivalent to the corresponding weighted third-derivative formula for this Fisher Hessian; its one-dimensional antecedent is contained in [11, Lemma 8]. □

6 Uniform smoothing and positive correction

The differential sandwich cannot be justified by formal cancellation alone. The following uniform smoothing estimate supplies its domain.

Lemma 6.1 (Two-derivative smoothing)**.** *For every fixed nonnegative integer s and k,*

$$\|T_h\|_{C^k(H^s\to H^{s+2})} \le C_{k,s}. \tag{32}$$

Proof. Put

$$K_h = (I - H_h)/h, \quad W_h = (b_h^{-2} - 1)/h, \quad v_h = q/b_h, \quad \widehat{A}_h = K_h + M_{W_h} + |v_h\rangle\langle v_h|.$$

There is an exact rank-one factorization

$$I - Q_h + h\Pi = hM_{b_h}\widehat{A}_h M_{b_h}. \tag{33}$$

The gap proves that $\widehat{A}_h$ has a uniform positive lower bound on L^2, hence a uniformly bounded inverse. The functions W_h, v_h and all fixed spatial and parameter derivatives are uniformly bounded: (13) gives $W_h = V + O(h)$.

The inverse is uniformly bounded on each fixed H^s. Indeed K_h commutes with spatial derivatives. Differentiating $\widehat{A}_h w = u$ j times gives the same operator on $\partial^j w$, with commutators involving only lower derivatives of w and smooth derivatives of W_h. The finite-rank commutators are bounded on L^2 with smooth ranges: move derivatives off w in scalar pairings by integration by parts. Induction using the L^2 inverse bound proves the H^j bounds. For smooth u, existence of these derivatives can first be justified at fixed h from

$$w = (1/h + W_h)^{-1}[u + H_h w/h - |v_h\rangle\langle v_h|w].$$

Its right side is smooth, and its denominator is positive for small h. The uniform estimates come from the derivative induction, not from this last formula. Density extends them to H^j inputs. Parameter derivatives of $\widehat{A}_h$ are uniformly bounded on each H^j, so inverse differentiation proves all fixed parameter versions.

Let $R_0 = (K_h + I)^{-1}$. On a Fourier mode with $\mu = 4\pi^2|n|^2$,

$$\frac{(1+\mu)e^{-h\mu/2}}{1 + (1 - e^{-h\mu/2})/h} \le C.$$

For $h\mu/2 \le 1$ use $1 - e^{-t} \ge ct$; otherwise use $te^{-t} \le C$. Hence $H_h R_0 : H^s \to H^{s+2}$ is uniformly bounded. The inverse identity

$$H_h\widehat{A}_h^{-1} = H_h R_0 + H_h R_0(I - M_{W_h} - |v_h\rangle\langle v_h|)\widehat{A}_h^{-1}$$

gives the same smoothing, including parameters, for its left side. From (33), exactly on the full space,

$$hQ_h(I - Q_h)^\dagger = M_{b_h}H_h\widehat{A}_h^{-1}M_{1/b_h} - \Pi. \tag{34}$$

The subtraction is necessary: $hQ_h(I - Q_h + h\Pi)^{-1}$ acts as the identity on q. Finally $(I + Q_h)^{-1}$ is uniformly bounded on every fixed H^s, with all fixed parameter derivatives. This follows from $Q_h - H_h = O_{C^k(H^s\to H^s)}(h)$ and a Neumann perturbation of the Fourier multiplier inverse $(I + H_h)^{-1}$. Multiply (34) by this plus inverse to obtain (32). □

Lemma 6.2 (Relative form control)**.** *Uniformly on the declared density classes, for every $u \in L^2$,*

$$\left|\langle u, T_h R T_h u\rangle\right| \le M\langle u, T_h u\rangle. \tag{35}$$

Proof. Lemma 6.1 and selfadjoint duality give compatible uniform maps $T_h : L^2 \to H^2$ and $T_h : H^{-2} \to L^2$. Interpolation gives $T_h : H^{-1} \to H^1$. For smooth v, positivity therefore implies

$$\|T_h^{1/2} v\|_2^2 = \langle v, T_h v \rangle \le C \|v\|_{H^{-1}}^2.$$

Density extends the square root to $H^{-1} \to L^2$; duality gives $T_h^{1/2} : L^2 \to H^1$ uniformly. Since the symmetric differential operator R is a bounded form $H^1 \to H^{-1}$,

$$|\langle T_h u, R T_h u \rangle| \le C \|T_h u\|_{H^1}^2 \le M \|T_h^{1/2} u\|_2^2.$$

This proves (35). The argument uses no parameter derivatives of a square root; it applies to the full-space zero extension and its moving kernel pointwise in the parameter. □

Proof of Theorem 1.2. The smoothing lemma makes RT_h uniformly bounded on L^2, and therefore $T_h R T_h$ is uniformly bounded. Its form $\langle T_h u, R T_h v \rangle$ is symmetric, proving that its extension is selfadjoint. It is compact since its left T_h is compact and RT_h is bounded.

For every $0 \le p < 1$,

$$\frac{hp}{1-p^2} + \frac{h}{6}\frac{1-p}{1+p} = \frac{h(p^2+4p+1)}{6(1-p^2)} \ge h/6. \tag{36}$$

Functional calculus and a uniform bound $\|T_h R T_h\| \le M$ give $C_h^0 \ge hI/6 - 2Mh^2 I \ge hI/12$ once $h \le 1/(24M)$, in addition to the previous small-h restrictions. The scalar expression in (36) equals

$$\frac{h}{6} + \frac{2h}{3} \sum_{j \ge 0} p^{2j+1} + \frac{h}{3} \sum_{j \ge 1} p^{2j}.$$

At $p = e^{-h\lambda}$ this is half the classical composite Simpson approximation to $\int_0^\infty e^{-t\lambda}\, \mathrm{d}t$. It also shows the $h/6$ identity term plus a compact remainder; the differential sandwich is compact as already proved.

