# Holomorphic certification on real subspaces

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Luminary Oddities

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**Abstract**

For a family of nonnegative Hermitian quartics on complexifications of real subspaces, we identify the exact failure set of eventual holomorphic squared-norm certification: the subspaces containing a sign vector. At fixed dimensions and positive perturbation, the least norm-multiplier degree grows uniformly as the inverse square of the distance to this set. This geometric law gives exact moment thresholds for random certificate degree. On Haar-distributed real $d$-planes in $\mathbb{R}^n$, the degree has tail order $M^{-(n-d)/2}$; it is finite almost surely but has infinite mean in codimensions one and two. Soft-constraint matrices with independent uniform entries on $[-1, 1]$ give the corresponding threshold in their number of rows. The probability estimates remain valid under changes of density bounded above and away from zero. The same classification yields coNP-hardness for already nonnegative integer quartics with exactly one negative coefficient eigenvalue and cubically bounded coefficients. A sharp uniform inertia threshold strengthens that coefficient restriction. Explicit hierarchy bounds supply global degree upper estimates, while logarithmic curvature gives the matching lower scale. In a weighted two-dimensional restriction we determine both an exact fixed-perturbation leading constant and the joint vanishing-perturbation crossover, including finite degree plateaus. The random and joint-limit results quantify how eventual certification can coexist with large or nonintegrable degree costs.

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## 1 The family and the main results

For a fixed multiplier degree, a holomorphic squared-norm certificate is a positive-semidefinite coefficient matrix. The union over all degrees raises a different question: does this certificate hierarchy eventually succeed? We study that question for quartics that are already pointwise nonnegative. Even when the coefficient matrix has only one negative direction, eventual success remains a coNP-hard decision.

The geometric mechanism also explains how the hierarchy becomes expensive. Real linear constraints reduce positivity to avoidance of the sign cube. The least certificate degree grows uniformly as the inverse square of the subspace distance to the nonstabilizing locus. Soft penalties for the constraints obey a corresponding law in their signed residual. A two-dimensional weighted restriction gives a complementary sharp result: its full coefficient matrices determine the exact leading constant. Thus the decision theorem and the quantitative results describe the same distinction between positivity of values and eventual positivity of coefficient matrices.

**Random certificate degree.** The distance law has a further consequence for typical inputs. Eventual certification holds almost surely under natural continuous constraint models, yet its degree need not have a finite mean. For Haar real $d$-planes and for a matrix with $r$ rows and independent uniform entries on $[-1,1]$, respectively, we prove

$$\Pr\{\mu_V(\rho) > M\} = \Theta(M^{-(n-d)/2}), \qquad \Pr\{\mu_B(\rho) > M\} = \Theta(M^{-r/2}).$$

The corresponding positive moments exist exactly below these exponents (Theorem 5.2). This is a distributional statement about the degree of the specified hierarchy; its geometric proof uses the uniform deterministic lower law and explicit global upper bounds.

Let $V \subset \mathbb{R}^n$ be a real linear subspace of dimension $d \geq 2$, and write $V_{\mathbb{C}} = V + iV$. Equip it with the inherited Hermitian metric. On $V_{\mathbb{C}}$ define

$$\begin{aligned} N(v) &= \sum_i |v_i|^2, \qquad p_2(v) = \sum_i v_i^2, \\ A(v) &= n\sum_i |v_i|^4 - |p_2(v)|^2 = \sum_{i<j} |v_i^2 - v_j^2|^2, \\ Q(v) &= N(v)^2 - |p_2(v)|^2, \qquad q_\rho = A + \rho Q, \quad \rho > 0. \end{aligned} \tag{1}$$

Both $A$ and $Q$ are nonnegative. A *holomorphic squared norm* means a finite sum $\sum_\nu |h_\nu|^2$ with holomorphic polynomial $h_\nu$. For a quartic define its norm stabilization degree by

$$\mu_V(\rho) = \min\{m \geq 0 : N^m q_\rho \text{ is a holomorphic squared norm on } V_{\mathbb{C}}\},$$

with value $+\infty$ when the set is empty.

The exact failure geometry is given by the following classification.

**Theorem 1.1** (Sign-cube classification)**.** *The following are equivalent:*

1. $V \cap \{-1,1\}^n = \varnothing$.
2. $q_\rho(v) > 0$ *for every nonzero* $v \in V_{\mathbb{C}}$.
3. $\mu_V(\rho) < \infty$.
4. $q_\rho$ *is a quotient of holomorphic squared norms.*
5. *There is a Hermitian polynomial* $B$*, nonzero in the intrinsic coordinate ring of* $V_{\mathbb{C}}$*, such that* $Bq_\rho$ *is a holomorphic squared norm.*

*These equivalences hold for every* $\rho > 0$.

The intrinsic qualification excludes ambient polynomials that vanish identically on the subspace. The last assertion permits multipliers of arbitrary degree, sign and vanishing order.

**Theorem 1.2** (One negative direction and bounded-coefficient hardness)**.** *At* $\rho = n - 1$*, the Hermitian coefficient matrix of* $q_\rho$ *in any independent linear coordinates on* $V_{\mathbb{C}}$ *has exactly one negative eigenvalue and all other eigenvalues positive.*

*Deciding whether some finite norm-power certificate exists is coNP-hard even for explicitly expanded, pointwise nonnegative, integer bihomogeneous quartics with exactly one negative coefficient eigenvalue. In the reduction, each coefficient has absolute value at most* $162n^3$*, where* $n$ *is the size of the ambient subspace construction. For the structured input given by signed triples defining that construction, the decision language is coNP-complete.*

A fixed-dimensional consequence also concerns succinct input: at $\rho = 1$, the family $b_L = 2+2^{-L}$ has rational coefficients of bit length $O(L)$ but requires stabilization degree $\Theta(4^L)$ (Corollary 6.1).

The classification has a uniform metric refinement. Let $P_V$ be orthogonal projection onto a real $d$-plane $V$, and define

$$\eta(V) = \min_{s\in\{-1,1\}^n} \frac{\|(I - P_V)s\|}{\sqrt{n}}, \qquad \mathcal{S}_{n,d} = \{W \in \mathrm{Gr}_d(\mathbb{R}^n) : W \cap \{-1,1\}^n \neq \varnothing\}.$$

Equip the real Grassmannian with $d_{\mathrm{op}}(V,W) = \|P_V - P_W\|_{\mathrm{op}}$.

**Theorem 1.3** (Uniform distance to the nonstabilizing locus)**.** *For $n \geq 3$, $2 \leq d < n$, and every real $d$-plane $V$,*

$$\eta(V) = \mathrm{dist}_{\mathrm{op}}(V, \mathcal{S}_{n,d}). \tag{2}$$

*Fix these dimensions and $\rho > 0$. There are constants $c, C, \varepsilon > 0$, depending only on $n, d, \rho$, such that*

$$c\eta(V)^{-2} \leq \mu_V(\rho) \leq C\eta(V)^{-2} \quad \textit{whenever} \quad 0 < \eta(V) < \varepsilon. \tag{3}$$

Thus the degree scale is uniform over the entire fixed-dimensional subspace family, including simultaneous approaches to several sign vectors. The next theorem treats constraints through a positive penalty. For a real matrix $B \in \mathbb{R}^{r\times n}$, where $n \geq 2$ and $r \geq 0$, set

$$F_{B,\rho}(v) = A(v) + \rho Q(v) + N(v)\|Bv\|^2, \qquad v \in \mathbb{C}^n, \quad \rho > 0, \tag{4}$$

$$\sigma(B) = \min_{s\in\{-1,1\}^n} \frac{\|Bs\|}{\sqrt{n}}. \tag{5}$$

Let $\mu_B(\rho)$ be the least $m \geq 0$ for which $N^m F_{B,\rho}$ is a holomorphic squared norm on $\mathbb{C}^n$, with value $+\infty$ when the set is empty.

