Joint Cutoffs and Negative-Spectrum Decay for Weighted Weil Operators

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Abstract

We obtain a joint finite-rank approximation law for the complete Birman–Schwinger family of a constrained theta-weighted Weil operator. At each fixed absolute error tolerance, the required rank is at most $C\exp(C\sqrt{\log(1/\varepsilon)})$ as the positive resolvent parameter ε tends to zero. The construction pays the complete paired arithmetic tail, high-frequency block and mixed localization errors. It gives a corresponding bound on the entire negative spectral projection and implies that the negative eigenvalue magnitudes decay as $C\exp[-c\log^2(n+1)]$. Thus the full negative part belongs to every Schatten class. We separate two arithmetic cutoff regimes. Effective simultaneous prime-phase returns give signed high-frequency saturation when the retained cutoff grows sufficiently slowly. In the faster regime of the rank construction, theta confinement and compressed-shift estimates instead give a strict high-block norm gap: its normalized upper bound is at most three quarters for every positive limiting arithmetic ratio, with a stronger branch above the theta-overlap threshold. Both arguments retain the original constraints, weighted norms and complete localization errors.

1 Introduction and results

The spectral effect of an arithmetic truncation depends on the resolvent parameter against which its error is measured. A fixed-cutoff approximation theorem therefore needs a joint parameter analysis before it gives information about eigenvalues accumulating at a threshold. We perform that analysis for the constrained theta-weighted realization introduced in [1]. The new recipient is the entire negative part of the original operator, rather than its output on a prescribed null space.

Write $H = A + B$ for the precise minimal-form realization defined in Section 2, with $A \geq 0$ and B bounded selfadjoint. For $\varepsilon > 0$ put

$$K_\varepsilon = (A+\varepsilon)^{-1/2}(-B)(A+\varepsilon)^{-1/2}, \qquad n_-(H;-\varepsilon) = \operatorname{rank} 1_{(-\infty,-\varepsilon)}(H).$$

The negative part is $H_- = \max(-H, 0)$ in the spectral calculus. The first theorem gives the main spectral conclusion. The second identifies a strict distinction between the fast cutoff family used in its proof and the slower saturation family.

Theorem 1.1 (Joint rank and negative-spectrum decay)**.** *For every fixed $0 < \eta \leq 1$ and $0 < \varepsilon \leq 1$, the complete operator K_ε has a selfadjoint finite-rank approximant with norm error less than η. Its rank is at most the explicit integer $N(\varepsilon,\eta)$ in (11). For each fixed η there is C_η such that*

$$N(\varepsilon,\eta) \leq C_\eta \exp\left(C_\eta\sqrt{\log(1/\varepsilon)}\right).$$

There are constants $C, c > 0$ such that

$$n_-(H; -\varepsilon) \le C \exp\left(C\sqrt{\log(1/\varepsilon)} \right), \tag{1}$$

$$\lambda_n(H_-) \le C \exp[-c \log^2(n+1)]. \tag{2}$$

Here the negative eigenvalue magnitudes are ordered nonincreasingly, repeated with multiplicity, and padded by zeros if finite. In particular $H_- \in \mathcal{S}_p$ for every $p > 0$.

For the regime comparison write $J_\varepsilon = M_s(A + \varepsilon)^{-1/2}$ and

$$K_{\mathrm{hi}}(M, \varepsilon, L) = J_\varepsilon^*(I - Q_L)V_M(I - Q_L)J_\varepsilon,$$

where V_M and its coefficient d_M are defined in Section 3.

Theorem 1.2 (Fast-regime high-block gap)**.** *Suppose $q \to \infty$, $\varepsilon = e^{-q}$, $M/q \to C > 0$, and $\log L/\sqrt{q} \to K$ with $0 < K < \infty$. Then*

$$\limsup_{q\to\infty} \frac{a(L)\|K_{\mathrm{hi}}(M, \varepsilon, L)\|}{d_M} \le \min\left\{ \frac{3}{4}, \frac{1}{2} + \frac{1}{4\sqrt{\pi C}} \right\} < 1. \tag{3}$$

The positive spectral edge obeys the same upper bound. For the explicit cutoff family of Theorem 1.1, $C = 1/\log 10$, and the stronger branch is $\frac{1}{2} + \frac{1}{4}\sqrt{\log 10/\pi}$.

The rank bound includes the growing retained arithmetic set and all localization remainders. The parameter η is fixed in the displayed asymptotic; the explicit formula governs its simultaneous variation. The finite-rank construction contains the actual resolvent rows, so the theorem is an approximation-rank statement rather than an estimate for the arithmetic cost of evaluating those rows.

The proof uses the frequency-first enclosure from [1], which we reproduce below in the exact form needed. An elementary Chebyshev estimate removes an unnecessary logarithm from the retained arithmetic coefficient. The form-domain Birman–Schwinger correspondence and finite-rank min–max then control the whole negative spectrum. These spectral tools and Schatten theory are classical; see [3, 4]. Related logarithmic-symbol spectral estimates include [5], whose finite-measure Dirichlet setting differs from the full-line theta-multiplication reference used here. Weighted arithmetic realizations and fixed-depth localization arguments also appear in [6]. The contribution here is the joint cutoff law and its quantitative consequence for H_- in the specified realization.

Section 6 proves saturation uniformly in a slow-growing arithmetic regime, paying every-interval prime-phase coverage and the original packet norm. Section 7 then proves Theorem 1.2: a smooth highpass transfers the theta-induced spatial confinement without a false sharp-cutoff commutator estimate, and medium and long translations have smaller correlations on that confined core. Thus the fast rank regime genuinely fails to saturate the fixed-cutoff high-band coefficient. This comparison concerns that retained block.