For the compact correction, its additional scalar term divided by the multiplier of T_h is exactly

$$\frac{(h/3)p(1-p)/[(1+p)(1+p^2)]}{hp/(1-p^2)} = \frac{(1-p)^2}{3(1+p^2)} \in [0, 1/3], \qquad 0 < p < 1. \tag{37}$$

Spectral calculus and Lemma 6.2 prove (9). The heat operator is injective and b_h is positive, so Q_h and T_h have no null vector on $q^\perp$. Choosing $2Mh^2 < 1$ gives strict positivity of $\widehat{C}_h$ there. The scalar added filter is continuous on the spectrum of Q_h and vanishes at zero. It is compact, as are T_h and $T_h R T_h$. This proves compactness and explains why no positive uniform L^2 floor can hold in infinite dimension.

The scalar base has the exact identity on $q^\perp$

$$T_h + \frac{h}{3} Q_h (I - Q_h)(I + Q_h)^{-1}(I + Q_h^2)^{-1} = \frac{4T_h - 2hQ_h^2(I - Q_h^4)^{-1}}{3}. \tag{38}$$

Both sides divided by T_h have multiplier $1 + (1-p)^2/[3(1+p^2)]$. This is a rational two-step Richardson filter constructed from one normalized operator. In general $Q_h^2 \ne Q_{2h}$: separately balancing the doubled heat time changes the scale. Thus (38) is not a positivity theorem for the raw two-bandwidth extrapolation T_h^{R}.

The two form completions are equivalent by (9). Applying Douglas's theorem to the bounded operators $T_h^{1/2}$ and $\widehat{C}_h^{1/2}$ in both directions gives their equal ranges. If $aT_h \le \widehat{C}_h \le bT_h$, then on that range the inverse-square-root norms satisfy $b^{-1/2}\|T_h^{-1/2} v\| \le \|\widehat{C}_h^{-1/2} v\| \le a^{-1/2}\|T_h^{-1/2} v\|$. Taking $a = 1 - 2Mh^2$ and $b = 4/3 + 2Mh^2$ gives uniform constants for small h.

We next prove accuracy without applying R to a rough tangent. Theorem 1.1 gives

$$T_h = G + h^2(2GRG - S/12) + O_{C^k(H^6\to L^2)}(h^4). \tag{39}$$

For the Cayley term, the exact identity between two Cayley filters uses only their plus inverses. Lemma 4.1 therefore gives

$$\frac{h}{6}\left[(I-Q_h)(I+Q_h)^{-1} - (I-E_h^*)(I+E_h^*)^{-1}\right] = O_{C^k(H^6\to L^2)}(h^6).$$

The reference scalar filter is $(h/6)\tanh(hS_h/2)$, and

$$(h/6)\tanh(hz/2) = h^2 z/12 + h^4 z^3 e(hz),$$

where e is bounded analytic on each strict right sector, removable at zero and decaying quadratically at infinity. The relative-insertion contour proof used for $m(hA_h)$ applies verbatim to $e(hA_h)$: bounded insertions give two resolvents, and $j \geq 1$ principal insertions cost $h^{2j}(1 + |\log h|)$. Compressing by J gives

$$\frac{h}{6}(I-Q_h)(I+Q_h)^{-1} = h^2 S/12 + O_{C^k(H^6\to L^2)}(h^4). \tag{40}$$

Replacing S_h by S costs only $O(h^4) : H^2 \to L^2$; the Cayley filter vanishes on the common kernel.

The exact sandwich identity is

$$T_h R T_h - GRG = (T_h - G)RT_h + GR(T_h - G).$$

Its right side is $O_{C^k(H^2\to L^2)}(h^2)$. For the first term, smoothing with $s = 2$ gives $RT_h : H^2 \to H^2$ uniformly and (15) applies. For the second, RG is bounded on L^2 by elliptic regularity, so its adjoint GR extends boundedly there. The product rule handles all parameters between the same fixed domains. Thus the Cayley term cancels $-h^2 S/12$ and the sandwich cancels $2h^2 GRG$ in (39). This proves the H^6 endpoint for C_h^0. Substitution of (4) at h and $2h$ proves that endpoint for T_h^{R}.

To pass to the compact correction, on the full fixed space exactly

$$\widehat{C}_h - C_h^0 = -\frac{h}{6} J(I-Q_h)^3(I+Q_h)^{-1}(I+Q_h^2)^{-1}J. \tag{41}$$

Indeed its Cayley addition is that of C_h^0 multiplied by $2Q_h(I+Q_h^2)^{-1}$, whose difference from I is $-(I - Q_h)^2(I+Q_h^2)^{-1}$. For each required integer $s \geq 0$,

$$I - Q_h = O_{C^k(H^{s+2}\to H^s)}(h).$$

This follows from the heat multiplier estimate for $I - H_h$ and $b_h - 1 = O_{C^k}(h)$ in the needed multiplier norms. Parameter derivatives of positive order preserve the factor h. Also $(I+Q_h^2)^{-1}$ is uniformly C^k bounded on every required H^s: the identity $Q_h^2 - H_h^2 = (Q_h - H_h)Q_h + H_h(Q_h - H_h)$ is $O_{C^k}(h)$ there, and a Neumann perturbation of the bounded Fourier multiplier $(I+H_h^2)^{-1}$ proves the assertion. Compose three increments through $H^6 \to H^4 \to H^2 \to L^2$ in (41). The plus inverses and smooth projections are uniformly bounded with every fixed parameter derivative, so the product rule gives $\widehat{C}_h - C_h^0 = O_{C^k(H^6\to L^2)}(h^4)$. No square-root differentiation is used for this estimate.