**Theorem 1.4** (Uniform degree law for soft constraints)**.** *The form $F_{B,\rho}$ is nonnegative for every real $B$. It is strictly positive away from zero, and has finite $\mu_B(\rho)$, exactly when $\sigma(B) > 0$. At $\sigma(B) = 0$, no nonzero polynomial multiplier makes $F_{B,\rho}$ a holomorphic squared norm.*

*Fix $n \geq 2$, $r \geq 0$, $\rho > 0$ and $K > 0$. There are constants $c, C, \varepsilon > 0$, depending only on these fixed parameters, such that*

$$c\,\sigma(B)^{-2} \leq \mu_B(\rho) \leq C\,\sigma(B)^{-2} \quad \textit{if} \quad \|B\|_{\mathrm{op}} \leq K, \quad 0 < \sigma(B) < \varepsilon. \tag{6}$$

The constants are uniform as the matrix rank changes and as several signed constraints approach zero together. The positive-penalty formulation makes the distance to failure explicit. Empty matrices and zero rows follow the same criterion. The positive assumption on $\rho$ accounts for the anisotropic obstruction; at $\rho = 0$, $A + N\|Bv\|^2$ is itself a holomorphic squared norm.

A complementary exact leading constant holds on the weighted hyperplane

$$H_b = \{v \in \mathbb{C}^3 : v_1 + v_2 + bv_3 = 0\}, \qquad b > 0.$$

Write $\mu_b(\rho)$ for its stabilization degree.

**Theorem 1.5** (Boundary degree law)**.** *For each fixed $\rho > 0$,*

$$\mu_2(\rho) = +\infty, \qquad \lim_{b\to 2,\ b\neq 2} (b-2)^2 \mu_b(\rho) = \frac{9}{2}\big(\sqrt{1+\rho} - 1\big)^2.$$

*The limit is obtained from matching bounds on the complete coefficient matrices.*

### 1.1 Relation to earlier work

Quillen and Catlin–D'Angelo prove finite stabilization of strictly positive Hermitian forms [13, 2]. D'Angelo's one-variable examples and Varolin's jet-pullback and basic-zero criteria describe obstructions at zeros [3, 17]. We apply that classical mechanism directly. D'Angelo [4] compares nonnegativity, plurisubharmonicity, Cauchy–Schwarz positivity and squared-norm representations for bihomogeneous families. Our central object is eventual norm-power certification on varying real subspaces. In the nonsingular coefficient regime of Theorem 1.2, Varolin's free-support criterion also identifies quotient existence with absence of nontrivial zeros.

The unweighted odd-coordinate hyperplane was treated in [15]: its least degree is at least $n-2$ for every positive perturbation, with a finite first-degree parameter window and a sharp five-coordinate endpoint. The present subspace classification includes even coordinates and arbitrary real linear constraints. Its complexity consequence uses an explicit sparse Boolean encoding while preserving nonnegativity and a single negative coefficient direction.

General quartic and tensor hardness concerns positivity, convexity, decomposition or norm problems [1, 10, 7]. Here the input is already nonnegative, and the decision asks whether a holomorphic certificate exists at *any* finite degree. Norm-power hierarchies and high stabilization degrees have substantial precedents [12, 2, 9, 6]; Theorem 1.5 determines the leading constant at the specified weighted-subspace boundary.

Gaubert and Qu [8, Section 4] prove hardness of strict Kraus-map positivity through rational Boolean encodings and a single-vector-to-bilinear feasibility transfer. Those mechanisms are important predecessors. The direct squared-modulus polynomial of their equations is already a holomorphic squared norm on every instance. Our construction instead preserves nonnegativity while making eventual certification equivalent to Boolean unsatisfiability, with one negative coefficient direction and polynomial coefficient magnitude. The anisotropic term and its local obstruction provide this interface.

Effective reciprocal-margin upper bounds were established by To and Yeung [16]; the quantitative theorem used here is Drouot–Zworski [6, Theorem 2]. Its bound supplies the upper bound in Theorem 1.4. Its lower bound uses a uniform anisotropic slice at a real minimum and the necessary plurisubharmonicity of the logarithm of a squared norm. The soft-constraint result tracks all bounded matrices through their smallest signed residual. Theorem 1.3 uses the same anisotropic calculation after restriction and identifies the residual with exact distance to the nonstabilizing locus; it quantifies the original subspace classification uniformly over all fixed-dimensional subspaces. For Theorem 1.5, complete finite coefficient matrices give the sharper leading constant; the limiting even trial sequence is proved by its explicit recurrence. Quadratic Toeplitz model energies and their squeezed-state descriptions are classical; see [5, Section 3.3]. Here complete finite row bounds and fixed finite trial limits supply the sharp boundary certificate.

The random-degree consequence uses classical probability and inverse-distance condition analysis. Amelunxen–Bürgisser [18] develop such analysis for Grassmannian ill-posedness in convex feasibility. Here the failure set is finite sign-line incidence, and the degree exponent comes from the matched inverse-square law. The explicit global upper bounds use the Hermitian hierarchy estimate of Lovitz–Johnston [12, Theorem 3.3]; the operator normalization is given in Proposition 5.1. The binary crossover is proved directly from complete finite matrices, allowing the perturbation and geometric distance to vanish together.

### 1.2 Proof strategy and organization

The local obstruction in Section 2 proves the sign-cube classification. The coefficient inertia in Section 3 and the Boolean encoding in Section 4 then give the restricted hardness theorem. Section 5 obtains the hard- and soft-constraint degree laws from a common positivity margin and logarithmic-curvature estimate. The explicit reciprocal-margin bound and the probability argument then give Theorem 5.2. Section 6 uses the complete coefficient matrices to determine the sharp binary constant and its two-parameter crossover.

## 2 Zeros and the local obstruction

**Lemma 2.1** (An anisotropic leading jet)**.** *Let $q$ be a Hermitian polynomial on a complex vector space, with zero constant and linear Taylor terms at $z_0$. Suppose a nonempty open set of directions $w$ has leading restriction*

$$q(z_0 + tw) = a|t|^2 - bt^2 - \bar{b}\,\bar{t}^2 + O(|t|^3), \qquad b \neq 0.$$

*Then no intrinsic nonzero polynomial $B$ has $Bq$ equal to a nonzero holomorphic squared norm.*

*Proof.* For a fixed nonzero $B$, some direction in the stated open set has $B(z_0 + tw) \not\equiv 0$. Otherwise every homogeneous Taylor coefficient of $B$ at $z_0$ vanishes on an open set and hence identically. Let $B_k(t, \bar{t})$ be the lowest nonzero term on this line. The leading product is $q_2 B_k$, a nonzero polynomial in $\mathbb{C}[t, \bar{t}]$.