The finite observation-norm estimate used for the smooth transfer was developed in the author's working note [7]; we reproduce its required upper proof in Lemma 2.2. The compact-window autocorrelation half bound has a direct predecessor in [8, Lemma 1] and [10, Lemmas 1 and 1′]. The square-zero numerical-radius estimate is a special case of the classical theorem of Haagerup and de la Harpe [9]; its elementary proof and the projection bounds are included. The prime number theorem used to evaluate the arithmetic mass is the classical unconditional theorem [2, §§25.16,27.2]. The new fast-regime conclusion is the transfer of these ingredients through the complete weighted receiver and its joint scales.

2 Exact realization and inherited budgets

Inner products are conjugate-linear first, and

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-iu\xi} f(u)\,\mathrm{d}u, \qquad \|f\|^2 = \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \frac{\mathrm{d}\xi}{2\pi}.$$

Use the full theta kernel

$$\psi(u) = 2\sum_{n\geq 1} (2\pi^2 n^4 e^{9u/2} - 3\pi n^2 e^{5u/2}) e^{-\pi n^2 e^{2u}}. \tag{4}$$

The locally uniformly differentiable series is even and strictly positive. For normalization, with $\vartheta(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}$ and $b(u) = e^{u/2}(\vartheta(e^{2u}) - 1)$, Poisson summation gives $b(u) - b(-u) = e^{-u/2} - e^{u/2}$ and $\psi = \frac{1}{2}(\partial_u^2 - 1/4)b$. Positivity follows from the individual terms on $u \geq 0$, then reflection.

Put $c(u) = \cosh(u/2)$, $w = \psi c$, $s = \sqrt{\psi/c}$, and

$$\mathcal{K} = \{h \in L^2(\mathbb{R}) : \langle \sqrt{w}, h\rangle = \langle \sqrt{w}\tanh(u/2), h\rangle = 0\}.$$

The two rows are smooth, in L^2, and linearly independent. The weighted unitary coordinate is $h = \sqrt{w}\, g$, with $\psi g = sh$.

Writing $\psi_0 = \Gamma'/\Gamma$, define

$$m_0 = \psi_0(1/4) - \log\pi < 0, \qquad a(\xi) = \Re\psi_0(1/4 + i\xi/2) - \psi_0(1/4).$$

The digamma series and asymptotic [2, §§5.7,5.11] give

$$a(\xi) = \sum_{j\geq 0} \frac{t^2}{(j+1/4)((j+1/4)^2 + t^2)}, \quad t = |\xi|/2, \qquad a(\xi) = \log|\xi| + O(1). \tag{5}$$

It is even, positive away from zero, and increasing in $|\xi|$.

On $\mathcal{C} = C_c^\infty(\mathbb{R}) \cap \mathcal{K}$ define

$$\mathfrak{a}[h] = \int a(\xi)|\widehat{sh}(\xi)|^2 \frac{\mathrm{d}\xi}{2\pi}.$$

The core is dense: compact smooth approximation followed by correction using two fixed compact smooth vectors with invertible row matrix proves this. The maximal graph map $h \mapsto a(D)^{1/2}(sh)$ is closed, so its restriction is closable. Let $\mathcal{V}$ be the closure of $\mathcal{C}$ in $\|h\|^2 + \mathfrak{a}[h]$, and let A be its associated nonnegative selfadjoint operator. This chosen core closure specifies the realization.

Let $S_b f(u) = f(u+b)$ and $c_n = \Lambda(n)/\sqrt{n}$. The complete bounded perturbation is

$$B = P_{\mathcal{K}} \left[m_0 M_s^2 - \sum_{n\geq 2} c_n M_s (S_{\log n} + S_{-\log n}) M_s \right] |_{\mathcal{K}}. \tag{6}$$

The paired series converges in norm by the estimate below. Thus $H = A + B$ is selfadjoint on $\mathrm{Dom}\, A$, with form domain $\mathcal{V}$.

Lemma 2.1 (Source bounds)**.** *The following complete-kernel bounds hold:*

$$\|s\|_\infty \le 1, \qquad \sup_{|u|>R} s(u) \le \sigma_R := \min\{1, 13e^{2R-(3/2)e^{2R}}\}, \tag{7}$$

$$\|B - B_M\| \le \delta_M := 640(M+1)^2/20^{M+1}, \tag{8}$$

$$|m_0| < 6, \qquad a(L) \ge \tfrac{1}{2}\log(2L) \quad (L \ge 1/2), \qquad a(\xi) \le 5 + \log(1+|\xi|). \tag{9}$$

Here B_M retains the same diagonal term and all $n \le M$.

Proof. The elementary full-series proof of the unit bound in [1, Lemma 2.1] gives the stronger explicit value $\psi(u) < 11161963/11500000 < 1$. We use this published estimate with exactly the normalization (4). Its companion envelope is

$$\psi(u) \le 8\pi^2 e^{9u/2-\pi e^{2u}}, \qquad s(u)^2 \le 16\pi^2 e^{4u-\pi e^{2u}} \quad (u \ge 0).$$

For completeness, the first follows by bounding the theta sum after its first exponential using

$$\sum_{n\ge1} n^4 q^{n-1} < 2, \qquad q = e^{-3\pi} < 1/100.$$

Division by $c(u) \ge e^{u/2}/2$ gives the second. Its square root, reflection and $\pi > 3$, $4\pi < 13$ give (7).

