Large negative subspaces for positive Euler deformations

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Running title: Negative subspaces for Euler products

Abstract

We construct positive Euler products on the ordinary primes whose prime-shift quadratic forms are negative on subspaces of growing dimension. The prime exponents can be uniformly as close to one as prescribed, agree with one on any prescribed finite prime prefix, and remain the same at every power of each prime. The corresponding logarithmic derivatives satisfy a classical subexponential prime number theorem remainder. For one fixed deformation and every sufficiently large support radius, the negative index is at least the support radius times a subexponential factor in its logarithm, with an exponentially large negative Rayleigh bound. The test functions are odd and satisfy two polar-cancellation conditions. The same construction works against each fixed Hermitian comparison form controlled by a finite Sobolev norm, with an onset depending on that form. Finite linear constraints and corrections are treated by dimension bounds; for general Hermitian repairs, only their positive index on the constructed space reduces the guarantee. The proof combines classical oscillatory templates with progression-free packet centers, then controls every mixed interaction and all higher prime powers. The result quantifies how large a signed-positivity obstruction can coexist with positive Euler structure, finite-prefix agreement and prime-counting control.

Keywords: positive Euler product; prime-weighted quadratic form; negative index; progression-free set; Sobolev regularization.

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1 The question and the main result

Positivity of Euler coefficients and positivity of an arithmetic quadratic form are different conditions. The first is local and multiplicative: each Euler factor has positive coefficients. The second compares a complete prime-shift operator with a fixed comparison form, including the interaction between all shifts. Prime number theorem estimates control a cumulative prime measure, whereas a quadratic form can detect coherent oscillations within that measure.

This paper asks how many independent negative directions can remain when the prime weights are positive, uniformly close to their classical values, and constrained by both a finite prime prefix and a prime number theorem remainder. The answer is quantitative. One fixed deformation of the ordinary primes has almost linearly many negative directions, measured in the support radius of the test functions, at every sufficiently large window. The same weights and test spaces work for each fixed comparison energy controlled by a finite Sobolev norm, with an eventual support threshold depending on that energy. Any Hermitian correction that factors through fewer linear coordinates than the dimension of a constructed negative space leaves a negative direction.

1.1 Arithmetic forms and negative index

For $f \in C_c^\infty(\mathbb{R})$, set

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(u) e^{-i\xi u}\,\mathrm{d}u, \qquad Mf(z) = \int_{\mathbb{R}} f(u) e^{zu}\,\mathrm{d}u, \qquad (\tau_t f)(u) = f(u-t),$$

and use the inner product $\langle f, g\rangle = \int_{\mathbb{R}} \overline{f(u)} g(u)\,\mathrm{d}u$. Our test space is

$$\begin{aligned} \mathcal{V} &= \{f \in C_c^\infty(\mathbb{R};\mathbb{C}) : f(-u) = -f(u), \quad Mf(1/2) = Mf(-1/2) = 0\}, \\ \mathcal{V}_L &= \{f \in \mathcal{V} : \operatorname{supp} f \subset [-L, L]\}. \end{aligned} \tag{1}$$

The two Mellin conditions will cancel a growing continuous kernel exactly.

Let $b = (b_p)_p$ be a bounded positive sequence indexed by the ordinary primes. Write

$$\Lambda_b(p^k) = b_p \log p \quad (k \geq 1), \qquad \Lambda_b(n) = 0 \quad \text{otherwise}, \qquad \psi_b(x) = \sum_{n \leq x} \Lambda_b(n). \tag{2}$$

The same b_p occurs at every power of p. Define the Hermitian arithmetic form

$$\mathfrak{c}_b[f] = 2 \operatorname{Re} \sum_{n \geq 2} \frac{\Lambda_b(n)}{\sqrt{n}} \langle f, \tau_{\log n} f \rangle. \tag{3}$$

Only finitely many terms contribute to each compactly supported test. For a Hermitian form $\mathfrak{a}$ on $\mathcal{V}$, the form to be studied is

$$\mathfrak{q}_b^{\mathfrak{a}}[f] = \mathfrak{a}[f] - \mathfrak{c}_b[f]. \tag{4}$$

We use the inhomogeneous Sobolev norm

$$\|f\|_{H^s}^2 = \frac{1}{2\pi} \int_{\mathbb{R}} (1 + \xi^2)^s |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi.$$

Call $\mathfrak{a}$ *Sobolev-controlled* if, for some finite $s \geq 0$ and $C_{\mathfrak{a}} < \infty$,

$$|\mathfrak{a}[f]| \leq C_{\mathfrak{a}} \|f\|_{H^s}^2 \qquad (f \in \mathcal{V}). \tag{5}$$

The constant is independent of the support of f. This class includes every fixed polynomially bounded Fourier multiplier, and in particular logarithmic multiplier energies.

For a Hermitian form q on a complex vector space E, its *negative index* is

$$\operatorname{ind}_-(q; E) = \sup\{\dim W : W \subset E, \ \dim W < \infty, \ q[f] < 0 \text{ for every } 0 \neq f \in W\}.$$

This is an algebraic form definition, so it applies directly to the spaces in (1).