Every nonzero squared norm of one-variable holomorphic polynomials has leading term $c|t|^{2\ell}$, $c > 0$: take the smallest vanishing order of its components and add the squared leading coefficients. Thus $q_2$ would divide $t^\ell \bar{t}^\ell$ in $\mathbb{C}[t, \bar{t}]$. Every divisor of a monomial is a monomial up to a constant, by unique factorization. The two nonzero pure-square coefficients of $q_2$ contradict this. □

*Proof of Theorem 1.1.* If $A(v) = 0$ and $v \neq 0$, all $v_i^2$ are equal and nonzero. Hence $v = c\epsilon$ with $c \neq 0$ and $\epsilon \in \{-1, 1\}^n$. Since $V$ is real, $c\epsilon \in V_{\mathbb{C}}$ implies $\epsilon \in V$. Conversely every such sign vector satisfies $A = Q = 0$. This proves the equivalence of the first two assertions. Strict positivity gives finite stabilization by [13, 2], and a norm-power certificate is a quotient certificate and a polynomial multiplier certificate.

Suppose now $\epsilon \in V \cap \{-1, 1\}^n$. For $w \in V_{\mathbb{C}}$, put

$$C = n\|w\|^2 - |\epsilon \cdot w|^2, \qquad D = n \sum_i w_i^2 - (\epsilon \cdot w)^2.$$

Expansion at the balanced-modulus vector $\epsilon$ gives

$$\begin{aligned} A(\epsilon + tw) &= 4C|t|^2 + O(|t|^3), \\ Q(\epsilon + tw) &= 2C|t|^2 - Dt^2 - \bar{D}\bar{t}^2 + O(|t|^3). \end{aligned}$$

There is a nonzero real $w \in V \cap \epsilon^\perp$, since $d \geq 2$. For this direction $C = D = n\|w\|^2 > 0$, so $D \neq 0$ on a nonempty open set of complex directions. Lemma 2.1 excludes every intrinsic nonzero polynomial multiplier and proves the remaining implication. □

**Corollary 2.2** (Hyperplanes)**.** *For $a \in \mathbb{R}^n_{>0}$, $n \geq 3$, the restriction to $\sum_i a_i v_i = 0$ has finite stabilization exactly when $\sum_i a_i \epsilon_i \neq 0$ for every sign vector. For the unweighted hyperplane this occurs exactly when $n$ is odd.*

The weighted condition is the absence of an exact signed partition. For example, $(1, 1, 2)$ has an obstruction in three coordinates, whereas $(1, 1, 1, 4)$ admits finite stabilization in four.

## 3 The coefficient inertia

Choose real orthonormal coordinates $z_1, \dots, z_d$ on $V$. Use the Fischer product $\langle z^\alpha, z^\beta \rangle_F = \alpha!\, \mathbf{1}_{\alpha=\beta}$. The coefficient operator of $N^2$ on holomorphic quadrics is $2I$, and that of $|p_2|^2$ is $|p_2\rangle\langle p_2|$, with $\|p_2\|_F^2 = 2d$. Thus

$$M_Q = 2I - |p_2\rangle\langle p_2|.$$

Its radial eigenvalue is $-2(d-1)$; every other eigenvalue is 2.

**Proposition 3.1** (Sharp uniform inertia threshold)**.** *For every real $d$-plane $V \subset \mathbb{R}^n$, $2 \le d \le n$, the coefficient matrix of $q_\rho$ has exactly one negative eigenvalue and all remaining eigenvalues positive whenever*

$$\rho > \frac{n-d}{d-1}.$$

*This strict threshold is sharp uniformly over $V$.*

*Proof.* Write $h = p_2$, $l_i(z) = v_i$, and $a_i = \|l_i\|^2 = (P_V)_{ii}$. Then $0 \le a_i \le 1$, $\sum_i a_i = d$ and $\sum_i a_i^2 \le d$. Reality of the coordinates gives $\langle h, l_i^2 \rangle_F = 2a_i$. Thus the coefficient operator is

$$M_\rho = n \sum_i |l_i^2\rangle\langle l_i^2| + 2\rho I - (1+\rho)|h\rangle\langle h|.$$

Its Rayleigh quotient on $h$, using $\|h\|_F^2 = 2d$, satisfies

$$\frac{\langle h, M_\rho h \rangle_F}{\|h\|_F^2} = \frac{2n}{d} \sum_i a_i^2 + 2\rho - 2d(1+\rho) \le 2(n-d) - 2\rho(d-1) < 0.$$

The first two terms of $M_\rho$ form a positive definite matrix. After congruence its rank-one subtraction is $I - (1+\rho)uu^*$. The strict negative direction therefore gives exactly one negative eigenvalue, and every other eigenvalue is positive.

For $V = \operatorname{span}(e_1, \dots, e_d)$ the radial eigenvalue is $2[n - d - \rho(d-1)]$. The other pure-square eigenvalues are $2(n+\rho)$, and the mixed-monomial eigenvalues are $2\rho$. At the stated threshold the radial eigenvalue is zero, proving sharpness. Independent coordinate changes preserve inertia by invertible congruence. Finally $\rho = n - 1$ lies strictly above the threshold because $(n-1)(d-1) - (n-d) = n(d-2) + 1 > 0$. This proves the inertia assertion in Theorem 1.2. □

## 4 Sparse Boolean encoding

Consider a not-all-equal three-literal instance with $k$ variables and $m$ clauses. A signed truth assignment is $x_i \in \{-1, 1\}$. For each clause define

$$l_c(x) = \sigma_{c,1} x_{i_1} + \sigma_{c,2} x_{i_2} + \sigma_{c,3} x_{i_3}, \qquad \sigma_{c,j} \in \{-1, 1\}.$$

Set $n = k + m$, and use the graph subspace

$$V = \{(x_1, \dots, x_k, l_1(x), \dots, l_m(x)) : x \in \mathbb{R}^k\}.$$

Its displayed coordinates are independent, and each embedding row has coefficient $\ell^1$ norm at most 3. Two unused variables may be added to ensure $k \ge 2$. The sum of three signs belongs to $\{-1, 1\}$ exactly when those signs are not all equal. Hence

$$V \cap \{-1, 1\}^n \neq \varnothing \quad \Longleftrightarrow \quad \text{the not-all-equal instance is satisfiable.} \tag{7}$$

For completeness, ordinary three-satisfiability reduces to these instances by a two-clause gadget. Introduce a global reference bit $f$ and one auxiliary bit $a_c$ for each clause $u \vee v \vee w$. Replace it by

$$\mathrm{NAE}(f, u, a_c), \qquad \mathrm{NAE}(v, w, \neg a_c).$$

Simultaneous complementation preserves every NAE predicate, so assume $f$ is false. If $u$ is false, the first predicate forces $a_c$ true, and the second is equivalent to $v \vee w$. If $u$ is true, the first allows either $a_c$, and the second can always be satisfied. This proves the reduction from classical three-satisfiability [11, 14].