For a shift $b \ge 0$, write $v = |u| + |u+b| \ge b$. The envelopes and arithmetic–geometric mean give

$$\begin{aligned}\|M_s S_b M_s\| &= \sup_u s(u)s(u+b)\\ &\le 16\pi^2 e^{2b-\pi e^b}.\end{aligned}$$

Indeed the intermediate exponent is at most $2v - \pi e^v$, decreasing for $v \ge 0$. Since $c_n \le 1$, both directions beyond M cost at most $32\pi^2 \sum_{n>M} n^2 e^{-\pi n}$. For $k = M+1 \ge 2$, use $e^{-\pi} < 1/20$ and

$$\sum_{n\ge k} n^2 e^{-\pi n} \le k^2 20^{-k} \sum_{j\ge0} (j+1)^2 20^{-j} < 2k^2 20^{-k}.$$

This proves (8) with $\pi^2 < 10$. The classical quarter-argument digamma value gives $|m_0| < 6$. Retaining the terms $0 \le j \le \lfloor L/2 - 1/4 \rfloor$ in (5) gives the lower bound in (9) by comparison with the reciprocal integral. For the upper bound, the summand decreases in its denominator variable. Its first term is at most4 and its remaining integral is $\frac{1}{2}\log(1+4\xi^2)$. Hence $a(\xi) \le 4 + \frac{1}{2}\log(1+4\xi^2) \le 5 + \log(1+|\xi|)$.

□

Lemma 2.2 (Finite observation bound)**.** *For the chosen constrained reference and $0 < \varepsilon \le 1$,*

$$j(\varepsilon) := \|M_s(A+\varepsilon)^{-1/2}\| \le \min\{\varepsilon^{-1/2}, 4\log(\log(1/\varepsilon)+4)\}.$$

Proof. We reproduce the upper estimate from [7]. Let $f = sh$ for $h \in \mathcal{V}$. For spatial and frequency cutoffs $R, \ell > 0$, direct integration gives $\|P_R Q_\ell\| \le \sqrt{2R\ell/\pi} =: \rho$. If $\rho < 1$, the identity

$$f = P_R Q_\ell f + (I - P_R)f + P_R(I - Q_\ell)f$$

and the full tails imply

$$(1-\rho)\|f\| \le \sigma_R\|h\| + a(\ell)^{-1/2}\sqrt{\mathfrak{a}[h]}.$$

Cauchy–Schwarz against the two energies therefore gives

$$\|sh\|^2 \le \frac{\sigma_R^2/\varepsilon + 1/a(\ell)}{(1-\rho)^2}(\mathfrak{a}[h] + \varepsilon\|h\|^2).$$

The statement extends from the core to its chosen closure, and spectral calculus identifies the best quotient with $j(\varepsilon)^2$. The first term of (5) gives $a(\xi) \ge 16\xi^2/(1+4\xi^2)$, hence $a(\xi) \ge (16/5)\xi^2$ for $|\xi| \le 1$. Put $q = \log(1/\varepsilon)$, $R = \log(q+4) > 1$, and $\ell = \pi/(8R) < 1$. Then $\rho = 1/2$. With $t = q + 4 \ge 4$, the theta tail gives

$$\sigma_R^2/\varepsilon \le 169t^4 e^{-3t^2+t-4} < 1.$$

The logarithmic derivative $4/t - 6t + 1$ is negative on $[4,\infty)$, and its endpoint value is $43264e^{-48} < 1$. It follows that

$$j(\varepsilon)^2 \le 4(1 + 20R^2/\pi^2) < 16R^2.$$

The independent cap $j \le \varepsilon^{-1/2}$ follows from $\|s\| \le 1$. □

3 A complete finite-rank enclosure

For $\varepsilon > 0$ put $R_\varepsilon = (A+\varepsilon)^{-1/2}$ and $J = M_s R_\varepsilon$. For unit $h \in \mathcal{K}$,

$$\|Jh\| \le \varepsilon^{-1/2}, \qquad \int a(\xi)|\widehat{Jh}(\xi)|^2 \frac{\mathrm{d}\xi}{2\pi} \le 1.$$

Define

$$V_M = -m_0 I + \sum_{2\le n\le M} c_n(S_{\log n} + S_{-\log n}), \quad d_M = |m_0| + 2\sum_{2\le n\le M} c_n.$$

Its Fourier multiplier is $v_M(\xi) = -m_0 + 2\sum c_n \cos(\xi \log n)$, and $\|V_M\| \le d_M$. Let Q_L denote the band projection and P_R the spatial projection. For $d \ge 0$ define

$$\widehat{E_d f}(\xi) = 1_{[-L,L]}(\xi)\sum_{k=0}^{d} \frac{(-i\xi)^k}{k!}\int_{-R}^{R} u^k f(u)\,\mathrm{d}u, \quad F_d = E_d J, \quad \mathcal{F} = F_d^* V_M F_d.$$

It is selfadjoint of rank at most $d+1$.

Lemma 3.1 (Full error)**.** *For $M \ge 1$, $R, L > 0$, and integer $d \ge 0$,*

$$\|K_\varepsilon - \mathcal{F}\| \le \frac{\delta_M}{\varepsilon} + \frac{d_M}{a(L)} + d_M\tau(2/\sqrt{\varepsilon} + \tau), \tag{10}$$

$$\tau = (\sigma_R + b_d)/\sqrt{\varepsilon}, \qquad b_d = \sqrt{2/\pi}\,\frac{(RL)^{d+3/2}}{(2d+3)(d+1)!}.$$

Proof. The arithmetic tail costs δ_M/ε. The retained sandwich is $J^*V_M J$. Since V_M commutes with Q_L, it is the sum of $(Q_L J)^* V_M (Q_L J)$ and $((I-Q_L)J)^* V_M((I-Q_L)J)$, with zero cross-frequency blocks. The high block has norm at most $d_M/a(L)$ by the energy bound. The real-phase Taylor remainder, integrated on the space–frequency rectangle, gives $\|Q_L P_R - E_d\| \le b_d$. Consequently

$$Q_L J - F_d = Q_L(I - P_R)J + (Q_L P_R - E_d)J$$

has norm at most τ, and $\|F_d\| \le \varepsilon^{-1/2} + \tau$. Expanding the difference of the two low-band signed products gives the last term of (10). This proves the bound on the entire receiving space.