Theorem 1.1 (Large negative subspaces)**.** *Fix $0 < \epsilon < 1$ and $P_0 \geq 2$. There are ordinary-prime weights b_p, constants $c, C, L_0 > 0$, and subspaces $W(L) \subset \mathcal{V}_L$, of dimensions $N(L)$, for every $L \geq L_0$, such that:*

1.

$$1 - \epsilon \leq b_p \leq 1 + \epsilon, \qquad b_p = 1 \quad (p \leq P_0), \tag{6}$$

and, for one $c_b > 0$ and one global implied constant,

$$\psi_b(x) = x + O\Big(xe^{-c_b\sqrt{\log x}}\Big) \qquad (x \geq 2). \tag{7}$$

2. The Euler product

$$Z_b(s) = \prod_p (1 - p^{-s})^{-b_p} = \sum_{n\geq 1} \frac{a_b(n)}{n^s}, \qquad \operatorname{Re} s > 1, \tag{8}$$

converges absolutely, has $a_b(n) > 0$, and satisfies $-Z_b'/Z_b = \sum_{n\geq 2} \Lambda_b(n)n^{-s}$. Moreover $a_b(n) = 1$ for $n \leq P_0$.

3.

$$N(L) \geq cL\exp(-C\sqrt{\log L}) \qquad (L \geq L_0). \tag{9}$$

There are constants $c_\epsilon, c_* > 0$ such that, for every fixed Sobolev-controlled Hermitian form $\mathfrak{a}$, all sufficiently large L satisfy

$$\mathfrak{q}_b^{\mathfrak{a}}[f] \leq -c_\epsilon e^{c_* L}\|f\|_2^2 \qquad (f \in W(L)). \tag{10}$$

The weights and subspaces are chosen independently of $\mathfrak{a}$; the eventual threshold in L may depend on the bound in (5)*.*

Consequently $\operatorname{ind}_-(\mathfrak{q}_b^{\mathfrak{a}}; \mathcal{V}_L) \geq cL\exp(-C\sqrt{\log L})$ *for every sufficiently large L.*

The universal choice of weights is important. Given any fixed Sobolev-controlled form $\mathfrak{a}_0$, adding a second such form R amounts to applying the theorem to $\mathfrak{a}_0 + R$; the deformation b and the spaces $W(L)$ stay the same. Section 6 also gives the precise dimension surviving finitely many linear constraints or a finite-rank correction.

1.2 Why a whole negative subspace is a stronger conclusion

Independent negative test vectors need not give a large negative index. For example, the form

$$q(z) = -|z_1 + \cdots + z_N|^2$$

is negative at every coordinate vector, but has negative index one. To establish (9), the entire mixed matrix must therefore be controlled, rather than only its diagonal entries.

Our tests are pairs of compact packets centered at $\pm t/2$. The self-correlation of the t-th pair detects a modulation band near the positive shift t. For two distinct packet centers t_i, t_k, the mixed correlations instead occur near

$$\frac{t_i + t_k}{2}, \qquad \frac{|t_i - t_k|}{2}.$$

Thus a three-term progression of centers creates an unwanted sum-midpoint interaction: the midpoint of two active centers is another active center. We remove this interaction with a progression-free set. The difference shift is smaller and need not avoid earlier active bands: its full contribution is exponentially small after normalization. This distinction permits fixed spacing within geometrically growing blocks and supplies a negative space at every sufficiently large support radius.

For example, in a block of centers $T + KB$, the set $B = \{0, 1, 3\}$ has pairwise midpoint labels $1/2, 3/2, 2$, all outside B. For $B = \{0, 1, 2\}$, the pair $0, 2$ has midpoint 1, which is an active band. A sufficiently large spacing K separates the sum-midpoint correlation intervals from the bands themselves. Dense classical progression-free sets then produce large negative spaces, while a complete Stieltjes estimate transfers their mixed matrix to ordinary primes.

1.3 Relation to earlier work

Positive oscillatory prime templates have a substantial history. Diamond, Montgomery and Vorhauer [5] constructed generalized prime systems with large oscillation; Broucke [3] develops the template method for prescribed zero-free contours. Broucke and Vindas [4] give a strong discrete approximation theorem by choosing generalized prime locations. The present construction keeps the ordinary prime locations fixed and assigns one positive exponent to every power of each prime. Classical PNT estimates provide the required transfer to those locations. The additional quantitative step is the complete mixed packet estimate, which turns a scalar oscillatory construction into a large negative subspace.

Negative index is an established object in the study of explicit-formula quadratic forms. Bombieri [2, Section 8] determines the inertia of finite zero-side approximations using the symmetries of the zero set. Here the index bound is obtained directly from the arithmetic shifts, with an

explicit support scale and a fixed finite prime prefix. The proof uses the classical progression-free construction of Behrend [1]; we include a short sphere-layer proof of the needed bound. The stronger quantitative constructions of Elkin and Green–Wolf [6, 7] are compatible with the argument, although their improved constants are not needed for (9).