In the graph coordinates take $\rho = n - 1$. Theorem 1.1 and (7) identify eventual certifiability with unsatisfiability. The construction expands in polynomial time. To check its numerical size, let $l_i$, $1 \le i \le n$, denote all embedding forms. Then

$$q_\rho = n \sum_i |l_i^2|^2 + \rho \sum_{i,j} |l_i l_j|^2 - (1+\rho) \Big| \sum_i l_i^2 \Big|^2.$$

Each $l_i$ has coefficient $\ell^1$ norm at most 3. The total coefficient $\ell^1$ norm is consequently at most

$$81n^2 + 81\rho n^2 + 81(1+\rho)n^2 = 162(1+\rho)n^2 = 162n^3.$$

Every coefficient is an integer, and there are $O(k^4)$ monomial pairs. The reduction remains polynomial when coefficients are written in unary.

In these graph coordinates the inherited multiplier is $N(x) = \sum_i |l_i(x)|^2$, a positive definite quadratic form. For the instances constructed here, eventual existence is independent of the positive definite multiplier norm. On sign-free instances, strict positivity and classical stabilization give a certificate for every such norm after a linear coordinate change. On sign-containing instances, the intrinsic polynomial-multiplier obstruction excludes every norm-power certificate. Thus the same reduction applies when the input convention fixes the standard coordinate norm.

Thus the coefficient-input problem is coNP-hard on the stated nonnegative, one-negative-direction promise class. For the structured signed-triple input, its complement has the Boolean satisfying assignment as a polynomial certificate, giving coNP-completeness. This membership statement concerns the structured language; coefficient-input hardness does not require recognizing the promise class.

### 4.1 A finite demonstration

The seven triples

$$123,\ 145,\ 167,\ 246,\ 257,\ 347,\ 356$$

form the Fano plane and have no two-coloring. Their graph-subspace quartic is therefore strictly positive and finitely stabilizable. Deleting the last triple gives a sign-feasible subspace and absolute multiplier obstruction. The accompanying exact-arithmetic verifier enumerates all colorings and reconstructs both $28 \times 28$ coefficient matrices. Their inertias are both $(27, 1, 0)$; their different stabilization behavior is determined by the coherent sign vectors, rather than by the coefficient inertia alone.

## 5 Uniform degree growth for hard and soft constraints

We first prove the analytic estimates for Theorem 1.4, then apply the same slice calculation to prove Theorem 1.3. Throughout this section $B$ is real. The penalty $N\|Bv\|^2$ is a holomorphic squared norm, so $F_{B,\rho} \ge 0$.

### 5.1 When a sign vector lies in the constraint kernel

If $A(v) = 0$ and $v \neq 0$, the zero-set argument above gives $v = cs$ for a sign vector $s$. At this vector

$$Q(cs) = 0, \qquad N(cs)\|Bcs\|^2 = n|c|^4\|Bs\|^2.$$

Consequently $\sigma(B) > 0$ gives strict positivity, and classical stabilization gives a finite norm-power certificate.

Suppose instead that $Bs = 0$. Choose a nonzero real $w \perp s$. On the complex line $s + zw$, where $z = x + iy$, the quadratic Taylor part is

$$\alpha|z|^2 + 4\rho n\|w\|^2 y^2, \qquad \alpha = 4n\|w\|^2 + n\|Bw\|^2 > 0. \tag{8}$$

The $x^2$ and $y^2$ coefficients differ, and hence the pure $z^2$ coefficient is nonzero. This remains true on an open set of complex directions around $w$. Lemma 2.1 excludes every nonzero polynomial multiplier. This proves the qualitative assertions.

### 5.2 A uniform positivity margin

Let

$$\delta(B) = \min_{v\in\mathbb{R}^n,\ \|v\|=1} (A(v) + \|Bv\|^2).$$

Assume $\|B\|_{\mathrm{op}} \leq K$. For a real unit vector $v$, choose $s_i = \mathrm{sign}(v_i)$, with either choice at zero, and put $u = s/\sqrt{n}$. Then

$$\begin{aligned} A(v) &= n\sum_i (v_i^2 - 1/n)^2 \\ &\geq \sum_i (|v_i| - 1/\sqrt{n})^2 = \|v - u\|^2. \end{aligned}$$

It follows that $\sigma(B) \leq \|Bu\| \leq \|Bv\| + K\sqrt{A(v)}$. Inserting a minimizing sign vector for the converse bound gives

$$\frac{\sigma(B)^2}{2\max\{1, K^2\}} \leq \delta(B) \leq \sigma(B)^2. \tag{9}$$

When $\delta(B) \leq 1$, it is also the exact minimum of $F_{B,\rho}$ on the complex unit sphere. Indeed rotate a vector by a scalar phase so that $v = x + iy$ has $x \cdot y = 0$, and put $a = \|x\|^2$, $b = \|y\|^2$. Homogeneity gives

$$A(x) + a\|Bx\|^2 \geq \delta(B)a^2, \qquad A(y) + b\|By\|^2 \geq \delta(B)b^2.$$

Because $B$ is real, direct expansion yields

$$\begin{aligned} A(v) + N(v)\|Bv\|^2 &\geq \delta(B)(a^2 + b^2) + 2ab + 2n\sum_i x_i^2 y_i^2 \\ &\quad + b\|Bx\|^2 + a\|By\|^2 \\ &\geq \delta(B)(a + b)^2. \end{aligned}$$

Adding $\rho Q \geq 0$ preserves the bound. A real minimizing unit vector attains equality in the resulting sphere minimum.

For a quartic $F = \sum c_{\alpha\beta} z^\alpha \bar{z}^\beta$, write

$$\Lambda(F)^2 = \sum_{|\alpha|=|\beta|=2} \frac{\alpha!\beta!}{4} |c_{\alpha\beta}|^2.$$

The coefficients of $F_{B,\rho}$, and hence $\Lambda(F_{B,\rho})$, are uniformly bounded for the fixed parameters and $\|B\|_{\mathrm{op}} \le K$. Drouot–Zworski [6, Theorem 2] bounds a sufficient multiplier exponent by a universal constant times

$$\frac{\Lambda(F_{B,\rho})}{\delta(B)} (n+2)^3 \log^3 n.$$

Together with (9), this proves the upper bound in (6) for small positive $\sigma(B)$.

### 5.3 The matching logarithmic-curvature obstruction

The upper bound uses the minimum value of the quartic. For a matching lower bound, we find a complex line along which the logarithm has large negative curvature. A norm-power certificate must compensate for that curvature, forcing a large multiplier degree. Take a real minimizing unit vector $v$, a real unit $w \perp v$, and abbreviate $\delta = \delta(B)$. Define

$$S_{22} = \sum_i v_i^2 w_i^2, \quad S_{13} = \sum_i v_i w_i^3, \quad S_{04} = \sum_i w_i^4,$$

$$b_0 = \|Bv\|^2, \quad b_1 = \langle Bv, Bw \rangle, \quad b_2 = \|Bw\|^2.$$

Stationarity of the real sphere minimum kills the linear term of the restriction to $v + zw$. Direct expansion gives the exact polynomial

$$\begin{aligned} q(z) := F_{B,\rho}(v + zw) &= \delta + \alpha x^2 + \beta y^2 + \gamma x(x^2 + y^2) + \eta (x^2 + y^2)^2, \\ \alpha &= 6nS_{22} - 2 + b_0 + b_2, \\ \beta &= 2nS_{22} + 2 + b_0 + b_2 + 4\rho, \\ \gamma &= 4nS_{13} + 2b_1, \qquad \eta = nS_{04} - 1 + b_2. \end{aligned} \tag{10}$$

The small constant term controls every coefficient required below:

$$|nS_{22} - 1| \le \sqrt{n\delta}, \quad b_0 \le \delta, \quad b_2 \le K^2, \quad |b_1| \le K\sqrt{\delta}.$$

Thus, uniformly as $\delta \downarrow 0$,

$$\alpha = 4 + b_2 + O_n(\sqrt{\delta}), \qquad \beta = 4 + b_2 + 4\rho + O_n(\sqrt{\delta}),$$

while $\gamma$ and $\eta$ remain bounded. In particular, $\alpha \ge 2$ and $\beta - \alpha \ge 2\rho$ once $\delta$ is sufficiently small, with a uniform onset.