□

4 Joint cutoff selection

We first sharpen the arithmetic coefficient with an elementary prime-power count.

Lemma 4.1 (Square-root arithmetic bound)**.** *For integer $M \ge 1$, $d_M \le 18\sqrt{M}$.*

Proof. Let $\Psi(x) = \sum_{n \le x} \Lambda(n)$. Every prime power $p^j \in (n, 2n]$ contributes one unit to the corresponding summand $\lfloor 2n/p^j \rfloor - 2\lfloor n/p^j \rfloor$ in the valuation of $\binom{2n}{n}$; all other summands are nonnegative. Thus

$$\Psi(2n) - \Psi(n) \le \log \binom{2n}{n} \le 2n \log 2.$$

Summing along powers of two gives $\Psi(2^r) < 2^{r+1} \log 2$. If $2^{r-1} < x \le 2^r$, then $\Psi(x) < 4x \log 2 < 3x$; at $x = 1$ it is zero. Stieltjes partial summation yields

$$\sum_{n \le M} \frac{\Lambda(n)}{\sqrt{n}} = \frac{\Psi(M)}{\sqrt{M}} + \frac{1}{2} \int_1^M \Psi(t) t^{-3/2} \, \mathrm{d}t \le 6\sqrt{M} - 3 \le 6\sqrt{M}.$$

Together with $|m_0| < 6$, this gives $d_M \le 6 + 12\sqrt{M} \le 18\sqrt{M}$.

□

For $0 < \varepsilon, \eta \le 1$ define the following positive quantities:

$$\begin{aligned}
k &= \max\left\{4, \left\lceil \frac{\log(1920/(\varepsilon\eta))}{\log 10} \right\rceil\right\}, & M &= k - 1, & D &= 18\sqrt{M}, \\
L &= \exp(6D/\eta), & X_0 &= \log \frac{312D}{\varepsilon\eta}, & R &= \tfrac{1}{2}\log(2X_0), \\
z &= RL, & b &= \frac{\varepsilon\eta}{24D}, & & \\
N(\varepsilon,\eta) &= \max\left\{1, \lceil 6z \rceil, \left\lceil \frac{\log(\sqrt{z}/b)}{\log 2} \right\rceil\right\}, & d &= N(\varepsilon,\eta) - 1. & & (11)
\end{aligned}$$

These formulas, including their integer ceilings, specify the joint approximation.

Proposition 4.2. *The cutoffs (11) make the full error (10) strictly less than η. For fixed η, $\log N(\varepsilon,\eta) = O_\eta(\sqrt{\log(1/\varepsilon)})$ as $\varepsilon \downarrow 0$.*

Proof. For integers $k \ge 4$, $k^2 \le 2^k$ by induction, starting at4. Thus $\delta_M \le 640\, 10^{-k} \le \varepsilon\eta/3$, which pays the arithmetic term. Next, $a(L) \ge \frac{1}{2}\log(2L) > 3D/\eta$ pays the high-frequency term by less than $\eta/3$.

Since $e^{2R} = 2X_0$,

$$\sigma_R \le 26X_0e^{-3X_0} \le 13e^{-X_0} = b,$$

using $2X_0 \le e^{2X_0}$. Write $n = d + 1 = N(\varepsilon, \eta)$. Stirling's elementary factorial lower bound $n! \ge (n/e)^n$ and $n \ge 6z$, $e < 3$ give

$$b_{n-1} \le \sqrt{z}(ez/n)^n \le \sqrt{z}\,2^{-n} \le b.$$

The last inequality is precisely the logarithmic ceiling in (11). Hence $\tau \le \eta\sqrt{\varepsilon}/(12D)$, and the last error term is at most

$$D\frac{\eta\sqrt{\varepsilon}}{12D}\left(\frac{2}{\sqrt{\varepsilon}} + \frac{\eta\sqrt{\varepsilon}}{12D}\right) = \frac{\eta}{6} + \frac{\eta^2\varepsilon}{144D} < \frac{\eta}{3}.$$

This proves the strict total error. All quantities in the formulas are finite, with $M \ge 3$, $X_0 > 1$, $R > 0$ and $z > 1$.

For the rank asymptotic put $u = \log(1/\varepsilon)$. At fixed η, M is comparable to u for large u, $D = O_\eta(\sqrt{u})$, $\log L = O_\eta(\sqrt{u})$ and $R = O_\eta(\log u)$. The third cutoff for n is $O_\eta(u + \log z)$, whereas $z = R\exp(6D/\eta)$ eventually dominates every fixed power of u. Thus $\log n = O_\eta(\sqrt{u})$. Enlarging the multiplicative constant on a remaining compact interval of ε proves the rank assertion in Theorem 1.1.

□

5 The entire negative part

Proof of the spectral conclusions. The preceding norm approximations, with arbitrary positive accuracy, imply that K_ε is compact selfadjoint. Fix $\eta = 1/2$ and a rank-N approximant $\mathcal{F}$ with error less than η. If the spectral subspace of K_ε for $(1, \infty)$ had dimension greater than N, it would intersect $\ker \mathcal{F}$ nontrivially. On that intersection its Rayleigh quotient is strictly greater than1, but $\mathcal{F}u = 0$ and the norm error bound make it at most η. This contradiction bounds that dimension by $N(\varepsilon, 1/2)$.