The proof proceeds as follows. Section 2 establishes the continuous cancellation and the exact packets. Section 3 constructs progression-free blocks and proves their support exclusions. Section 4 verifies the global PNT remainder and the Euler coefficients. Section 5 controls the full actual-prime matrix and completes the proof of Theorem 1.1.

2 Main-term cancellation and test packets

The continuous prime measure gives a tractable starting point. The two polar conditions cancel its growing part exactly. We then construct test packets satisfying those conditions and choose a modulation phase that makes each diagonal modulation contribution positive.

For a real locally bounded v on $[0,\infty)$, write

$$\mathfrak{c}_v[f] = 2\,\mathrm{Re}\int_0^\infty e^{t/2}v(t)\langle f,\tau_t f\rangle\,\mathrm{d}t. \tag{11}$$

This is the continuous prime-measure comparison obtained from $d\psi(x) = dx$ after the change of variable $t = \log x$. The support of f makes the integral finite.

Lemma 2.1 (Exact main-measure cancellation)**.** *Let K be convolution by $e^{-|u|/2}$. For $f \in \mathcal{V}$,*

$$\mathfrak{c}_1[f] = -\langle f, Kf\rangle, \qquad 0 \le K \le 4I. \tag{12}$$

Proof. The two-sided kernel in (11) with $v = 1$ is $e^{|u-w|/2}$. Use the identity

$$e^{|x|/2} = 2\cosh(x/2) - e^{-|x|/2}.$$

The first term integrates to $\overline{Mf(1/2)}Mf(-1/2) + \overline{Mf(-1/2)}Mf(1/2) = 0$. The remaining term is $-\langle f, Kf\rangle$. The Fourier multiplier of K is $(\xi^2 + 1/4)^{-1}$, which lies in $[0,4]$. □

Fix $r > 0$, a real even nonnegative nonzero $\phi \in C_c^\infty(-r,r)$, and a real cutoff $\chi \in C_c^\infty(-3r,3r)$

such that

$$0 \le \chi \le 1, \qquad \chi = 1 \text{ on } [-2r, 2r]. \tag{13}$$

Write $\Phi = M\phi$. For $\omega \ge 1$, define

$$g_\omega(u) = -\omega^{-2}\left(\frac{d^2}{du^2} - \frac{1}{4}\right)[\phi(u)\cos(\omega u)], \qquad G_\omega = Mg_\omega. \tag{14}$$

Lemma 2.2 (Packet identities and bounds)**.** *The function g_ω is real, even and supported in $[-r, r]$. It satisfies*

$$G_\omega(1/2) = G_\omega(-1/2) = 0, \tag{15}$$

$$G_\omega(1/2 + i\omega) = \frac{1 - i/\omega}{2}\{\Phi(1/2) + \Phi(1/2 + 2i\omega)\} \longrightarrow \frac{\Phi(1/2)}{2} > 0, \tag{16}$$

$$\|g_\omega\|_2^2 \longrightarrow \frac{\|\phi\|_2^2}{2}, \qquad \|g_\omega\|_{H^s} \le C_s(1 + \omega)^s \quad (s \ge 0). \tag{17}$$

The constants in (17) *may depend on s and ϕ.*

Proof. Integration by parts in the Mellin transform multiplies a function by $z^2 - 1/4$ when the differential operator in (14) is applied. This proves (15). Since

$$M[\phi\cos(\omega u)](z) = \{\Phi(z + i\omega) + \Phi(z - i\omega)\}/2,$$

substitution of $z = 1/2 + i\omega$ proves the equality in (16). The remaining oscillatory integral tends to zero by the Riemann–Lebesgue lemma applied to $e^{u/2}\phi(u)$. The limit is positive because ϕ is nonnegative and nonzero.

Expansion of the differential operator gives

$$g_\omega = \phi\cos(\omega u) + 2\omega^{-1}\phi'\sin(\omega u) - \omega^{-2}(\phi'' - \phi/4)\cos(\omega u).$$

The last two terms tend to zero in L^2; the squared norm of the first tends to $\|\phi\|_2^2/2$. For the Sobolev bound at every real $s \ge 0$, use

$$\widehat{g}_\omega(\xi) = \frac{\xi^2 + 1/4}{2\omega^2}\{\widehat{\phi}(\xi - \omega) + \widehat{\phi}(\xi + \omega)\}.$$

After setting $\xi = \zeta \pm \omega$, the squared weighted factor is bounded by $C_s(1+\omega^2)^s(1+\zeta^2)^{s+2}$. The Schwartz decay of $\widehat{\phi}$ proves (17). □

Let $c_0 > 0$ be a constant in the classical PNT bound

$$\psi(x) = \sum_{n \le x} \Lambda(n) = x + E(x), \qquad |E(x)| \le Cxe^{-c_0\sqrt{\log x}} \quad (x \ge 2). \tag{18}$$

This standard de la Vallée Poussin form is recalled, for example, in [3, Introduction]. Choose once and for all

$$0 < d < c_0/4, \qquad \omega_t = e^{d\sqrt{t}}. \tag{19}$$

For a sufficiently large center t, put

$$f_t = \tau_{t/2} g_{\omega_t} - \tau_{-t/2} g_{\omega_t}. \tag{20}$$

These are real odd elements of $\mathcal{V}$, supported in $[-t/2 - r, t/2 + r]$. Their L^2 norms are bounded above and below by fixed positive constants for large t. Choose a phase α_t so that

$$\mathrm{Re}\{e^{i\alpha_t} G_{\omega_t}(1/2 + i\omega_t)^2\} = -|G_{\omega_t}(1/2 + i\omega_t)|^2. \tag{21}$$

Equation (16) makes this choice possible with a nonzero value for every sufficiently large t.