At the L^2 endpoint, (15) and the triangle inequality give $O(h)$ for Richardson. For C_h^0, its Cayley term is $O(h)$ and its bounded sandwich is $O(h^2)$, so the same endpoint holds. The right side of (41) is $O_{C^k(L^2\to L^2)}(h)$, which proves the L^2 endpoint for $\widehat{C}_h$ as well. Interpolation between the fixed-space endpoints gives (10) with all fixed parameter derivatives.

To prove sharpness put $f = 1$. Then $R = 0$, $S = -\Delta/2$ and $Q_h = e^{-hS}$. On an eigenmode of S with eigenvalue λ, the errors of Richardson and the positive correction respectively are $hF_{\mathrm{R}}(h\lambda)$ and $hF_{+}(h\lambda)$, where

$$F_{\mathrm{R}}(t) = \frac{2}{3\sinh t} - \frac{1}{3\sinh(2t)} - \frac{1}{2t} = -7t^3/180 + O(t^5),$$
$$F_{+}(t) = \frac{1}{2\sinh t} + \frac{1}{6}\tanh(t/2) - \frac{1}{2t} = t^3/360 + O(t^5).$$

Choose a fixed positive t with both functions nonzero and lattice frequencies

$$n_h = (\mathrm{round}\,\sqrt{t/(2\pi^2 h)}, 0, \ldots, 0).$$

Then $h\lambda_{n_h} \to t$ and their H^r norm is of order $h^{-r/2}$. Each operator error is at least $c_r h^{1+r/2}$ for all small h. A fixed nonzero mode additionally gives the nonzero fourth-order coefficients $-7\lambda^3/180$ and $\lambda^3/360$. This proves sharpness for these two tensors. For $\widehat{C}_h$, (38) agrees with ordinary Richardson at constant density because $Q_h^2 = Q_{2h}$ and $R = 0$. Its error function is therefore F_{R} and its nonzero fourth-order coefficient is also $-7\lambda^3/180$. The same lattice modes establish sharpness for every $r \in [0, 6]$. □

7 Family geometry and limits

Proof of Corollary 1.4. Since $\langle \eta_i, q\rangle = \int \partial_i f = 0$, all tangent contractions are on the stated complement. Joint smoothness provides the finitely many spatial and parameter bounds on η_i required by the H^6 estimates. The product rule and Theorems 1.1– 1.2 give the stated matrix expansions in C^k. The inverse weighted-Laplacian form is strictly positive on nonzero tangents. Immersion and compactness give a uniform positive lower bound for g^G. Uniform convergence then makes the corrected matrices positive and uniformly invertible. Matrix inversion is smooth on this set; the coordinate Christoffel formula uses one metric derivative and the curvature formula uses two. The rates therefore persist in C^{k-1} and C^{k-2}, respectively. □

In density coordinates the compact positive construction is explicitly

$$\widehat{C}_{h,f} = T_{h,f} + \frac{h}{3}P_h(I - P_h)(I + P_h)^{-1}(I + P_h^2)^{-1} - \frac{h^2}{6}T_{h,f}D_H T_{h,f}$$

on $L^2_0(f)$, with the bounded sandwich interpretation proved above. The local Hessian of V is the density-specific mobility defect; the scalar extrapolation filter alone does not remove it.

The topology qualifications are essential. The original tensor has an order-two bias, and fourth-order corrected accuracy requires H^6 inputs. No fourth-order full L^2 estimate is possible for these constructions. The compact correction preserves the original tensor's completed tangent form space and square-root range up to equivalent norms, while the Simpson comparison has a noncompact identity floor. Raw two-bandwidth Richardson is positive at constant density. Beyond the restricted analytic construction of Appendix B, effective discretized or sample-based guarantees, joint vanishing-bandwidth/geodesic approximation, and extension to vanishing densities or boundary-domain fixed costs require separate arguments. The fixed-regularization discretization theory of [13] concerns its stated fixed-regularization regime.

A A certified nonconstant computation

We give a finite continuum certificate for metric values, rather than using differences between numerical grids as an error estimate. General normalized-diffusion refinement has established precedents [16]; the calculation below concerns the corrected inverse tensor and its differential sandwich. The certificate concerns metric values.

A.1 An exact scale and rational residuals

On the circle put

$$w_\theta = 1 + \theta_1 \cos(2\pi x) + \theta_2 \cos(4\pi x), \qquad v = H_h w_\theta,$$
$$Z = \int w_\theta v, \qquad f_{h,\theta} = w_\theta v / Z, \qquad b = \sqrt{w_\theta / v}.$$

For $|\theta_1| + |\theta_2| < 1$, these are positive and $bH_h(b\sqrt{f}) = \sqrt{f}$ exactly. This is the classical symmetric Schrödinger endpoint product construction [17]. Its Markov operator and actual parameter scores are

$$Pu = H_h(wu)/v, \qquad Y_i = \partial_i \log f = w_i/w + v_i/v - Z_i/Z.$$

Here and below $w = w_\theta$. The density varies with h: comparisons are always with $G[f_{h,\theta}]$, not $G[f_{0,\theta}]$. Joint smoothness and positivity on compact parameter sets allow the uniform analytic estimates above to apply along this family.