The map $h = (A + \varepsilon)^{-1/2}u$ is a bijection from $\mathcal{K}$ onto the chosen form domain $\mathcal{V}$, with its form topology, and

$$\langle h, (H + \varepsilon)h\rangle_{\text{form}} = \|u\|^2 - \langle u, K_\varepsilon u\rangle.$$

Transporting finite-dimensional strictly negative form subspaces in both directions proves equality of the two negative indices. The spectral theorem identifies the form index of $H + \varepsilon$ with $\operatorname{rank} 1_{(-\infty,-\varepsilon)}(H)$. This proves (1) from Proposition 4.2, without replacing a form-domain statement by a finite compression.

Since B is bounded and $A \ge 0$, H is bounded below. Finiteness of every negative-depth projection makes its negative spectrum discrete with finite multiplicities and possible accumulation only at zero. If the count is bounded by $A_0\exp(B_0\sqrt{\log(1/\varepsilon)})$, take, for large n,

$$\varepsilon_n = \exp\left[-\left(\frac{\log(n/(2A_0))}{B_0}\right)^2\right].$$

The number of eigenvalue magnitudes strictly larger than ε_n is at most $n/2$, so the nth magnitude is at most ε_n. Adjusting constants for finitely many remaining indices proves (2), including finite or empty spectra by zero padding. Every positive power of this bound is summable: for

fixed $p > 0$, eventually $cp \log^2(n+1) > 2\log(n+1)$. Hence $\sum_n \lambda_n(H_-)^p < \infty$, proving the stated Schatten conclusion.

□

6 Effective phase returns and growing-cutoff sharpness

For the retained operator define

$$K_{\mathrm{hi}}(M,\varepsilon,L) = J^*(I-Q_L)V_M(I-Q_L)J, \qquad b_M(L) = \max\left\{0, \sup_{|\xi|>L} \frac{v_M(\xi)}{a(\xi)}\right\}.$$

The energy estimate gives $\|K_{\mathrm{hi}}\| \le d_M/a(L)$ and $b_M(L) \le d_M/a(L)$.

Theorem 6.1 (Uniform growing-cutoff saturation)**.** *Let $X = \log\log\log L$ for sufficiently large L, and let $\omega(L) \ge \log 2$ be any specified envelope satisfying $\omega(L) = o(X)$. Uniformly over integers $2 \le M \le e^{\omega(L)}$ and $0 < \varepsilon \le 1$,*

$$\frac{a(L)\sup\sigma(K_{\mathrm{hi}}(M,\varepsilon,L))}{d_M} \longrightarrow 1, \qquad \frac{a(L)\|K_{\mathrm{hi}}(M,\varepsilon,L)\|}{d_M} \longrightarrow 1.$$

The scalar caps also satisfy $a(L)b_M(L)/d_M \to 1$ uniformly in that range.

6.1 Explicit every-interval prime-phase coverage

Lemma 6.2. *Let $p_1,\dots,p_m$ be the primes at most $M \ge 2$, and put $P = \prod p_j$. For $0 < \rho < 1$ define*

$$h = \lceil 256(m+1)^2\rho^{-4}\rceil, \qquad T = 8P^h(2h+1)^m.$$

Every real interval of length T contains N such that $\max_{p\le M}|e^{iN\log p}-1| < \rho$.

Proof. The nonnegative trigonometric polynomial

$$F(t) = \prod_{p\le M}\left(\frac{1+\cos(t\log p)}{2}\right)^h$$

has nonnegative Fourier coefficients summing to1. Its constant coefficient is

$A_h = [\binom{2h}{h}/4^h]^m \ge (2h+1)^{-m}$. Every other frequency is $\sum k_p \log p$, with $|k_p| \le h$. Unique factorization makes it nonzero. Write it as $\log(A/B)$, where A, B are distinct positive integers at most P^h. The inequality $\log(A/B) \ge (A-B)/A$ for $A > B$ gives absolute frequency at least P^{-h}. Thus the average of F on any interval of length T differs from A_h by at most $2P^h/T = 1/[4(2h+1)^m]$.

If one prime phase has distance at least ρ from1, then $F(t) \le (1-\rho^2/4)^h$. This is less than $1/[2(2h+1)^m]$. To check the constants put $B_0 = (m+1)^2\rho^{-4} \ge 1$. Then $h \le 257B_0$, $2h+1 \le 515B_0$, and

$$h\rho^2/4 \ge 64(m+1)^2\rho^{-2} > \log 2 + m\log(2h+1).$$

For the last comparison use $\log 515 < 7$, $\log(m+1) \le m+1$ and $\log(1/\rho) \le 1/\rho$ to bound its right side by $(1+13m+2m^2)\rho^{-2}$. The exponential inequality $1-x \le e^{-x}$ now proves the claim. The average lower bound $3/[4(2h+1)^m]$ exceeds the proposed pointwise failure ceiling, so some point of every interval has all phases within ρ.