Lemma 2.3 (Diagonal modulation)**.** *Suppose a band centered at t is the only modulation band meeting the positive-shift correlation of f_t away from zero, and no band meets its correlation near zero. If that band is $\epsilon\chi(u-t)\cos(\omega_t(u-t)+\alpha_t)$, then*

$$\mathfrak{c}_v[f_t] - \mathfrak{c}_1[f_t] = 2\epsilon e^{t/2} |G_{\omega_t}(1/2 + i\omega_t)|^2. \tag{22}$$

Proof. For real even $g = g_{\omega_t}$, write $c_g(u) = \langle g, \tau_u g \rangle$. Near the positive shift t, the autocorrelation of $\tau_{t/2} g - \tau_{-t/2} g$ is $-c_g(u-t)$. Its other positive-shift component is near zero. The cutoff equals one on the entire cross-correlation interval. For $z = 1/2 + i\omega_t$,

$$\int_{\mathbb{R}} e^{zu} c_g(u)\, \mathrm{d}u = Mg(z) Mg(-z) = G_{\omega_t}(z)^2,$$

where evenness was used in the last equality. The contribution to $\mathfrak{c}_v - \mathfrak{c}_1$ is therefore $-2\epsilon e^{t/2}\,\mathrm{Re}(e^{i\alpha_t}G_{\omega_t}(z)^2)$. Apply (21). □

3 Choosing packet centers to control interactions

Lemma 3.1 (Classical dense progression-free sets)**.** *There is an absolute constant C such that every sufficiently large integer M has a subset $B \subset \{0,\dots,M-1\}$ with*

$$|B| \geq M e^{-C\sqrt{\log M}}, \qquad a+c=2b,\ a,b,c\in B \implies a=b=c. \tag{23}$$

Proof. This is the classical sphere construction of Behrend [1]. We include a sufficient estimate. Set $h=\lfloor\sqrt{\log M}\rfloor$ and $q=\lfloor M^{1/h}/2\rfloor$. For large M,

$$q\geq 2, \qquad (2q)^h \leq M, \qquad q \geq M^{1/h}/4.$$

Partition $\{0,\dots,q-1\}^h$ by the integer $\sum_\nu x_\nu^2$. Some sphere layer S has size at least $q^h/[h(q-1)^2+1]$. Encode a point by

$$x \longmapsto \sum_{\nu=0}^{h-1} x_\nu (2q)^\nu.$$

The encoding is injective and lies below M. In an equation $a+c=2b$, both sides have digits below $2q$; there are no carries, so the equation holds coordinatewise. On a sphere, strict convexity of the squared norm forces the three coordinate vectors to coincide. Finally,

$$\frac{q^h}{h(q-1)^2+1} \geq \frac{M}{2h\,4^h M^{2/h}} \geq M e^{-C\sqrt{\log M}},$$

after enlarging C. □

Fix one integer $K > 20r+2$. Choose a sufficiently large integer T_1, and set

$$T_j = 16^{j-1}T_1, \qquad K_j = K, \qquad M_j = \left\lfloor \frac{T_j}{4K} \right\rfloor. \tag{24}$$

Take $T_1 \geq 8K$ so large that M_1 is in the range of Lemma 3.1, T_1 exceeds $\log P_0 + 3r$ and the packet

threshold in (16), and $T_1 > 100r + 10$. For B_j supplied by that lemma, define

$$\mathcal{T}_j = \{T_j + Kb : b \in B_j\}, \qquad N_j = |B_j|. \tag{25}$$

Every center in the block lies in $[T_j, 5T_j/4]$. Let U_{j-1} be the largest earlier center plus $3r$, with $U_0 = 0$. Our choices give

$$T_j > 8U_{j-1} + 100r. \tag{26}$$

Indeed for $j \geq 2$, the right side is at most $10T_{j-1} + 124r < 16T_{j-1}$. All centers in the entire construction have a fixed positive minimum spacing, and the modulation bands of radius $3r$ are disjoint.

Lemma 3.2 (Support geometry)**.** *Let $t_i \neq t_k$ belong to the same block of scale T_j. The interval of radius $2r$ about the sum midpoint $S = (t_i + t_k)/2$ is disjoint from every modulation interval $[t - 3r, t + 3r]$, for every center t in every block. The difference midpoint $D = |t_i - t_k|/2$ satisfies*

$$K/2 \leq D \leq T_j/8.$$

Its interval of radius $2r$ meets no current or future band, but may meet earlier bands. For the diagonal packet f_{t_i}, the only band meeting its positive-shift correlation away from zero is its own band. No band meets its correlation near zero.