Set

$$t^2 = \frac{2(\alpha + \beta)}{\beta(\beta - \alpha)} \quad \text{and evaluate at} \quad x = 0,\ y = t\sqrt{\delta}.$$

The numbers $t$ stay bounded above and away from zero. Writing $\Delta_{\mathbb{R}^2} = \partial_x^2 + \partial_y^2$, differentiation of (10) gives, with uniform remainders,

$$\begin{aligned} q\Delta_{\mathbb{R}^2} q - |\nabla_{\mathbb{R}^2} q|^2 &= \delta\{2(\alpha + \beta) + 2\beta(\alpha - \beta)t^2\} + O(\delta^2) \\ &= -2\delta(\alpha + \beta) + O(\delta^2), \\ q &= \delta\{1 + \beta t^2 + O(\delta)\}. \end{aligned}$$

Consequently some uniform $c_0 > 0$ satisfies

$$\partial_z \partial_{\bar{z}} \log q \le -\frac{c_0}{\delta}. \tag{11}$$

The logarithm of a positive holomorphic squared norm is plurisubharmonic. Along this line $N(v + zw) = 1 + |z|^2$, and $\partial_z \partial_{\bar{z}} \log N = (1 + |z|^2)^{-2} \le 1$. A certificate for $N^m F_{B,\rho}$ would therefore imply, at the specified point,

$$0 \le m\partial_z \partial_{\bar{z}} \log N + \partial_z \partial_{\bar{z}} \log q \le m - \frac{c_0}{\delta}.$$

Hence $m \ge c_0/\delta \ge c_0/\sigma(B)^2$, by (9). This proves the lower bound and completes Theorem 1.4. When $r = 0$ or $B = 0$, its positive-residual regime is empty and the qualitative sign-null alternative applies.

### 5.4 Distance to the nonstabilizing subspaces

*Proof of Theorem 1.3.* Write $u = s/\sqrt{n}$ for a normalized sign vector. If $u \in W$, then

$$\|(I - P_V)u\| = \|(P_W - P_V)u\| \le \|P_W - P_V\|_{\mathrm{op}}.$$

Taking the minimum over signs and the infimum over $W \in \mathcal{S}_{n,d}$ gives one direction of (2). For the other, choose a minimizing sign $u$, put $p = P_V u$, and set

$$W = (V \cap p^{\perp}) \oplus \mathrm{span}_{\mathbb{R}}\{u\}.$$

Here $p \neq 0$: averaging $\|P_V s/\sqrt{n}\|^2$ over all signs gives $d/n > 0$, so a minimizing-distance sign has nonzero projection. For $w \in V \cap p^{\perp}$, $\langle w, u\rangle = \langle w, p\rangle = 0$, so $W$ is a real $d$-plane containing $u$. The two planes have this common $(d-1)$-plane, while the angle between their remaining directions $p/\|p\|$ and $u$ has sine $\eta(V)$. Hence $\|P_V - P_W\|_{\mathrm{op}} = \eta(V)$, proving the equality.

Let

$$\delta_V = \min_{v \in V,\ \|v\|=1} A(v),$$

where the minimum is over real vectors. For such $v$, the normalized coordinate sign $u$ gives $A(v) \ge \|v - u\|^2 \ge \eta(V)^2$. Conversely, for a minimizing sign $u$, normalize its projection: $v = P_V u/\|P_V u\|$. Since $\|v - u\|^2 = 2(1 - \sqrt{1 - \eta(V)^2}) \le 2\eta(V)^2$ and $|v_i^2 - u_i^2| \le 2|v_i - u_i|$,

$$\eta(V)^2 \le \delta_V \le 8n\eta(V)^2. \tag{12}$$

The preceding phase-rotation argument applies inside $V_{\mathbb{C}}$: the orthogonal real and imaginary parts remain in $V$. When $\delta_V \le 1$, it proves that the complex-sphere minimum of $q_\rho$ is exactly $\delta_V$. In intrinsic real orthonormal coordinates the quartic coefficient norm is bounded uniformly over $V$. The cited quantitative stabilization theorem in dimension $d$ therefore gives $\mu_V(\rho) \le C/\delta_V \le C/\eta(V)^2$.

For the lower bound choose a real minimizing unit $v \in V$ and a real unit $w \in V \cap v^{\perp}$, possible since $d \ge 2$. Stationarity within $V$ removes the linear term on $v + zw$. Formula (10) now holds with $B = 0$ and $\delta = \delta_V$. Its coefficients satisfy

$$\alpha = 4 + O_n(\sqrt{\delta_V}), \qquad \beta = 4 + 4\rho + O_n(\sqrt{\delta_V}),$$

and the cubic and quartic coefficients are uniformly bounded. The calculation leading to (11) consequently gives $\partial_z \partial_{\bar{z}} \log q_\rho(v + zw) \le -c_0/\delta_V$ at the specified point, uniformly over $V$. Since $N(v + zw) = 1 + |z|^2$, logarithmic plurisubharmonicity again forces

$$\mu_V(\rho) \ge c_0/\delta_V \ge \frac{c_0}{8n\eta(V)^2}.$$

This proves (3). □

### 5.5 Explicit reciprocal-margin upper bounds

For a bihomogeneous quartic $F$ on $\mathbb{C}^D$, let $H_F$ be its Hermitian operator on $\mathrm{Sym}^2\,\mathbb{C}^D$, normalized by $F(z) = \langle z^{\otimes 2}, H_F z^{\otimes 2}\rangle$. Write $\lambda(F) = \min_{\|z\|=1} F(z)$. The Hermitian HSOS hierarchy of Lovitz–Johnston [12, Theorem 3.3] gives the following explicit form of the upper bound.