□

6.2 Packets with their original weighted norm

Fix once and for all $f_0 \in C_c^\infty(-1,1)$ with $\|f_0\| = 1$, and write $D_0 = \|f_0'\|$. For $R \geq 1$ let $f_R(u) = R^{-1/2} f_0(u/R)$. Then

$$\int_{|\xi|>r} |\widehat{f}_R(\xi)|^2 \frac{\mathrm{d}\xi}{2\pi} \leq D_0^2/(R^2 r^2). \tag{12}$$

On $|u| \leq R$ the first positive theta term gives

$$s(u)^2 \geq e^{-4e^{2R}}, \qquad \|f_R/s\| \leq e^{2e^{2R}}. \tag{13}$$

Indeed $4x^2 - 6x \geq 2x^2$ for $x \geq \pi > 3$ implies $\psi(u) \geq 2\pi^2 e^{9u/2 - \pi e^{2u}}$ for $u \geq 0$. Divide by $\cosh(u/2) \leq e^{u/2}$, use $\pi < 4$, and reflect.

Choose fixed compact smooth repair vectors χ_0, χ_1 with invertible two-row matrix U. The exact identities

$$\sqrt{w}/s = \cosh(u/2), \qquad \sqrt{w}\tanh(u/2)/s = \sinh(u/2)$$

make the constraint integrals of $e^{iNu} f_R/s$ those of $e^{iNu} f_R$ against cosh and sinh. One integration by parts therefore gives a correction $r_{N,R}$ with $v_{N,R} = e^{iNu} f_R/s - r_{N,R} \in \mathcal{C}$ and

$$\|r_{N,R}\| + \|s r_{N,R}\| + \sqrt{\mathfrak{a}[r_{N,R}]} \leq C_0 e^{R/2}\sqrt{R}/N =: \kappa. \tag{14}$$

For an explicit fixed constant, put $g_j = \|\chi_j\| + \|s\chi_j\| + \sqrt{\mathfrak{a}[\chi_j]}$ and take

$$C_0 = 2(1/2 + D_0)\|U^{-1}\|_{\mathrm{op}} \Big(\sum g_j^2\Big)^{1/2}.$$

The derivative L^1 norms of each row integrand are at most $\sqrt{2}(1/2 + D_0)e^{R/2}\sqrt{R}$; multiplication by U^{-1} and Cauchy–Schwarz against the fixed vector norms prove (14). The form expression on a repair vector here denotes the same ambient nonnegative Fourier integral; it is defined for compact smooth functions without imposing the two rows.

The symbol upper bound in Lemma 2.1 and Plancherel give the uniform energy estimate

$$\mathfrak{a}[e^{iNu} f_R/s] \leq \log(1+N) + 6 + D_0^2/R^2.$$

Use $1 + |N + \zeta| \leq (1+N)(1+|\zeta|)$ and $\log(1+|\zeta|) \leq 1 + \zeta^2$ for this comparison. Form Cauchy–Schwarz then gives

$$\begin{aligned} \mathfrak{a}[v_{N,R}] &\leq [\sqrt{\log(1+N) + 6 + D_0^2/R^2} + \kappa]^2, \\ \|v_{N,R}\| &\leq e^{2e^{2R}} + \kappa. \end{aligned} \tag{15}$$

The inverse-theta input norm remains in the second bound.

6.3 A joint Rayleigh bound and the uniform limit

Fix $0 < \theta \leq 1/8$ and let

$$k_M = \lceil \log_2 M \rceil, \quad \rho = \theta/(2k_M), \quad r = \theta/(2\log M), \quad R = \max\{1, D_0/(r\sqrt{\theta})\}.$$

Choose the return $N \in [L+r+1, L+r+1+T]$ from Lemma 6.2. Each retained nonzero channel is $n = p^k$, $k \le k_M$. Thus $|e^{iN\log n} - 1| \le \theta/2$, and the additional phase on a window $|t| \le r$ is at most $\theta/2$. Consequently $v_M(N+t) \ge (1-\theta)d_M$ throughout that window. It lies entirely above L, and (12) puts at least $1-\theta$ of the carrier mass there. On the rest of the high band use $v_M \ge -d_M$. Applying the correction bound yields

$$\langle s v_{N,R}, (I - Q_L) V_M (I - Q_L) s v_{N,R} \rangle \ge d_M(1 - 3\theta - 2\kappa - \kappa^2).$$

No frequency mass outside the central window is assigned an unproved positive sign.

Put

$$\kappa_L = C_0 e^{R/2}\sqrt{R}/L, \quad E_L = \log(1 + L + r + 1 + T) + 6 + D_0^2,$$

$$\mathcal{D}_L = (\sqrt{E_L} + \kappa_L)^2 + \varepsilon(e^{2e^{2R}} + \kappa_L)^2.$$

When $\kappa_L \le 1/8$, the numerator below is positive. Normalizing $v_{N,R}$ by the actual reference form, as $u = (A+\varepsilon)^{1/2} v_{N,R}/\sqrt{\mathfrak{a}[v_{N,R}] + \varepsilon\|v_{N,R}\|^2}$, and using (15) gives

$$\sup \sigma(K_{\mathrm{hi}}) \ge d_M(1 - 3\theta - 2\kappa_L - \kappa_L^2)/\mathcal{D}_L. \tag{16}$$

Proof of Theorem 6.1. Take $\theta = [(1+\omega(L))/X]^{1/3}$ for large L, so $\theta \to 0$ and eventually $\theta \le 1/8$. The preceding choices give

$$R = O(\sqrt{X(1+\omega(L))} + 1) = o(X).$$

Since the number of retained primes is at most M, $\log P \le m \log M$, and the explicit phase budget gives

$$\log T \le C(M+1)^3(\log M + 1)^5 \theta^{-4} \le e^{O(\omega(L))}(1+\omega(L))^5 X^{4/3}.$$

The logarithm of this last bound is $o(X)$, so $\log T = e^{o(X)}$, whereas $\log L = e^{e^X}$. Therefore $T/L \to 0$ and $\kappa_L \to 0$ uniformly over the declared range. The normalization term is also paid uniformly for $0 < \varepsilon \le 1$, because

$$\log\left(\frac{e^{4e^{2R}}}{\log L}\right) = 4e^{2R} - \log\log L = e^{o(X)} - e^X \longrightarrow -\infty.$$

Thus $\mathcal{D}_L/\log L \to 1$ uniformly. Combine (16) with $a(L)/\log L \to 1$ and the universal upper $d_M/a(L)$ to prove both operator limits. For the scalar cap, the same returns give $v_M(N) \ge (1-\theta)d_M$, $N/L \to 1$ and $a(N)/a(L) \to 1$. This proves its lower limit; its upper bound is again $d_M/a(L)$.