Proof. The positive-shift mixed correlation has exactly the two possible centers S, D, since the packets are supported near $\pm t_i/2$ and $\pm t_k/2$. For a current center $t_\ell = T_j + Kb_\ell$,

$$|S - t_\ell| = \frac{K}{2}|b_i + b_k - 2b_\ell| \geq K/2 > 5r$$

by progression-freeness. The sum midpoint is above all previous bands and below all future bands by (26). The stated range for D follows from the center spacing and the block width. Its upper endpoint is below the current bands with the required margin. Earlier bands are allowed there and will be bounded in Section 5. For a diagonal, the positive-shift correlation is confined to radius $2r$ about zero and its own center. Distinct centers have spacing greater than $5r$, and the first center is greater than $5r$. □

Define the single function and prime weights

$$v(u) = 1 + \epsilon \sum_{t \in \bigcup_j \mathcal{T}_j} \chi(u-t)\cos(\omega_t(u-t)+\alpha_t), \qquad b_p = v(\log p). \tag{27}$$

The sum is locally finite, and the bands are disjoint. Consequently $1-\epsilon \le v \le 1+\epsilon$, and (6) holds.

4 The global prime estimate and all Euler powers

Lemma 4.1 (A global weighted PNT estimate)**.** *For the function v in* (27)*,*

$$\sum_{n \le x} \Lambda(n) v(\log n) = x + O\Big(x e^{-(d/2)\sqrt{\log x}}\Big) \qquad (x \ge 2).$$

The implied constant is independent of the final band or endpoint.

Proof. For a center t, integrate one oscillatory band against dx. In logarithmic coordinates, a complete or partial band has the form

$$\int e^u \chi(u-t)\cos(\omega_t(u-t)+\alpha_t)\,\mathrm{d}u.$$

One integration by parts, including the partial-band endpoint, bounds this by $Ce^t/\omega_t = Ce^{t-d\sqrt{t}}$. Integration against dE, with E as in (18), costs

$$C(1+\omega_t)e^{t-c_0\sqrt{t}}.$$

Indeed the boundary term has this bound, and the total variation of the band factor is $O(1+\omega_t)$. Shifting t by at most $3r$ changes the exponent by a bounded amount. As $d < c_0/2$, both contributions are $O(e^{t-d\sqrt{t}})$.

The function $F(t) = t - d\sqrt{t}$ has $F'(t) \ge 1/2$ eventually. If successive centers have spacing at least $H > 0$, completed-band bounds sum backwards with ratio at most $e^{-H/2}$. A partial final band obeys the same bound. Between bands use monotonicity of F. Thus there is one estimate for every x, even though the number of centers in a block grows. The unweighted PNT error is smaller. Weakening d to $d/2$ absorbs fixed initial and endpoint constants. □

Lemma 4.2 (The complete higher-power correction)**.** *Replacing* $\Lambda(p^k)v(k\log p)$ *by* $b_p\log p$ *at all* $k\geq 2$ *has cumulative absolute cost*

$$O(\sqrt{x}\log^2 x)\qquad (n\leq x). \tag{28}$$

On tests whose shifts satisfy $n\leq X$*, the difference of the complete paired arithmetic forms has absolute value at most*

$$C(1+\log^2 X)\|f\|_2^2. \tag{29}$$

Proof. Each changed coefficient is at most $2\epsilon\log p$ in absolute value. At most $\log_2 x$ exponents contribute, each with bases at most $\sqrt{x}$, which gives (28). For the form estimate, use $|\langle f,\tau_t f\rangle|\leq \|f\|_2^2$. The square terms are bounded by

$$\sum_{p\leq\sqrt{X}}\frac{\log p}{p}\leq\sum_{2\leq n\leq\sqrt{X}}\frac{\log n}{n}=O(1+\log^2 X).$$

For all powers $k\geq 3$ together, use the convergent majorant

$$\sum_{n\geq 2}\frac{(\log n)n^{-3/2}}{1-n^{-1/2}}.$$

This bounds the entire operator difference, including all its mixed entries on any subspace. □

Proposition 4.3. *The weights* (27) *satisfy parts (1) and (2) of Theorem 1.1.*

Proof. The prefix and positivity statements follow from disjoint bands. Lemma 4.1 and (28) give (7), since $\sqrt{x}\log^2 x=o(xe^{-(d/2)\sqrt{\log x}})$. One may take a smaller fixed $c_b>0$ to include all initial values with one constant.

For $\operatorname{Re} s>1$, the series

$$\sum_p b_p\sum_{k\geq 1}\frac{p^{-ks}}{k}$$

is absolutely convergent. Its exponential defines (8) and permits termwise logarithmic differentiation. Each factor has the binomial expansion

$$(1-z)^{-b_p}=\sum_{k\geq 0}\frac{(b_p)_k}{k!}z^k,$$

whose coefficients are positive because $b_p > 0$. Multiplication over prime factors gives $a_b(n) > 0$. The absolute Dirichlet series at a real $\sigma > 1$ is bounded by $\zeta(\sigma)^{1+\epsilon}$. Every prime factor of $n \le P_0$ has exponent $b_p = 1$; all local coefficients at these primes are one. Hence $a_b(n) = 1$ for $n \le P_0$. $\square$

5 Full interaction estimates and negative subspaces

A negative value for each test packet is insufficient: every linear combination in the proposed subspace must remain negative. We normalize the packets so that the useful diagonal contribution has fixed size, then bound the complete matrix of errors and off-diagonal interactions. The row sums pay for the growing number of packets.