Let $\mathcal{R}_z = (I - zP)^{-1}$ for $z = -1, i$, and let $\mathcal{R}_1$ be the inverse on $L^2_0(f)$. Scalar functional calculus gives

$$T = \frac{h}{2}(\mathcal{R}_1 - \mathcal{R}_{-1}), \qquad C_{\text{base}} = \frac{h}{6}(3\mathcal{R}_1 - 5\mathcal{R}_{-1} + 2\,\mathrm{Re}\,\mathcal{R}_i), \tag{42}$$

where C_{base} is the scalar part of $\widehat{C}_{h,f}$. In particular

$$g^{\widehat{C}}_{ij} = \langle Y_i, C_{\text{base}} Y_j \rangle_f - \frac{h^2}{6} \int f V''(TY_i)'(TY_j)'. \tag{43}$$

The polynomial $F_i = wvY_i = w_i v + w v_i - (Z_i/Z)wv$ has degree at most four and zero mean. For any trigonometric polynomial s, the trial $x = s/w$ has the exact continuum residual

$$Y_i - (I - zP)x = E/(wv), \qquad E = F_i - vs + zwH_h s. \tag{44}$$

Thus every residual Fourier coefficient is obtained by finite, unaliasing polynomial operations. All generated modes are retained. Project a trial to $L^2_0(f)$ by replacing s with $s - cw$, $c = \int vs/Z$. At $z = 1$ this leaves the residual unchanged; at the other values it is recomputed after projection.

A.2 A computable continuum error bound

Write

$$w_0 = 1 - |\theta_1| - |\theta_2|, \quad v_0 = 1 - |\theta_1| e^{-2\pi^2 h} - |\theta_2| e^{-8\pi^2 h}, \quad w_1 = 1 + |\theta_1| + |\theta_2|.$$

Reversibility and heat Poincaré comparison give the whole-space gap

$$I - P \geq \delta I \text{ on } L^2_0(f), \qquad \delta = (w_0/w_1)^2 (1 - e^{-2\pi^2 h}). \tag{45}$$

Indeed the Dirichlet form is $(2Z)^{-1} \iint w(x)w(y)k_h(x-y)(u(x)-u(y))^2$. It bounds w_0^2/Z times the unweighted heat form, whereas $\int fu^2 \leq (w_1^2/Z)\,\mathrm{Var}_{dx}(u)$ for $\int fu = 0$. This proves (45), including the unresolved Fourier complement.

For (44), Parseval gives

$$\|r\|_f \leq \frac{\|E\|_2}{\sqrt{Z w_0 v_0}}, \qquad \|r'\|_f \leq \frac{\|E'\|_2 + L\|E\|_2}{\sqrt{Z w_0 v_0}}, \quad L = \frac{\|w'\|_\infty}{w_0} + \frac{\|v'\|_\infty}{v_0}. \tag{46}$$

Use the sum of squared real and imaginary polynomial norms for complex E. No discrete residual is substituted for these continuum norms. The following estimate pays for the differential sandwich.

Lemma A.1 (Explicit residual smoothing)**.** *Set*

$$\mathcal{F} = 2(\|v'\|_\infty/v_0)^2 + \frac{2w_1}{v_0 h}, \quad K_-^2 = \frac{2\mathcal{F}h^2}{\delta} + \frac{h^2}{\delta^2}, \quad K_+ = h\sqrt{1+2\mathcal{F}}.$$

For $r \in L_0^2(f)$,

$$\|hP\mathcal{R}_1 r\|_{H^1(f)} \le K_-\|r\|_f, \qquad \|hP\mathcal{R}_{-1} r\|_{H^1(f)} \le K_+\|r\|_f.$$

The constant K_- remains bounded as $h \downarrow 0$ on compact positive parameter sets.

Proof. The Markov kernel is $p(x,y) = k_h(x-y)w(y)/v(x)$. The wrapped Gaussian score satisfies $\int (k_h')^2/k_h \le 1/h$: Cauchy–Schwarz on its Gaussian translates and unfolding a period reduce this to Gaussian Fisher information. Consequently $\int p|\partial_x \log p|^2 \le \mathcal{F}$. Subtracting $u(x)$ inside the differentiated Markov integral and using reversibility yields

$$\|(Pu)'\|_f^2 \le 2\mathcal{F}\langle u, (I-P)u\rangle_f.$$

Apply this to $u = \mathcal{R}_1 r$, using $\langle u, (I-P)u\rangle_f \le \delta^{-1}\|r\|_f^2$ and $\|u\|_f \le \delta^{-1}\|r\|_f$. For $u = \mathcal{R}_{-1} r$, both $\|u\|_f \le \|r\|_f$ and $\langle u, (I-P)u\rangle_f \le \|r\|_f^2$ follow from $0 \le P \le I$. These prove the two estimates. Since $\delta \asymp h$ and $\mathcal{F} = O(h^{-1})$, the final assertion follows. □

For each column let $r_{1,j}, r_{-1,j}, r_{i,j}$ be the residuals of its three trial inverses, and put $U_j = (h/2)(x_{1,j} - x_{-1,j})$. Define the computable bounds

$$e_j = \tfrac{1}{2}\big[h(\|r_{1,j}\|_{H^1(f)} + \|r_{-1,j}\|_{H^1(f)}) + K_-\|r_{1,j}\|_f + K_+\|r_{-1,j}\|_f\big],$$
$$d_j = \tfrac{h}{6}\big(3\|r_{1,j}\|_f/\delta + 5\|r_{-1,j}\|_f + 2\|r_{i,j}\|_f\big).$$

Then $\|TY_j - U_j\|_{H^1(f)} \le e_j$ and the error in a symmetrized trial value of (43) is at most

$$\frac{\|Y_i\|_f d_j + \|Y_j\|_f d_i}{2} + \frac{h^2}{6}\|V''\|_\infty(e_i\|U_j'\|_f + e_j\|U_i'\|_f + e_i e_j). \tag{47}$$

For the first claim use $\mathcal{R}_1 = I + P\mathcal{R}_1$ and $\mathcal{R}_{-1} = I - P\mathcal{R}_{-1}$; retaining the identity terms accounts for the residual derivatives. The base error uses inverse norms $\delta^{-1}, 1, 1$ in (42). Expanding the last integral in (43) proves the remaining terms of (47). This is a direct value certificate; no additional adjoint solve is needed.

