Growing-Index Recovery and Tail Scales for Signed Measures

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Abstract

We study uniform recovery of negative directions by polynomial squares when the requested negative index and the tail resource vary. For a shrinking negative probability measure κ, write h_m for its degree-m monic squared norm and $L = \log(R/h_m)$. Under a gap from its containing interval, we prove that the minimax degree for recovering $m+1$ negative directions is asymptotic to $L^2/(8cd^2)$ in a Gaussian background class. In a separately specified full exponential-moment class, its logarithm is asymptotic to $\pi L/(2cd)$. No additional restriction on the growth of the index is needed: the negative measure may be atomic, continuous, singular or mixed, and its support concentration pays the dimension costs. Adaptive rational damping and a subleading Fourier extension retain the sharp constants. Without a support gap, a local mass bound $A_0 R r^q$ and tails $\exp(c|x|[\log(e+|x|)]^\beta)$ give $\log D = \Theta(R^{1/[q(\beta+1)]})$ for every fixed $\beta \geq 0$ and $q > 0$. For a uniformly superlinear tail scale H, we obtain $D = \Theta(F^{-1}(R^{1/q}))$, where $F(t) = t/H^{-1}(t)$, including every pure power greater than one. The proofs distinguish these signed-measure recovery conclusions from their classical orthogonal-polynomial and localization ingredients.

1 Introduction

A negative part of a signed measure need not be visible to polynomial squares of modest degree. The number of directions recovered depends on the geometry of that negative part, the permitted background, and the cost of controlling a polynomial on the full real line. Fixed-index results do not automatically remain valid when the requested index grows: constants hidden in a polynomial multiplier or a series tail can then affect the leading degree scale.

We address two forms of this uniformity problem. First, an arbitrary negative probability collapses into a small interval while the requested index increases. Its intrinsic monic squared norm, rather than an atom-count or minimum-separation proxy, remains the controlling statistic throughout the concentration regime, with no additional index-growth hypothesis. Second, when the background reaches the negative atom, local mass control and the tail scale together determine the worst recovery degree. We give matching laws for logarithmically strengthened exponential tails and for a class of uniformly superlinear scales. All statements concern the original signed square form, including mixed terms.

The predecessors [1, 2] study sharp degree and collision barriers. The fixed-index geometric and local-concentration treatment [3] provides the immediate foundation. The present packet results replace a fixed requested index by a jointly varying one without an additional growth restriction. The main receiving step damps the growing polynomial family rationally, pays its Fourier variation using the automatic bound $m = o(L)$, and trims only $o(L)$ extra bandwidth before polynomial approximation. The tail results complete the pure-power range and treat nearcritical classes that

do not have a uniform superlinear power lower bound. These changes are statements about uniform recovery over specified classes; the definitions of Christoffel functions and the special-function identities are classical.

Classical orthogonal-polynomial extremality and kernel theory underlie the lower bounds [4, 6]. The Laplace Fourier law, convolution rules, Hermite and Meixner–Pollaczek generating functions used in the packet receiver are classical; the coupled growing-family construction is the step that pays uniformity. The compact multipliers used for the tail laws belong to established fast-decreasing-polynomial theory [7, 8]. Levin–Lubinsky already establish the corresponding nonsingular nearcritical origin-kernel scales [9]. Our local-mass conclusions retain arbitrary signed or singular backgrounds in the upper class, arbitrary $q > 0$, and an explicit original-measure construction with its complete approximation budget.

We use ordinary polynomial degree throughout: $P \in \mathcal{P}_N$ has degree at most N, while its square has degree at most $2N$. A negative direction is strictly negative; a null square is not counted as negative. The constants in the local-mass order statements may depend on the declared fixed class data. A theta or spectral-form application requires a separate objective-preserving receiving map.

2 Growing indices for shrinking negative probabilities

For a real signed measure with all absolute moments, write

$$\mathcal{Q}_\mu(P) = \int P^2 \, \mathrm{d}\mu, \qquad n_s(\mu) = \inf\{N : \mathrm{ind}_-(\mathcal{Q}_\mu|_{\mathcal{P}_N}) \geq s\}.$$

Here $\mathcal{P}_N$ is the space of real polynomials of ordinary degree at most N, and $\inf \varnothing = \infty$. Negative index counts strictly negative directions; a null square does not count. All estimates below apply to the complete original square form, including mixed terms.

Fix $c, d > 0$. At each parameter value let $\varepsilon > 0$, $m \geq 1$, $R \geq 1$, and let κ be a probability supported in $[-\varepsilon, \varepsilon]$ with at least $m + 1$ distinct support points. Define its monic square norm

$$h_m = h_m(\kappa) = \min_{\substack{A \text{ monic} \\ \deg A = m}} \int A^2 \, \mathrm{d}\kappa > 0, \qquad L = \log(R/h_m), \qquad s = m + 1. \tag{1}$$

We distinguish two classes. For a fixed $C_G > e^{cd^2}$, the Gaussian class consists of

$$\mu = -\kappa + \nu, \quad \operatorname{supp}\nu \subset \{|x| \geq d + \varepsilon\}, \quad \|\nu\|_{\mathrm{TV}} \leq R, \quad \int e^{cx^2} \, \mathrm{d}|\nu| \leq C_G R. \tag{2}$$

For a fixed $C_E > e^{cd}$, the exponential class instead consists of

$$\mu = -\kappa + \nu, \quad \operatorname{supp}\nu \subset \{|x| \geq d + \varepsilon\}, \quad \int e^{c|x|} \, \mathrm{d}|\mu| \leq C_E(1 + R). \tag{3}$$

Remainders may be signed in both classes. There is no additional remainder-variation bound in (3). The supports of κ and ν are disjoint, so $|\mu| = \kappa + |\nu|$. Let $D_s^G(R; \kappa, \varepsilon)$ and $D_s^E(R; \kappa, \varepsilon)$ be the suprema of n_s over the respective classes. The positive lower constructions below ensure that these classes are nonempty for all sufficiently small ε.