Fix a block and abbreviate $T = T_j$, $N = N_j$. List its centers as $t_1, \dots, t_N$, put

$$\omega_{\max} = e^{d\sqrt{5T/4}}, \qquad w_i = e^{t_i/4}, \qquad F_z = \sum_{i=1}^{N} z_i \frac{f_{t_i}}{w_i}, \quad z \in \mathbb{C}^N.$$

The packets have disjoint supports within the block. Thus

$$\|F_z\|_2^2 \le Ce^{-T/2}\|z\|_{\ell^2}^2. \tag{30}$$

For every fixed finite s, the triangle inequality and Cauchy–Schwarz give

$$\|F_z\|_{H^s}^2 \le C_s N e^{2sd\sqrt{5T/4}-T/2}\|z\|_{\ell^2}^2 = o(1)\|z\|_{\ell^2}^2, \tag{31}$$

because $N \le M_j \le T$. For $s = 0$, the sharper estimate (30) is available.

5.1 The continuous matrix

Let

$$\kappa_0 > 0, \qquad 2|G_{\omega_t}(1/2 + i\omega_t)|^2 \ge \kappa_0$$

for all centers, increasing the first center if necessary. Lemma 2.3 gives a normalized modulation entry at least $\kappa_0\epsilon$ on the diagonal of $\mathfrak{c}_v - \mathfrak{c}_1$. For distinct centers, Lemma 3.2 gives $v = 1$ on the sum-midpoint region. On the difference region of fixed width about $D = |t_i - t_k|/2$, the polarized correlation is uniformly bounded by Lemma 2.2, and $|v - 1| \le \epsilon$. Its normalized modulation entry

is therefore bounded in absolute value by

$$C\epsilon e^{D/2-(t_i+t_k)/4} = C\epsilon e^{-\min(t_i,t_k)/2} \le C\epsilon e^{-T/2}. \tag{32}$$

This includes every earlier band that meets that region. The real symmetric matrix on the real packet basis gives the Hermitian form on complex coefficients as well. Row-sum control of its off-diagonal part and Lemma 2.1 consequently give

$$-\mathfrak{c}_v[F_z] \le -\kappa_0\epsilon\|z\|_{\ell^2}^2 + C\epsilon N e^{-T/2}\|z\|_{\ell^2}^2 + 4\|F_z\|_2^2. \tag{33}$$

5.2 Transfer from the continuous measure to actual primes

We first compare $\mathfrak{c}_v$ with the arithmetic form using $\Lambda(n)v(\log n)$, before making the consistent-power replacement. For real basis packets define the polarized correlation

$$h_{ik}(u) = \langle f_{t_i}, \tau_u f_{t_k}\rangle + \langle f_{t_k}, \tau_u f_{t_i}\rangle.$$

The corresponding real symmetric matrix defines the Hermitian form on complex coefficients as well. Uniformly in the block,

$$|h_{ik}| \le C, \qquad |h'_{ik}| \le C(1+\omega_{\max}). \tag{34}$$

These follow from Cauchy–Schwarz and the L^2 and H^1 bounds in Lemma 2.2. Its support for $i \ne k$ is confined to the two intervals in Lemma 3.2.

Lemma 5.1 (Normalized mixed PNT error)**.** *For some $\delta > 0$, the operator norm of the full normalized error matrix is bounded by*

$$CNe^{-\delta\sqrt{T}} + CNe^{-T/2}. \tag{35}$$

In particular it tends to zero as $j \to \infty$.

Proof. For $i \ne k$, $v = 1$ on the sum-midpoint region. Near the sum midpoint $S = (t_i + t_k)/2$, use

the test function

$$x^{-1/2}h_{ik}(\log x)$$

in Stieltjes integration against $dE(x)$. Its derivative is

$$x^{-3/2}\{h'_{ik}(\log x) - h_{ik}(\log x)/2\}.$$

The endpoints vanish by smooth compact support. On this fixed logarithmic-width interval, (18) and (34) bound the error by

$$C(1+\omega_{\max})e^{S/2-c_0\sqrt{S}}.$$

Division by $w_i w_k = e^{S/2}$, and $S \geq T$, give at most $Ce^{-\delta\sqrt{T}}$, since $d\sqrt{5/4} < c_0$.

Near the difference midpoint $D = |t_i - t_k|/2$, use the elementary consequence $\psi(x) \leq Cx$ of (18). Since $D \geq K/2 > 2r+1$,

$$\sum_{|\log n - D| \leq 2r} \frac{\Lambda(n)}{\sqrt{n}} \leq e^{-(D-2r)/2}\psi(e^{D+2r}) \leq Ce^{D/2}.$$

The continuous integral of $x^{-1/2}\,\mathrm{d}x$ over the same region has the same bound. Both v and h_{ik} are uniformly bounded, so their arithmetic–continuous discrepancy, after normalization, is at most

$$Ce^{D/2-(t_i+t_k)/4} \leq Ce^{-T/2}.$$

This estimate includes earlier modulation bands without differentiating them or assuming that $v = 1$ there. The sum and difference regions are disjoint, so their contributions can be estimated separately.