A.3 Interval results and reproduction

Take $(\theta_1, \theta_2) = (1/5, 1/10)$. The supplement `validated_metric_example.py` freezes even trigonometric trials as exact decimal rationals and evaluates (46)– (47) with `mpmath` 1.3.0 interval arithmetic at 60 decimal digits. The 75-digit trial solve is only a proposal; the residual certificate does not assume its accuracy. Rational denominators are enclosed away from zero, and all interval endpoints are rendered outward by exact integer division.

Quadrature uses 192 equispaced nodes and the strip $|\operatorname{Im} x| \le a = 1/16$. On this strip

$$|w| \ge 1 - \tfrac{1}{5}\cosh(2\pi a) - \tfrac{1}{10}\cosh(4\pi a) > 0,$$

and the corresponding bound for v is larger. Finite weighted sums $\sum_n |s_n| e^{2\pi a|n|}$ bound polynomial strip suprema and their derivatives. Rational product bounds therefore give an explicit strip bound M for each integrand. The trapezoidal error is bounded by $2M/(e^{2\pi aJ} - 1)$: contour shifting bounds Fourier coefficients, and the quadrature error is their sum at nonzero multiples of J. The resulting radius is added to (47). For evaluating V'', set $l = (w'/w + v'/v)/2$ and use $V'' = l''' /2 + (l')^2 + ll''$.

Independently, if B_i is the odd periodic primitive of $\partial_i f$, then $f\phi_i' = -B_i$ solves the weighted Poisson equation; its constant flux vanishes by parity. Hence $g^G_{ij} = \int B_i B_j / f$, evaluated with its own quadrature enclosure. At $h = 0.000625$, the intervals, rounded outward further for display, are

ij	$g^{\widehat{C}}_{ij}$	g^G_{ij}
11	[0.05599852219998459, 0.05599852219998460]	[0.05599852221282253, 0.05599852221282254]
12	[0.00018731713055807, 0.00018731713055809]	[0.00018731698780002, 0.00018731698780003]
22	[0.01155401521838808, 0.01155401521838810]	[0.01155402112826470, 0.01155402112826472]

The certified Gershgorin lower bound for this corrected matrix exceeds 0.01136669808. Table 1 gives the matching-density 22 bias at five bandwidths; N is the maximum trial Fourier frequency. Complete intervals,

h	N	certified $g^{\widehat{C}}_{22} - g^G_{22}$
0.01	20	$[-1.535438, -1.535437]\ 10^{-4}$
0.005	20	$[-1.607374, -1.607373]\ 10^{-5}$
0.0025	20	$[-1.277676, -1.277675]\ 10^{-6}$
0.00125	28	$[-8.949553, -8.949552]\ 10^{-8}$
0.000625	28	$[-5.909877, -5.909876]\ 10^{-9}$

Table 1: Continuum value enclosures; each row uses its own $f_{h,\theta}$.

residual budgets and rational trial coefficients are in `validation_results.json`. These finite values illustrate the analytic theorem. Exact normalization isolates metric-solve error here, so no general normalization-solver, empirical, numerical C^2 or runtime guarantee is inferred. The core convergence and positivity proofs are analytic and independent of this calculation.

B Prescribed analytic densities and logarithmic resolution

The preceding example has an h-dependent density and an exact scale. We now include normalization for a prescribed density which stays fixed as $h \downarrow 0$. The smallness and analyticity assumptions below are substantial. Exponential spectral approximation and uniform nonlocal limits are established principles [18]; bandwidth-independent normalization iteration also predates this work [3]. The result here concerns their specific integration with the corrected inverse tensor and its differential term. At fixed regularization, Sinkhorn information-matrix approximation already has logarithmic tolerance dependence [19]; band-limited heat-kernel transport with complexity estimates is also established [20]. We control the joint vanishing-bandwidth resolution of a particular corrected local tensor, rather than uniformly approximating the entire narrow heat kernel with logarithmic bandwidth.

For $n \in \mathbb{Z}^d$ write $|n|_1 = \sum_j |n_j|$, and define the Banach Fourier algebras

$$\|u\|_{A_\rho} = \sum_n |\widehat{u}(n)| e^{\rho|n|_1}, \qquad \|u\|_{B_\rho} = \sum_n (1 + 2|n|_1) |\widehat{u}(n)| e^{\rho|n|_1}.$$

Products are contractive in these norms: both weights are submultiplicative. Fourier projections and H_h are contractions. If $0 < \rho' < \rho$ and $|\alpha| = j \geq 1$, then

$$\|\partial_x^\alpha u\|_{A_{\rho'}} \leq (2\pi)^j \left(\frac{j}{e(\rho - \rho')} \right)^j \|u\|_{A_\rho}. \tag{48}$$

It follows by maximizing $t^j e^{-(\rho-\rho')t}$.