**Proposition 5.1** (An explicit finite degree bound)**.** *If $\lambda(F) > 0$, then*

$$\mu(F) \le \max\left\{0, \left\lceil \frac{8(D-1)\|H_F\|_{\mathrm{op}}}{\lambda(F)} \right\rceil - 3\right\}.$$

*Here the exponent is defined by the multiplier $\|z\|^{2m}$. For the restricted family and soft-penalty family, respectively, put*

$$L_{n,d,\rho} = \max\{n+\rho, \rho(d-1)\}, \qquad L_{n,\rho,K} = \max\{n+\rho+K^2, \rho(n-1)\}.$$

*Then, for $\eta(V) > 0$ and $\sigma(B) > 0$, $\|B\| \le K$,*

$$\mu_V(\rho) \le \max\left\{0, \left\lceil \frac{8(d-1)L_{n,d,\rho}}{\eta(V)^2} \right\rceil - 3\right\},$$
$$\mu_B(\rho) \le \max\left\{0, \left\lceil \frac{16(n-1)\max\{1,K^2\}L_{n,\rho,K}}{\sigma(B)^2} \right\rceil - 3\right\}.$$

*Proof.* For $k = m+2$, the normalized coefficient operator of $\|z\|^{2(k-2)}F$ is $H_{F,k} = \Pi_k(H_F \otimes I^{\otimes(k-2)})\Pi_k$ on $\mathrm{Sym}^k\,\mathbb{C}^D$. The cited Hermitian hierarchy theorem, including its proof by the quantum de Finetti estimate, gives

$$\lambda(F) - \lambda_{\min}(H_{F,k}) \le \frac{8(D-1)\|H_F\|_{\mathrm{op}}}{k+1}.$$

Thus $m+3$ at least the displayed ratio makes the complete coefficient operator positive semidefinite. The smallest nonnegative integer meeting this sufficient condition gives the first bound. In the Fischer normalization used in Section 3, $H_F = M_F/2$: the coherent monomials are $\sqrt{2!/\alpha!}\,z^\alpha$, whereas the Fischer orthonormal monomials are $z^\alpha/\sqrt{\alpha!}$.

For $A$ in ambient coordinates, put $\omega = \sum_i e_i \otimes e_i$ and let $P$ project onto their span. Then $H_A = nP - |\omega\rangle\langle\omega|$, so $0 \preceq H_A \preceq nI$. Its restriction to $\mathrm{Sym}^2 V_{\mathbb{C}}$ is an isometric compression. Since $V$ is real, $H_Q$ there has eigenvalues 1 and $1-d$. Consequently $\|H_{q_\rho}\| \le L_{n,d,\rho}$. The orthogonal phase decomposition used above yields $\lambda(q_\rho|V_{\mathbb{C}}) \ge \min\{\delta_V, 1\} \ge \eta(V)^2$. For the soft family, the penalty operator is $\Pi_2(I \otimes B^*B)\Pi_2$, between 0 and $K^2 I$. Hence $\|H_{F_{B,\rho}}\| \le L_{n,\rho,K}$. The same phase decomposition and (9) give

$$\lambda(F_{B,\rho}) \ge \min\{\delta(B), 1\} \ge \frac{\sigma(B)^2}{2\max\{1,K^2\}}.$$

For the final inequality use $\sigma(B) \le K$; its right side is at most $1/2$. Substitution proves the bounds. □

### 5.6 Random constraints and degree moments

The distance laws also describe the distribution of certificate degree. The random variables here are the exponent of the stated norm-power hierarchy.

**Theorem 5.2** (Degree tails and exact moment thresholds)**.** *Fix $n \geq 3$, $2 \leq d < n$ and $\rho > 0$, and let $V$ be Haar-uniform on $\mathrm{Gr}_d(\mathbb{R}^n)$. Then $\mu_V(\rho) < \infty$ almost surely and, as $M \to \infty$,*

$$\Pr\{\mu_V(\rho) > M\} = \Theta_{n,d,\rho}(M^{-(n-d)/2}).$$

*For every $p > 0$,*

$$\mathbb{E}[\mu_V(\rho)^p] < \infty \quad \Longleftrightarrow \quad p < \frac{n-d}{2}.$$

*The same tail exponent and moment threshold hold for any subspace law whose density relative to Haar is bounded above and below by positive constants.*

*For soft constraints fix $n \geq 2$, $r \geq 1$ and $\rho > 0$, and let the entries of $B \in \mathbb{R}^{r\times n}$ be independent uniform random variables on $[-1, 1]$. Then $\mu_B(\rho) < \infty$ almost surely and*

$$\Pr\{\mu_B(\rho) > M\} = \Theta_{n,r,\rho}(M^{-r/2}), \qquad \mathbb{E}[\mu_B(\rho)^p] < \infty \Longleftrightarrow\ p < r/2.$$

*These conclusions are likewise preserved by densities bounded above and below relative to the uniform matrix-box law.*

*Proof.* For each fixed normalized sign vector $u = s/\sqrt{n}$, rotational invariance and a Gaussian representation of a uniform sphere point give

$$X_u = \|(I - P_V)u\|^2 \sim\ \mathrm{Beta}((n-d)/2, d/2).$$

Indeed $X_u$ is the ratio of a sum of $n - d$ independent standard-normal squares to the sum of all $n$ squares. Let $F$ be this beta distribution function. A fixed sign event and a union bound over the $2^{n-1}$ sign lines give, for $0 < \varepsilon < 1$,

$$F(\varepsilon^2) \leq \Pr\{\eta(V) \leq \varepsilon\} \leq \min\{1, 2^{n-1}F(\varepsilon^2)\}.$$

The beta density is a positive constant times $x^{(n-d)/2-1}(1-x)^{d/2-1}$, so this probability is $\Theta(\varepsilon^{n-d})$. Independence between sign events is not required. In particular $\eta(V) > 0$ almost surely. The uniform lower law in Theorem 1.3 and the global upper in Proposition 5.1 imply the asserted tail by substituting $\varepsilon$ equal to fixed positive multiples of $M^{-1/2}$.

For the soft model, $\|B\| \leq \|B\|_F \leq \sqrt{rn}$ deterministically. For each fixed sign vector, the coordinates of $Bs/\sqrt{n}$ are independent copies of $n^{-1/2}\sum_{j=1}^n U_j$, with independent $U_j$ uniform on $[-1, 1]$. The scalar density is continuous, bounded, and positive at zero. Integration on a small $r$-ball gives

$$\Pr\{\|Bs\|/\sqrt{n} \leq \varepsilon\} = \Theta(\varepsilon^r).$$

A fixed-event lower bound and a union bound over all signs therefore give $\Pr\{\sigma(B) \leq \varepsilon\} = \Theta(\varepsilon^r)$. Apply Theorem 1.4 and the global explicit upper as before.

These degree variables are Borel measurable. For each fixed $m$, the coefficient operator of the multiplied quartic is continuous in local orthonormal frames on the Grassmannian, and polynomial-continuous in $B$ for the soft model. Positive semidefiniteness is a closed condition. Since multiplying a squared norm by $N$ preserves that property, $\{\mu \leq m\}$ is precisely this closed condition. For $p > 0$, Tonelli's theorem gives

$$\mathbb{E}[\mu^p] = p\int_0^\infty M^{p-1}\Pr\{\mu > M\}\,dM.$$

The stated power tails prove both moment equivalences, including logarithmic divergence at equality. Bounded positive density changes preserve the probability estimates by comparison. □

Thus eventual success is almost sure, while the mean degree diverges for codimension one or two in the hard model and for one or two rows in the bounded soft model. These continuous probability models describe hierarchy degree rather than the running time of an algorithm or a distribution of finite-bit inputs.

The explicit soft-penalty calculation in Appendix A illustrates the uniform lower law with computable necessary degrees.