For a diagonal entry, the cross region is near t_i. The differentiated test now contains vh_{ii}. The derivative is $v'h_{ii} + vh'_{ii}$, whose size is $O(1+\omega_{t_i})$, with no product of derivatives. The normalized error is $O(e^{-(c_0-d)\sqrt{t_i}})$. The self region has fixed logarithmic width near zero. Only finitely many actual prime powers occur there; their contribution and the continuous integral are $O(1)$ by the bound on h_{ii}, so their normalized error is $O(e^{-T/2})$.

These estimates bound every normalized entry, with one constant independent of i, k, j. The norm of a Hermitian matrix is bounded by the maximum absolute row sum. There are N entries

in each row, which proves (35). □

5.3 Completion of the negative-subspace estimate

All shifts of a test in the block satisfy $\log n \le 5T/4 + 2r$. Lemma 4.2 and (30) therefore bound the entire consistent-power replacement by

$$CT^2 e^{-T/2}\|z\|_{\ell^2}^2 = o(1)\|z\|_{\ell^2}^2. \tag{36}$$

For any fixed $\mathfrak{a}$ satisfying (5), (31) gives

$$|\mathfrak{a}[F_z]| = o(1)\|z\|_{\ell^2}^2.$$

More explicitly, the preceding bounds give

$$\mathfrak{q}_b^{\mathfrak{a}}[F_z] \le [-\kappa_0\epsilon + \mathcal{E}_{\mathfrak{a}}(T)]\|z\|_{\ell^2}^2,$$

where, for a constant C independent of the block and constants C_s from (31), one may take

$$\mathcal{E}_{\mathfrak{a}}(T) = C\{Ne^{-\delta\sqrt{T}} + (N + T^2 + 1)e^{-T/2}\} + C_{\mathfrak{a}}C_s N e^{2sd\sqrt{5T/4}-T/2}.$$

Here $s, C_{\mathfrak{a}}$ are any fixed bounds in (5); constants absorb the fixed packet and cutoff choices. The first group pays the complete arithmetic transfer, difference interactions, main continuous term and higher powers. The last term pays the comparison energy. Since $N \le T$, a sufficient block condition is $\mathcal{E}_{\mathfrak{a}}(T) \le \kappa_0\epsilon/2$. This condition displays explicitly the dependence of the eventual threshold on the comparison energy.

Combining this with (33), Lemma 5.1 and (36), for all sufficiently large j we obtain the complete estimate

$$\mathfrak{q}_b^{\mathfrak{a}}[F_z] \le -\frac{\kappa_0\epsilon}{2}\|z\|_{\ell^2}^2. \tag{37}$$

Every term in the actual-prime form is included in this estimate.

Proof of Theorem 1.1. Parts (1) and (2) were proved in Proposition 4.3. Set

$$L_j = \frac{5T_j}{8} + r, \qquad W_j = \operatorname{span}_{\mathbb{C}}\{f_t : t \in \mathcal{T}_j\}.$$

The packets are nonzero and disjointly supported within the block, so $\dim W_j = N_j$, and $W_j \subset \mathcal{V}_{L_j}$. Equations (30) and (37) give

$$\mathfrak{q}_b^{\mathfrak{a}}[f] \le -c_\epsilon e^{T_j/2}\|f\|_2^2 \qquad (f \in W_j).$$

The block construction, weights and spaces did not use $\mathfrak{a}$; only the eventual threshold uses its Sobolev bound.

Since K is fixed and $T_j \ge 8K$, $T_j/(8K) \le M_j \le T_j$. Lemma 3.1 gives

$$N_j \ge cT_j e^{-C\sqrt{\log T_j}}.$$

For every $L \ge L_1$, let $j(L)$ be the largest index with $L_j \le L$, and set $W(L) = W_{j(L)}$. These spaces are contained in $\mathcal{V}_L$; no cross-block combinations are needed. Geometric growth gives

$$L < L_{j+1} = 10T_j + r, \qquad (L-r)/10 < T_j \le (8/5)(L-r).$$

Thus T_j and L are uniformly comparable at all large radii, which proves (9). The Rayleigh estimate above gives

$$\mathfrak{q}_b^{\mathfrak{a}}[f] \le -c_\epsilon e^{T_j/2}\|f\|_2^2 \le -c'_\epsilon e^{L/20}\|f\|_2^2 \qquad (f \in W(L)),$$

after absorbing $e^{-r/20}$ into the constant. This proves (10), for example with $c_* = 1/25$. Finally $j(L) \to \infty$, so the eventual block threshold for each fixed $\mathfrak{a}$ becomes an eventual threshold in L. □

6 Finite constraints, corrections and examples

Corollary 6.1 (Constraint and repair rank)**.** *Let $b, W(L), N(L)$ be as in Theorem 1.1, and let $\mathfrak{a}$ be fixed and Sobolev-controlled. For linear maps $\ell_L : \mathcal{V}_L \to \mathbb{C}^{r(L)}$, all sufficiently large L, with*

threshold depending on $\mathfrak{a}$, satisfy

$$\operatorname{ind}_-(\mathfrak{q}_b^{\mathfrak{a}}; \mathcal{V}_L \cap \ker \ell_L) \geq N(L) - r(L)$$

whenever the right side is positive. For these L, the same bound holds after adding any Hermitian correction that factors through ℓ_L. For a fixed finite number of constraints and a fixed finite-rank correction, arbitrarily large negative subspaces and arbitrarily negative L^2 Rayleigh quotients remain.