Theorem B.1 (Fixed prescribed densities)**.** *Fix $d \geq 1$, $h_0 = 1/100$, $\rho, \eta > 0$ and a compact real parameter box $K \subset \mathbb{R}^p$. Suppose $f_\theta = 1 + g_\theta$ has integral one, is independent of h, and extends holomorphically to the complex polydisc neighborhood K_η with*

$$\sup_{\theta \in K_\eta} \|g_\theta\|_{B_\rho} \leq 1/8. \tag{49}$$

For real parameters assume real Fourier symmetry. For every fixed integer $k \geq 0$, there is a symmetric finite matrix construction M_h with

$$\|M_h - g^{\widehat{C}_h[f]}\|_{C^k(K)} \leq h^4, \qquad \|M_h - g^{G[f]}\|_{C^k(K)} \leq C_k h^4$$

for $0 < h \leq h_0$, in the entrywise matrix norm. The first bound refers to exact evaluation of the finite construction; a separately enclosed evaluation error at most h^4 changes it to $2h^4$. The Fourier numerator degrees in each coordinate, normalization and root iteration counts, and quadrature resolution in each coordinate are $O_k(1+\log(1/h))$. Thus total retained modes and quadrature nodes are $O_k((1+\log(1/h))^d)$, with constants depending on d.

Finite execution requires exact or certified input Fourier coefficients on K_η, with the norm/tail information in (49)*; nonpolynomial inputs are not assumed to arrive with a free coefficient oracle.*

We prove the theorem and give sufficient integer choices. The proof separates normalization, the shrinking inverse, density/score changes, root truncation and integration. In particular it does not replace normalization at f silently by normalization at a different density.

B.1 Normalization and its finite approximation

Set

$$r = \frac{1 - 1/\sqrt{2}}{2}, \quad \lambda = 2r < 1, \quad L_h = (I + H_h)^{-1}.$$

The Fourier multiplier of L_h has modulus at most one, so it is a contraction on both algebras. The map

$$\Phi_h(u) = L_h(g - uH_h u) \tag{50}$$

preserves the closed B_ρ ball of radius r, since $1/8 + r^2 = r$, and is λ-Lipschitz there. Its unique fixed point is holomorphic in parameters by uniform holomorphic iteration. Put

$$w = 1 + u_h, \quad v = H_h w, \quad wv = f, \quad Z = \int wv = 1.$$

For real parameters $w, v > 0$, and $b = \sqrt{w/v}$ is the exact scale. Thus this construction normalizes the prescribed fixed density.

Let P_N retain $|n|_\infty \leq N$, and form

$$t_0 = 0, \qquad t_{j+1} = \mathsf{P}_N \Phi_h(t_j). \tag{51}$$

Every iterate has degree at most N in each coordinate and norm at most r. Comparing the exact and projected fixed points and then iterating the contraction gives, for $\rho_1 = 3\rho/4$,

$$\|u_h - t_L\|_{B_{\rho_1}} \leq \varepsilon_N := \frac{re^{-(\rho-\rho_1)N}}{1-\lambda} + r\lambda^L. \tag{52}$$

Indeed the exact fixed-point tail is at most $re^{-(\rho-\rho_1)N}$; the remaining difference is absorbed by $1 - \lambda$. The iteration error from zero is at most $r\lambda^L$. Only the retained input coefficients enter (51); certified coefficient uncertainty adds its separately enclosed budget.

Write $w_t = 1 + t_L$, $v_t = H_h w_t$, $Z_t = \int w_t v_t$ and $f_t = w_t v_t / Z_t$. Generally $f_t \neq f$ and $Z_t \neq 1$. Their changes will be paid for below. On $K_{3\eta/4}$, parameter Cauchy estimates give

$$\|w_i\|_{B_{\rho_1}}, \|(w_t)_i\|_{B_{\rho_1}} \leq B_0 := 4r/\eta, \qquad \|w_i - (w_t)_i\|_{B_{\rho_1}} \leq 4\varepsilon_N/\eta. \tag{53}$$

B.2 A uniform scaled inverse on a fixed complex neighborhood

For this step let $w = 1 + u$ be any complex element with $\|u\|_{B_{\rho_1}} \le R = 1/5$. Set $v = H_h w$ and $Z = \int wv$. With $U = 1 + R$, $z_0 = 1 - 2R - R^2$, we have

$$\|1/w\|_{A_{\rho_1}}, \|1/v\|_{A_{\rho_1}} \le (1 - R)^{-1}, \qquad |Z| \ge z_0 > 0.$$

These follow from geometric series and $|Z - 1| \le 2R + R^2$.

The transformed root equation is again

$$\mathcal{L}_z(w)s = vs - zwH_h s = F, \qquad x = s/w, \quad F = wvY, \quad z \in \{1, -1, i, -i\}.$$

Put $e_n = e^{-2\pi^2 h|n|_2^2}$ and $\mathcal{D}_z = \mathrm{diag}(1 - ze_n)$. For $z = 1$ omit the zero Fourier row and column and choose $s_0 = 0$. The full zero row vanishes identically, and $F_0 = 0$ is exactly compatibility. Since $|w_0| \ge 1 - R$, this gauge is transverse to the true kernel $\mathrm{span}(w)$. Write $\mathcal{L}_z = \mathcal{D}_z(I + \mathcal{E}_z(w))$. The constant perturbation u_0 is retained in $\mathcal{E}_z$. For $n = m + a \ne 0$,

$$\frac{|e_a - e_m|}{1 - e_n} \le \max\left(1, \frac{\left||a|_2^2 - |m|_2^2\right|}{|n|_2^2}\right) \le 1 + 2|a|_1. \tag{54}$$

For the first inequality use $|e^{-A} - e^{-B}| \le 1 - e^{-|A-B|}$ and $(1 - e^{-at})/(1 - e^{-as}) \le \max(1, t/s)$. For the second use $m = n - a$ and $|n|_2 \ge 1$. Weighted column summation therefore gives

$$\|\mathcal{E}_1\|_{A_{\rho_1} \to A_{\rho_1}} \le R, \qquad \|\mathcal{E}_z\|_{A_{\rho_1} \to A_{\rho_1}} \le 2R \quad (z = -1, i, -i). \tag{55}$$

The latter uses $|1 - ze_n| \ge 1$ and $|e_a - ze_m| \le 2$. The B norm controls coefficients; the inverse bounds are on A, not on B with an unaccounted shift weight.