□

7 Fast cutoffs and a strict high-block gap

The proof uses medium as well as disjoint spatial translations. We give the projection facts before transferring them to the original weighted output.

Lemma 7.1 (Projected compact-core correlations)**.** *Let F be supported in $I = [-R, R]$ and let Q be an orthogonal projection on $L^2(\mathbb{R})$. For $b > R$,*

$$|\langle F, QS_bF\rangle| \le \tfrac{3}{4}\|F\|^2.$$

For $b > 2R$ the upper coefficient can be replaced by $1/2$. If Q commutes with translations, the same bounds apply to $\langle QF, S_bQF\rangle$.

Proof. The compression $A_b = P_I S_b P_I$ on $L^2(I)$ is a contraction. For $b > R$ it satisfies $A_b^2 = 0$, since $u, u+b, u+2b$ cannot all belong to I. The compact-window half bound is also recorded in [8, Lemma 1] and [10, Lemmas 1 and 1′]; it is the order-two case of [9, Theorem 1]. Here is a direct proof. Decompose $L^2(I) = \ker A_b \oplus (\ker A_b)^\perp$. Since $\operatorname{Ran} A_b \subset \ker A_b$, for $F = (x, y)$ one has

$$|\langle F, A_bF\rangle| \le \|x\|\|y\| \le \tfrac{1}{2}\|F\|^2.$$

Now $2Q - I$ is unitary, so

$$|\langle F, QS_bF\rangle| \le \tfrac{1}{2}(|\langle F, S_bF\rangle| + \|F\|^2) \le \tfrac{3}{4}\|F\|^2.$$

If $b > 2R$, F and S_bF are orthogonal. For normalized orthogonal vectors F, G put $\alpha = \langle F, QF\rangle$, $\beta = \langle G, QG\rangle$. The squared cross term is bounded both by $\alpha\beta$ and $(1-\alpha)(1-\beta)$, using projection Cauchy–Schwarz and $\langle F, QG\rangle = -\langle F, (I-Q)G\rangle$. At least one product is at most $1/4$, according as $\alpha + \beta \le 1$ or $\alpha + \beta \ge 1$. This proves the stronger half bound. Finally, commutation and $Q^2 = Q$ give $\langle QF, S_bQF\rangle = \langle F, QS_bF\rangle$. □

Fix a spatial margin $\delta > 0$. For large q put

$$R_w = \tfrac{1}{2}\log(q/\pi), \qquad R_0 = R_w + \delta/2, \qquad R_1 = R_w + \delta.$$

Choose real even $\chi_q \in C_c^\infty$, with $0 \le \chi_q \le 1$, equal to1 on $[-R_0, R_0]$ and supported in $[-R_1, R_1]$. A fixed transition of width $\delta/2$ gives $\operatorname{Lip}\chi_q \le C_\delta$ independently of q. Fix real even $\eta_0 \in C_c^\infty$ with $0 \le \eta_0 \le 1$, equal to1 on $[-1/2, 1/2]$ and zero outside $[-1, 1]$. Let B_L have Fourier multiplier $1 - \eta_0(\xi/L)$, and set

$$k(x) = \frac{1}{2\pi}\int e^{ix\xi}\eta_0(\xi)\,\mathrm{d}\xi, \qquad C_{\eta_0} = \int |x||k(x)|\,\mathrm{d}x < \infty.$$

The multiplier is a contraction, vanishes below $L/2$, and satisfies $(I - Q_L)B_L = I - Q_L$ exactly. The identity part of B_L commutes with χ_q; the remaining commutator kernel, up to sign, is

$$Lk(L(u-v))(\chi_q(v) - \chi_q(u)).$$

Its row and column absolute integrals are each at most $C_\delta C_{\eta_0}/L$. The integral Schur test proves

$$\|[B_L, \chi_q]\| \le C_\delta C_{\eta_0}/L. \tag{17}$$

This estimate concerns the smooth multiplier B_L.

For any unit input $h \in \mathcal{K}$, let $v = (A+\varepsilon)^{-1/2}h$, $f = sv$, $g = B_Lf$ and $F = \chi_q g$. The original input satisfies $\|v\| \le e^{q/2}$ when $\varepsilon = e^{-q}$. The sharper envelope in the proof of Lemma 2.1 gives

$$\sup_{|u|>R} s(u) \le 4\pi e^{2R - (\pi/2)e^{2R}}.$$

At R_0 this bounds $\|(1-\chi_q)f\|$ by $4qe^{\delta}\exp[-q(e^{\delta}-1)/2]$. Commuting B_L and χ_q with (17), and using Lemma 2.2 for $\|f\|$, yields the full error

$$\|g-F\| \le e_q := 4qe^{\delta}e^{-q(e^{\delta}-1)/2} + \frac{4C_\delta C_{\eta_0}\log(q+4)}{L}. \tag{18}$$

Both terms are uniform in the unit input. The two different norms have distinct roles: $e^{q/2}$ pays the original spatial tail, and the smaller observation norm pays only the commutator. Since g vanishes in the frequency interval $[-L/2, L/2]$, its complete reference energy gives

$$(I-Q_L)g = (I-Q_L)f, \qquad \|g\|^2 \le 1/a(L/2), \qquad \|F\| \le \|g\|. \tag{19}$$

Thus the sharp highpass output is compared to a confined smooth-highpass core without assuming that the sharp projection itself preserves support.