# 6 A sharp fixed-dimensional degree law

We now specialize to a two-dimensional family to determine the leading constant. Weighted row estimates give an upper bound for the complete coefficient matrix; finite trial vectors supply the matching lower bound. Fix $\rho > 0$, $b > 0$, and put $t = 4/b^2$, $c = 2+t$. Real orthonormal coordinates on $H_b$ are

$$v_1 = \frac{Y}{\sqrt{c}} + \frac{X}{\sqrt{2}}, \quad v_2 = \frac{Y}{\sqrt{c}} - \frac{X}{\sqrt{2}}, \quad v_3 = -\frac{2Y}{b\sqrt{c}}.$$

Then $N = |X|^2 + |Y|^2$, and

$$q_b = \alpha|X|^4 + \beta|Y|^4 + \chi|XY|^2 + \eta(X^2\bar{Y}^2 + \bar{X}^2Y^2), \tag{13}$$

where

$$\alpha = \frac{1}{2}, \quad \beta = \frac{2(t-1)^2}{(2+t)^2}, \quad \chi = \frac{12}{2+t} + 2\rho, \quad \eta = \frac{3}{2+t} - 1 - \rho.$$

These coefficients follow by expanding $3\sum|v_i|^4 - |X^2+Y^2|^2 + \rho Q$. They are rational functions of $b^2$.

## 6.1 The coefficient matrices

For $\ell = m+2$, use the ordinary monomials $X^jY^{\ell-j}$. The coefficient matrix of $N^m q_b$ has diagonal

$$D_j = \alpha\binom{m}{j-2} + \chi\binom{m}{j-1} + \beta\binom{m}{j}$$

and entry $\eta\binom{m}{j}$ between $j$ and $j+2$; out-of-range binomial coefficients are zero. In particular, its principal minor at indices $0, 2$ gives the necessary condition

$$\beta[\alpha + \chi m + \beta m(m-1)/2] \geq \eta^2. \tag{14}$$

The proof below uses the complete matrices for matching bounds. At $b = 2$, Theorem 1.1 gives infinite degree.

## 6.2 Matching bounds and the exact constant

Write $e_b = |\eta|$, $\chi_b = \chi$, and $\beta_b = \beta$. Near $b = 2$, $e_b > 0$, $\chi_b > 2e_b$, and

$$r_b = \frac{\chi_b - \sqrt{\chi_b^2 - 4e_b^2}}{2e_b} \in (0,1), \qquad \kappa_b = e_b r_b.$$

Thus $\chi_b = e_b(r_b + r_b^{-1})$, and

$$\kappa_b \longrightarrow \kappa_0 = \frac{4 + 2\rho - \sqrt{(4+2\rho)^2 - 4\rho^2}}{2} = (\sqrt{1+\rho} - 1)^2. \tag{15}$$

We prove $\beta_b\mu_b(\rho) \to \kappa_0$.

Let $H_{b,\ell}$ be the coefficient matrix after congruence by $\operatorname{diag}(\binom{\ell}{j}^{-1/2})$ and multiplication by $\ell(\ell-1)$. With $v = \ell - j$, its complete entries are

$$(H_{b,\ell})_{jj} = \alpha j(j-1) + \chi_b jv + \beta_b v(v-1),$$
$$(H_{b,\ell})_{j,j+2} = \eta\sqrt{(j+1)(j+2)v(v-1)}.$$

It is positive semidefinite exactly when the exponent $m = \ell - 2$ is a certificate.

For the upper bound choose positive weights $d_0 = d_1 = 1$ and

$$\frac{d_{j+2}}{d_j} = r_b\sqrt{\frac{j+1}{j+2}}.$$

The two weighted off-diagonal row sums are bounded by

$$e_b r_b(j+1)v + \frac{e_b}{r_b}j(v+2) = \chi_b jv + \kappa_b v + \frac{2e_b}{r_b}j.$$

Indeed the exact square-root factors are $\sqrt{v(v-1)} \le v$ and $\sqrt{(v+1)(v+2)} \le v+2$. Missing endpoint entries only reduce the sum. The diagonal minus this weighted row sum is therefore at least

$$\begin{aligned}&\alpha j(j-1) + \beta_b v(v-1) - \kappa_b v - 2e_b j/r_b \\ &\quad = \ell[\beta_b(\ell-1) - \kappa_b] + j[-\beta_b(2\ell-1) + \kappa_b - \alpha - 2e_b/r_b] + (\alpha+\beta_b)j^2. \end{aligned} \tag{16}$$

Fix $\delta > 0$ and take $\ell_b = \lceil(\kappa_b + \delta)/\beta_b\rceil$. The linear coefficient in (16) is bounded below by $-C_\delta$ uniformly near $b = 2$. Every row slack is at least

$$\ell_b(\delta - \beta_b) - \frac{C_\delta^2}{4\alpha} > 0$$

eventually. Congruence by $\operatorname{diag}(d_j)$ now gives a diagonally dominant positive semidefinite matrix. Hence $\mu_b \le \ell_b - 2$, and $\limsup \beta_b\mu_b \le \kappa_0 + \delta$. Letting $\delta \downarrow 0$ proves the upper limit.

For the lower limit first note that $\mu_b \to \infty$. Otherwise a subsequence has one fixed exponent, and its block (14) has determinant tending to $-\rho^2$, a contradiction. Consider a subsequence with $\beta_b\mu_b \to u < \infty$, and put $\ell_b = \mu_b + 2$. For every fixed finite set of indices, the entries of $H_{b,\ell_b}/\ell_b$ tend to those of $uI + J$, where

$$J_{jj} = (4+2\rho)j, \qquad J_{j,j+2} = -\rho\sqrt{(j+1)(j+2)}.$$

This is only a finite-entry limit; no infinite-operator domain assertion is needed.

Set $r = \lim r_b$, and define

$$a_{2k} = r^k\frac{\sqrt{(2k)!}}{2^k k!}, \qquad a_{2k+1} = 0.$$

The squares satisfy $a_{2k}^2 \le r^{2k}$, so all polynomially weighted square sums converge. The recurrence gives, entry by entry,

$$(Ja)_j = [(4+2\rho)j - \rho r(j+1) - \rho j/r]a_j = -\kappa_0 a_j$$

on even indices, including $j = 0$; odd entries vanish. All associated quadratic-form series converge absolutely. Thus the finite truncations $a^{[K]} = (a_0, \dots, a_{2K})$ obey

$$\frac{\langle a^{[K]}, Ja^{[K]} \rangle}{\|a^{[K]}\|^2} \longrightarrow -\kappa_0.$$

If $u < \kappa_0$, choose one finite truncation with Rayleigh quotient below $-u$. The finite-entry limit then gives a negative quadratic form for $H_{b,\ell_b}/\ell_b$, contradicting its positive semidefiniteness. Therefore $u \geq \kappa_0$, proving the lower limit. Finally $\beta_b/(b-2)^2 \to 2/9$ and (15) prove Theorem 1.5.

**Corollary 6.1** (Binary input size)**.** *At fixed $\rho = 1$ and $b_L = 2 + 2^{-L}$, the binary quartic* (13) *has bounded rational coefficients of bit length $O(L)$, and its least norm multiplier degree is $\Theta(4^L)$.*

*Proof.* The rational functions in (13) have fixed degree and stay bounded near $b = 2$. The degree law gives the conclusion. □

The sharp binary law also has a direct geometric normalization. For the real plane underlying $H_b$, the sign-distance formula gives, for $b$ sufficiently close to 2,

$$\eta(H_b)^2 = \frac{(b-2)^2}{3(2+b^2)}, \qquad \lim_{b \to 2,\ b \neq 2} \eta(H_b)^2 \mu_b(\rho) = \frac{1}{4}(\sqrt{1+\rho} - 1)^2.$$

Indeed its normal is $(1, 1, b)$, and the smallest absolute signed normal sum near $b = 2$ is $|b - 2|$. Substitution into Theorem 1.5 proves the displayed limit. This expresses its exact constant in the same distance used in Theorem 1.3.