Set $c_0 = (1 - e^{-2\pi^2 h_0})/h_0$. Then

$$\|h\mathcal{D}_1^{-1}\| \le c_0^{-1}, \quad \|h\mathcal{L}_1^{-1}\| \le [c_0(1 - R)]^{-1}, \quad \|\mathcal{L}_z^{-1}\| \le (1 - 2R)^{-1} \ (z = -1, i, -i).$$

To select invariant mean zero, replace s/w by $s/w - \int vs/Z$. This map from s has norm at most

$$C_P = (1 - R)^{-1} + U/z_0. \tag{56}$$

All estimates hold on the same complex parameter neighborhood. The map $w \mapsto \mathcal{E}_z(w)$ is linear, with difference norm at most $\|w - \widetilde{w}\|_{B_{\rho_1}}$ for $z = 1$, and twice that bound for the stable roots. The resolvent identity for $I + \mathcal{E}$ therefore gives uniform Lipschitz bounds for $h\mathcal{L}_1^{-1}$. For example, at fixed compatible F its difference is at most

$$\frac{\|w - \widetilde{w}\|_{B_{\rho_1}} \|F\|_{A_{\rho_1}}}{c_0(1 - R)^2}.$$

This cancellation avoids an unnecessary h^{-1} normalization-error loss. It also supplies the complex bounds needed for parameter differentiation; a real spectral gap alone would not suffice.

B.3 An explicit stability and integration ledger

We give conservative constants to make every cutoff below definite. For w_i of norm one, the actual score forcing is

$$Y_i = w_i/w + H_h w_i/v - Z_i/Z, \qquad F_i = w_i v + wH_h w_i - (Z_i/Z)wv, \quad |Z_i| \le 2U\|w_i\|_{A_{\rho_1}}.$$

Define

$$K_f = U^2/z_0, \qquad K_Y = 2/(1-R) + 2U/z_0,$$
$$K_F = 2U(1+U^2/z_0), \qquad K_X = C_P K_F \max\{[c_0(1-R)]^{-1}, h_0/(1-2R)\}.$$

Thus every mean-zero scaled root $h\mathcal{R}_z Y_i$ has A norm at most $K_X\|w_i\|_{B_{\rho_1}}$; the same bound holds for the finite projected iterates below. Put $\sigma = \rho/4$, $\rho_2 = \rho/2$, $\rho_3 = \rho/4$, and

$$D_j = (2\pi)^j [j/(e\sigma)]^j \quad (1 \le j \le 4), \qquad L_0 = -\log(1-R),$$
$$K_H = d^2 \left[\tfrac{1}{2} D_4 L_0 + (D_2^2 + D_1 D_3) L_0^2\right].$$

To verify K_H, let $\ell = (\log w + \log v)/2$ up to its spatially constant normalization. Then $V = (\Delta\ell + \nabla\ell\cdot\nabla\ell)/2$ and (48) bounds each Hessian entry by K_H/d. Summing products with two gradients bounds the full integrand by $dK_H D_1^2$ times their undifferentiated norms.

The holomorphic extension of the scalar base uses all four roots:

$$T = \tfrac{h}{2}(\mathcal{R}_1 - \mathcal{R}_{-1}), \qquad C_{\text{base}} = \tfrac{h}{6}(3\mathcal{R}_1 - 5\mathcal{R}_{-1} + \mathcal{R}_i + \mathcal{R}_{-i}).$$

On the real slice this agrees with (42). Taking a real part on complex parameters would not be holomorphic. The complete metric expression is

$$g_{ij}(w) = \int f_w Y_i C_{\text{base}} Y_j - \frac{h^2}{6} \int f_w \nabla(TY_i)^{\mathsf{T}} \operatorname{Hess} V \nabla(TY_j), \quad f_w = wv/Z. \tag{57}$$

Define

$$B_* = K_f \left[\tfrac{5}{3} K_Y K_X + \tfrac{h_0^2}{6} dK_H D_1^2 K_X^2\right],$$
$$B_Q = B_* B_0^2, \qquad C_R = 2B_Q, \qquad C_S = B_* \left[B_0^2/(R-r) + 8B_0/\eta\right],$$
$$A_k = k! \max(1, 2/\eta)^k.$$

Here B_* bounds the A_{ρ_2} metric integrand for unit parameter directions, uniformly on the complex B_{ρ_1} ball of radius R. Cauchy's estimate in that Banach ball bounds its derivative in w, at norm at most r, by $B_*/(R-r)$. The expression is bilinear in w_i, w_j. Applying (53) therefore gives the explicit transfer bound

$$\sup_{K_{3\eta/4}} |g_{ij}(w) - g_{ij}(w_t)| \le C_S \varepsilon_N. \tag{58}$$

This includes the changes in density, mass, scores, reciprocal factors and invariant mean. Parameter Cauchy estimates on radius $\eta/2$ multiply this bound by at most A_k for derivatives of order at most k. The ledger gives a conservative bound.