Define the medium-and-long and long arithmetic masses

$$S_1 = \sum_{\substack{2\le n\le M\\ \log n > R_1}} c_n, \qquad S_2 = \sum_{\substack{2\le n\le M\\ \log n > 2R_1}} c_n.$$

Apply Lemma 7.1 with $Q = I-Q_L$. Each short paired channel costs at most $2c_n\|F\|^2$, each medium channel costs $\frac{3}{2}c_n\|F\|^2$, and each long channel costs $c_n\|F\|^2$. The constant term costs $(-m_0)\|F\|^2$. Therefore

$$|\langle F, (I-Q_L)V_M(I-Q_L)F\rangle| \le (d_M - \tfrac{1}{2}S_1 - \tfrac{1}{2}S_2)\|F\|^2.$$

The coefficient is positive, since $S_1, S_2 \le \sum c_n$. The bounded signed multiplier and (18)–(19) now imply the whole-input norm estimate

$$\|K_{\mathrm{hi}}\| \le \frac{d_M - \frac{1}{2}S_1 - \frac{1}{2}S_2}{a(L/2)} + d_M e_q\left(\frac{2}{\sqrt{a(L/2)}} + e_q\right). \tag{20}$$

Indeed the difference of the g and F quadratic forms is bounded by the second term, including all mixed terms. The identity in (19) returns this to the original high block, and a uniform absolute quadratic bound on a selfadjoint operator bounds its norm.

Proof of Theorem 1.2. The classical unconditional prime number theorem gives

$$\Psi(x) = \sum_{n\le x}\Lambda(n) = x + o(x);$$

see [2, 25.16.3 and 27.2.3]. Abel summation gives

$$A_1(x) := \sum_{n\le x} c_n = \frac{\Psi(x)}{\sqrt{x}} + \frac{1}{2}\int_1^x \Psi(t)t^{-3/2}\,\mathrm{d}t = 2\sqrt{x} + o(\sqrt{x}).$$

For the error, split the integral at any fixed threshold beyond which $|\Psi(t)-t| \le \tau t$, bound the remainder by a constant times $\tau\sqrt{x}$, then send τ to zero. No uniform quantitative PNT error is required.

Since $M \sim Cq$, $d_M \sim 4\sqrt{Cq}$. Also $e^{R_1} = e^{\delta}\sqrt{q/\pi}$, so $A_1(e^{R_1}) = O_\delta(q^{1/4})$. Consequently $S_1/d_M \to 1/2$ for every fixed $C > 0$. Put $A_\delta = e^{2\delta}/\pi$. If $C > A_\delta$, subtraction of the two prefix asymptotics gives $S_2/d_M \to \frac{1}{2}(1-\sqrt{A_\delta/C})$. If $C < A_\delta$, this mass is eventually zero. At equality it

is still $o(d_M)$, since the positive part of the difference of the two prefix asymptotics is $o(\sqrt{q})$. Thus in all cases

$$\frac{S_2}{d_M} \longrightarrow \tfrac{1}{2}(1-\sqrt{A_\delta/C})_+.$$

Under $\log L \sim K\sqrt{q}$, $a(L)/a(L/2) \to 1$ and $\sqrt{a(L)}e_q \to 0$: the first error term decays exponentially in q at fixed δ, and the second is $O_\delta(\log q/L)$. Multiplying (20) by $a(L)/d_M$ therefore gives

$$\limsup \frac{a(L)\|K_{\mathrm{hi}}\|}{d_M} \le \frac{3}{4} - \frac{1}{4}(1-\sqrt{A_\delta/C})_+.$$

The target is independent of δ and this holds for every fixed positive δ. Sending $\delta \downarrow 0$ after the q limsup proves (3), including $C = 1/\pi$. The positive spectral edge is at most the norm. For the explicit cutoffs (11) at fixed η, $M/q \to 1/\log 10$ and $\log L/\sqrt{q} \to 108/(\eta\sqrt{\log 10})$, which proves the stated specialization. □

8 Parameter regimes and further applications

Theorem 1.1 selects $M \asymp \log(1/\varepsilon)$ and $\log L \asymp \sqrt{\log(1/\varepsilon)}$ at fixed approximation tolerance. Theorem 6.1 proves high-block saturation in the slower range $\log M = o(\log\log\log L)$. Theorem 1.2 now shows a strict failure of saturation throughout the fast rank scale, including the critical theta-overlap ratio. Thus extending the fixed-cutoff sharpness constant to that scale would be incorrect. The high-block upper coefficient there has a genuine reserve; the finite bound (20) states how its medium and long arithmetic masses enter before the asymptotic evaluation.

The optimal approximation rank depends on the low-frequency map and complete arithmetic remainder as well as the retained-block gap. A natural further refinement is a joint bound on the arithmetic sum before separately maximizing its channel correlations. Such a bound must preserve the actual confined smooth-highpass range and its full weighted input norm.

The new negative-spectrum conclusion concerns H_- itself. It uses no identification of an entire kernel or derivative-null closure. Structured finite signed certificates can address individual lower thresholds through a different reference and computational task. The rank theorem and the joint saturation theorem provide analytic information about the full operator family across parameter ranges.

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