### 6.3 Joint approach to the zero-perturbation boundary

The fixed-$\rho$ law does not describe limits in which the anisotropic perturbation also vanishes. Set

$$\delta = \frac{1-t}{2+t}, \qquad t = \frac{4}{b^2}.$$

The exact binary coefficients become

$$\alpha = \tfrac{1}{2}, \qquad \beta = 2\delta^2, \qquad \chi = 4(1+\delta) + 2\rho, \qquad \eta = \delta - \rho, \qquad e = |\eta|.$$

The following bounds use the complete finite coefficient matrices.

**Proposition 6.2** (Finite crossover bounds)**.** *For $b \neq 2$ and $\rho > 0$, degree zero holds exactly when*

$$b > 2, \qquad 0 < \rho \leq 2\delta.$$

*If $\chi \geq 1/2$ and $0 < e < 1/16$, then*

$$\mu_b(\rho) \leq \left\lceil \frac{e^2}{\beta\chi(1-16e)} \right\rceil.$$

*If $e = 0$, then $\mu_b(\rho) = 0$.*

*Proof.* At degree zero the middle diagonal is $\chi > 0$ and the endpoint $2 \times 2$ block has determinant $\alpha\beta - \eta^2 = \delta^2 - (\delta - \rho)^2$. For positive $\rho$ this is nonnegative exactly as stated.

For the upper bound put $\ell = m+2$ and remove index zero from the complete normalized matrix $H_{b,\ell}$, leaving $R$ on $1, \dots, \ell$. With $v = \ell - j$, its diagonal satisfies

$$D_j = \tfrac{1}{2}j(j-1) + \chi jv + \beta v(v-1) \geq \tfrac{1}{2}j(\ell-1) \geq \tfrac{1}{4}j\ell.$$

The absolute off-diagonal row sum is at most

$$e[(j+2)v + j(v+2)] \leq 4ej\ell \leq 16eD_j.$$

Omitting the edge to index zero only reduces this sum. Hence $R \succeq (1-16e)\operatorname{diag}(D_j) > 0$. The sole coupling from index zero is to index two, with squared modulus $2e^2\ell(\ell-1)$, while the top diagonal is $\beta\ell(\ell-1)$. For $\ell \geq 3$,

$$(R^{-1})_{22} \leq \frac{1}{(1-16e)D_2} \leq \frac{1}{2\chi(1-16e)(\ell-2)}.$$

Thus the Schur complement is nonnegative whenever $\beta\chi(1-16e)(\ell-2) \geq e^2$. The displayed ceiling chooses such an $m \geq 1$. If $e = 0$, the original matrix is already diagonal and positive semidefinite. □

**Corollary 6.3** (Diverging joint regime and finite degree plateaus)**.** *If $\rho \to 0$, $b \to 2$, $b \neq 2$, and $\rho/|b-2| \to \infty$, then*

$$\mu_b(\rho) \sim \frac{9}{8}\frac{\rho^2}{(b-2)^2}.$$

*Alternatively, let $\delta \to 0$, $\delta \neq 0$, with $\rho > 0$ and $\rho/\delta \to c \in \mathbb{R}$. If $c(c-2)/8 \notin \mathbb{N} \cup \{0\}$, then eventually*

$$\mu_b(\rho) = \max\{0, \lceil c(c-2)/8 \rceil\}.$$

*Proof.* More generally suppose $e \to 0$, $\Lambda = e^2/(\beta\chi) \to \infty$, with $\chi$ bounded above and below by positive constants. Proposition 6.2 gives $\limsup \mu_b/\Lambda \leq 1$. The necessary minor (14) implies

$$1 \leq \frac{\alpha}{\chi\Lambda} + \frac{\mu_b}{\Lambda} + \frac{e^2}{2\chi^2}\left(\frac{\mu_b}{\Lambda}\right)^2.$$

The upper bounds the ratio; the first and last terms tend to zero. Therefore $\mu_b/\Lambda \to 1$. Under the first claimed regime, $\delta \sim (b-2)/3$, $e \sim \rho$ and $\chi \to 4$, proving its constant.

For the second regime fix $m$ and $\ell = m+2$. The block $R$ in the preceding proof converges to the positive diagonal with entries $D_j^0 = \frac{1}{2}j(j-1) + 4j(\ell-j)$. In particular $(R^{-1})_{22} \to (1+8m)^{-1}$. After division by $\beta\ell(\ell-1)$, the top Schur complement tends to

$$1 - \frac{(1-c)^2}{1+8m}.$$

It is eventually positive if $(1-c)^2 < 1+8m$, and eventually negative if the inequality is reversed. The first strict admissible integer is exactly the displayed plateau value; the excluded parameters are precisely the zero-limit Schur complements. The strict inequalities settle all smaller integers as well. □

The exact degree-zero phase and the finite plateaus separate a vanishing anisotropic perturbation from the fixed-positive-perturbation boundary. At the excluded plateau endpoints, higher-order terms determine the degree; the diverging law above is proved uniformly in both parameters.

# 7 Reproducibility

All results above are analytic. The included standard-library Python programs provide exact finite controls for the Boolean gadget, full integer coefficient matrices, inertia, soft-penalty curvature identities, exact subspace-projector geometry and the binary stabilization matrices, including the new joint-limit finite regimes. For example, at $\rho = 1, b = 9/4$, the exact least exponent is 13; complete exact-matrix checks certify that exponent and reject 12. These values illustrate the certificate construction; the limiting exponent is proved by the two analytic bounds.

# A An explicit soft-penalty example

Let $n = 3$, $\rho = 1$, and $B_\epsilon = (\epsilon, 0, 0)$, where $0 < \epsilon \le 1$. Then $\sigma(B_\epsilon) = \epsilon/\sqrt{3}$. On $v = (1,1,1) + z(0,1,-1)$, with $e = \epsilon^2$,

$$N = 3 + 2|z|^2,$$
$$F_{B_\epsilon,1} = 3e + (24 + 2e)x^2 + (48 + 2e)y^2 + 2(x^2 + y^2)^2.$$

At $z = i\epsilon$,

$$\partial_z \partial_{\bar{z}} \log F_{B_\epsilon,1} = \frac{-468 + 78e + 4e^2}{e(51 + 4e)^2}, \qquad \partial_z \partial_{\bar{z}} \log N = \frac{6}{(3 + 2e)^2}.$$

The resulting exact necessary bound is

$$\mu_{B_\epsilon}(1) \ge \frac{(468 - 78e - 4e^2)(3 + 2e)^2}{6e(51 + 4e)^2} \sim \frac{78}{289\epsilon^2}. \tag{17}$$

For $\epsilon = 1/10$ and $1/100$, it gives the integer lower bounds 28 and 2700, respectively. This example displays a computable necessary degree inside the uniform law; Section 6 determines a leading constant by complete coefficient matrices for a weighted restriction.

## Data availability

The analytic results are proved in the article. The editable source and verification code accompany this manuscript. The earlier public version is at the Luminary Research Archive.

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