B.4 Finite root solves and the total error budget

For w_t use one fixed Fourier cube and iterate on the scaled unknown $a = hs$:

$$a_0 = 0, \qquad a_{j+1} = \mathsf{P}_M[h\mathcal{D}_z^{-1} F_i - \mathcal{E}_z(w_t) a_j]. \tag{59}$$

Omit $n = 0$ for $z = 1$. Products are formed at their full degree before projection. The contraction is at most $2R$. The exact scaled solution is uniformly bounded on A_{ρ_1}. Comparing projected and exact fixed points on A_{ρ_2} and iterating gives a root error at most

$$K_X B_0 \xi, \qquad \xi = \frac{e^{-\sigma M}}{1-2R} + (2R)^J,$$

after division and invariant-mean projection. For the scaled polynomial this projection is explicitly $a \leftarrow a - w_t \int v_t a / Z_t$. The numerator degree is at most $\max(M, N)$ in each coordinate. An unprojected iteration could increase degree by N at every step; (59) avoids that logarithm-squared degree.

Use the direct root combinations in (57), and symmetrize the finite matrix. The error in T is at most $K_X B_0 \xi$, and in the scalar base at most $5K_X B_0 \xi/3$. Paying the additional strip loss from ρ_2 to ρ_3 for gradient errors, the sandwich difference is bounded by the sum of one-error/one-field products. Both exact and trial root norms are at most $K_X B_0$, so the entrywise error is at most $C_R \xi$. This proves the root part of the ledger without an adjoint solve or nonpolynomial forcing oracle.

All finite metric integrands have A_{ρ_3} norm at most B_Q on $K_{3\eta/4}$. For a product trapezoidal rule with Q nodes in each coordinate, the error is at most $B_Q e^{-\rho_3 Q}$. Indeed only Fourier coefficients in $Q\mathbb{Z}^d \setminus \{0\}$ alias into the integral, and their $|n|_1$ is at least Q. This weighted-norm version of exponential periodic quadrature [21] avoids an additional lattice-sum constant. Consequently, with exact finite evaluation,

$$\|M_h - g^{\widehat{C}_h[f]}\|_{C^k(K)} \le A_k \left[C_S \varepsilon_N + C_R \xi + B_Q e^{-\rho_3 Q} \right]. \tag{60}$$

For example choose positive integers satisfying

$$e^{-\sigma N} \le \frac{(1-\lambda)h^4}{6A_k C_S r}, \qquad \lambda^L \le \frac{h^4}{6A_k C_S r},$$
$$e^{-\sigma M} \le \frac{(1-2R)h^4}{6A_k C_R}, \qquad (2R)^J \le \frac{h^4}{6A_k C_R},$$
$$e^{-\rho_3 Q} \le \frac{h^4}{3A_k B_Q}.$$

Taking ceilings of the corresponding logarithms, with a lower bound of one, makes every choice $O_k(1 + \log(1/h))$ and proves the first bound in Theorem B.1. Theorem 1.2 gives the second. Parameter estimates are applied to holomorphic functions with cutoffs fixed at each h, not by differentiating the integer cutoffs or a scalar norm bound. Initially the second bound holds for sufficiently small h. It extends, with a larger C_k, to $0 < h \le h_0$: on any remaining compact positive h interval the normalization, root bounds and smooth coefficients above uniformly bound both exact metrics and their fixed parameter derivatives.

For polynomial input, the construction uses only finite convolutions, projections, heat multipliers, nonzero scalar divisions and rational point evaluations. Input-enclosure error and arithmetic error must be added separately to (60). At a fixed h the finite operations and strict reciprocal/inverse margins permit validated precision to be increased until the chosen finite-evaluation budget is met. Numerical parameter-jet validation likewise requires enclosures of those jets, not just a real-parameter scalar interval. No uniform bit-complexity statement follows. For nonpolynomial input the coefficient/tail information stated in the theorem remains an explicit input requirement.

B.5 A fixed-density certificate and finite-family geometry

On the circle take

$$f_\theta = 1 + \theta_1 \cos(2\pi x) + \theta_2 \cos(4\pi x), \qquad |\theta_1| \le .02, \quad |\theta_2| \le .005.$$

With $\rho = .2$ and complex thickening $\eta = .0005$, $3(.0205)e^{.2} + 5(.0055)e^{.4} < 1/8$, so the theorem applies to these fixed prescribed densities. The supplement `fixed_normalization_example.py` supplies interval normalization certificates at $(.02, .005)$ and $h = .01, .00125, .00015625$, using cutoff 32 and 28 iterations. For a polynomial trial t in the radius-r ball, the full continuum fixed-point residual gives

$$\|u_h - t\|_{B_\rho} \le \|\Phi_h(t) - t\|_{B_\rho}/(1-\lambda).$$

Every generated mode through twice the cutoff is included. At $h = .00125$ the resulting w error is below $3.787\,10^{-42}$, and the sup error in $b = \sqrt{w/H_h w}$ is below $6.023\,10^{-42}$. An independent cosine-convolution interval reconstruction includes the omitted modes, which account for almost the entire residual. The complete rational trials and bounds are in `fixed_normalization_results.json`. This validates normalization at the stated points, not a full fixed-density metric, parameter-box or curvature computation.

For this fixed family an explicit limiting Gram margin is

$$g_\theta^G(c,c) \geq \frac{|c|^2}{32\pi^2(1.025)}.$$

The density perturbation is $\zeta = c_1\cos(2\pi x) + c_2\cos(4\pi x)$. The variational inverse weighted-Laplacian form is at least $\|\zeta\|^2_{\dot{H}^{-1}}/1.025$, and its two modes give this bound. For general immersed finite families compactness supplies a positive limiting Gram margin. Sufficiently accurate symmetric M_h are therefore positive for small h; inversion and the coordinate formulas give fourth-order connection and intrinsic-curvature approximation in C^{k-1} and C^{k-2} for $k \geq 2$. These are finite-family analytic consequences, not ambient curvature, global distance, numerical curvature certificates or a general-density normalization theorem without (49).

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