Proof. Rank-nullity gives $\dim(W(L) \cap \ker \ell_L) \geq N(L) - r(L)$. The estimate (10) holds on this intersection. A correction $H(\ell_L f, \ell_L f)$, for any Hermitian form H on the finite coordinate space, vanishes there. If $r(L)$ is fixed, $N(L) - r(L) \to \infty$. □

Corollary 6.2 (Positive-index repair bound)**.** *Fix $\mathfrak{a}$ as in Theorem 1.1. Let R_L be any Hermitian form on $\mathcal{V}_L$, and let $r_+(L)$ be the positive index of its restriction to $W(L)$. For any linear map $\ell_L : \mathcal{V}_L \to \mathbb{C}^{d(L)}$, at every sufficiently large L,*

$$\operatorname{ind}_-(\mathfrak{q}_b^{\mathfrak{a}} + R_L; \mathcal{V}_L \cap \ker \ell_L) \geq N(L) - r_+(L) - d(L)$$

whenever the right side is positive. A subspace of at least this dimension retains the Rayleigh bound (10) *for $\mathfrak{q}_b^{\mathfrak{a}} + R_L$. The threshold depends only on the original fixed comparison energy and construction, and is uniform in the choices of R_L and ℓ_L.*

Proof. On the finite-dimensional space $W(L)$, diagonalize R_L using the inherited L^2 inner product. Its nonpositive spectral subspace U_L has dimension $N(L) - r_+(L)$. Rank-nullity gives $\dim(U_L \cap \ker \ell_L) \geq N(L) - r_+(L) - d(L)$. On this intersection $R_L[f] \leq 0$, so the unchanged estimate (10) proves both assertions. □

With no additional rows, only the repair's positive index is lost; this can be smaller than its rank. If a repair factors through the same rows as the constraints, Corollary 6.1 instead uses their common kernel and pays those rows only once.

Corollary 6.3 (Fixed differential and logarithmic energies)**.** *The weights in Theorem 1.1 work*

simultaneously for each fixed Fourier multiplier

$$\mathfrak{a}_a[f] = \frac{1}{2\pi}\int_{\mathbb{R}} a(\xi)|\widehat{f}(\xi)|^2\,\mathrm{d}\xi$$

with real symbol satisfying $|a(\xi)| \le C(1+\xi^2)^s$ *for a finite* s*. They also work after adding any fixed multiple of a finite Sobolev norm to such a form.*

Proof. The symbol bound gives (5). Sums of two Sobolev-controlled forms satisfy the same condition after increasing the exponent and constant. □

Logarithmic symbols are a useful case: their energy can grow with packet frequency, but that growth is still paid in the whole-space estimate (31). For example, each fixed symbol

$$a(\xi) = \operatorname{Re}\frac{\Gamma'}{\Gamma}\left(\alpha + \frac{i\xi}{2}\right) + c, \qquad \alpha > 0, \quad c \in \mathbb{R},$$

has logarithmic growth and belongs to Corollary 6.3. Such multipliers appear in archimedean terms of explicit formulas; see [2, Section 2]. The theorem treats the chosen multiplier and the complete deformed arithmetic form in the same original test norm.

Remark 6.4 (An explicit elementary center family). The sphere construction is needed only for the almost-linear index scale. If one instead uses the integers below 3^h whose base-3 digits lie in $\{0,1\}$, the same no-carry argument makes them progression-free. Their cardinality is 2^h. Choosing h with $3^h \le M_j < 3^{h+1}$, and selecting one block for each support radius as above, gives the fully explicit alternative

$$N(L) \ge cL^{\log 2/\log 3}$$

at every sufficiently large L. All analytic estimates and repair conclusions are unchanged.

6.1 Reproducible finite geometry

The companion program `verify_geometry.py` uses integer and rational arithmetic to check finite sphere-layer sets, their midpoint exclusions, and all mixed-support separations in three explicit blocks. It includes the progression-free example $\{0,1,3\}$ and the collision example $\{0,1,2\}$ from the introduction. The additional program `verify_geometric_windows.py` checks fixed-spacing

geometric blocks, the all-support interpolation inequalities and the exact normalized difference exponent. It includes an adverse example in which a difference midpoint lies at an earlier active center; that interaction is bounded by (32), rather than declared absent. The analytic PNT, Sobolev and all-prime-power estimates are proved in the text; the programs supply finite reproducible examples of their geometric input.

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