Theorem 2.1 (Growing-index packet recovery)**.** *Along every sequence of the declared data with $\varepsilon \to 0$, one has*

$$\frac{D_{m+1}^G(R; \kappa, \varepsilon)}{L^2} \longrightarrow \frac{1}{8cd^2}, \tag{4}$$

$$\frac{\log D_{m+1}^E(R; \kappa, \varepsilon)}{L} \longrightarrow \frac{\pi}{2cd}. \tag{5}$$

The integer m, the contrast R, and the negative probability may vary jointly. There is no density, fixed atom count, lower bound on atomic weights, or comparable-spacing assumption. No additional growth restriction on the index is imposed.

These are minimax statements at the exact negative probability, not asymptotics for every background. The gap is from the entire containing interval, hence from the negative convex hull. A gap merely from each point of a disconnected negative support is not substituted for it. The fixed-index probability-packet and atomic results of [3] have constants depending on the index; allowing the index to grow requires the adaptive Fourier construction below. In particular, $m \log(m+2) = o(L)$ is not required. The orthogonal polynomial, Christoffel and approximation tools themselves are classical [4, 5, 6, 1, 2].

2.1 Negative-side norms and a reference-gap estimate

Let A_j be the monic orthogonal polynomial of degree j for κ, and put $h_j = \int A_j^2 \,\mathrm{d}\kappa$, $0 \le j \le m$. The support hypothesis makes the moment inner product nondegenerate through degree m. Orthogonality makes A_j the monic norm minimizer: subtracting A_j from another monic polynomial leaves a lower-degree polynomial orthogonal to it. Its roots lie in the negative convex hull. Indeed, if it had fewer than j sign-changing roots in that hull, multiplication by the product of those factors would give a lower-degree polynomial whose product with A_j has one sign and nonzero integral, contradicting orthogonality. This standard argument also applies to finite positive support when there are at least $j+1$ support points.

The trial polynomial xA_{j-1} gives

$$h_0 = 1, \qquad h_j \le \varepsilon^2 h_{j-1}. \tag{6}$$

In particular, for $\varepsilon \le 1$, $h_j \ge h_m$ for $j \le m$ and

$$h_m \le \varepsilon^{2m}, \qquad L \ge 2m \log(1/\varepsilon). \tag{7}$$

Thus $L \to \infty$ and $m = o(L)$ follow from total concentration. The adaptive damping below pays its growing-index costs using only this automatic consequence.

We also need a degree-uniform estimate inside a reference gap. Let ρ be an infinite-support positive probability on $[D^2, \infty)$, with all moments. Its orthonormal polynomials are p_j, and

$$K_{\rho,l}(t,t) = \sum_{j=0}^{l} p_j(t)^2 = \sup_{0 \ne F \in \mathcal{P}_l} \frac{|F(t)|^2}{\int F^2 \,\mathrm{d}\rho}.$$

Let τ_* be the even lift $X = \pm\sqrt{v}$, $v \sim \rho$, with fair signs, and write $K_l = K_{\rho,l}(0,0)$.

Lemma 2.2 (Evaluation inside a reference gap)**.** *If $0 \le u < D^2$, then*

$$K_{v\rho,l}(u,u) \le D^{-2} K_{\rho,l}(u,u) \le D^{-2} K_l.$$

Consequently, for $Q \in \mathcal{P}_N$ and $|x| \le b < D$,

$$|Q(x)|^2 \le (1 + b^2/D^2) K_{\lfloor N/2 \rfloor} \int Q^2 \,\mathrm{d}\tau_*. \tag{8}$$

Proof. The first inequality follows from $v \ge D^2$ and the variational formula. Every zero r_i of p_j lies in $[D^2, \infty)$, so $|p_j(u)/p_j(0)| = \prod_i (1 - u/r_i) \le 1$. This proves the second inequality. Write $Q(x) = F(x^2) + xH(x^2)$. The reference squared norm is $\int F^2 \,\mathrm{d}\rho + \int vH^2 \,\mathrm{d}\rho$. Apply the two evaluation estimates at $u = x^2$ and Cauchy–Schwarz on this orthogonal direct sum. Enlarging the degree of H to $\lfloor N/2 \rfloor$ only increases its bound. □

The lower bounds use the following elementary index observation. If the original square form is nonnegative on a codimension-m subspace of $\mathcal{P}_N$, its negative index is at most m: every subspace of dimension $m+1$ intersects that nonnegative subspace nontrivially. For $N < m$, ambient dimension alone gives the same exclusion.

2.2 A common compact Fourier window

The exponential upper needs an actual polynomial approximation to a common nonpolynomial target. For $D, T > 0$ define the classical window

$$g_{D,T}(x) = \frac{\cos(T\sqrt{x^2 - D^2})}{\cosh(TD)}. \tag{9}$$

The numerator denotes its entire even power series. This real function is the Fourier transform of a symmetric positive probability on $[-T, T]$; this normalization was used in [1]. Here is a direct positivity proof, included to avoid inferring it from bandwidth. Put

$$M_x = \begin{pmatrix} ix & D \\ D & -ix \end{pmatrix}, \qquad M_x^2 = (D^2 - x^2)I.$$

Its exponential has trace $2\cosh(T\sqrt{D^2 - x^2})$. Expand in ordered time integrals of the diagonal and off-diagonal parts. Only even numbers $2j$ of off-diagonal factors contribute to the trace; all their weights are positive. Their phases are $e^{ix\int_0^T \epsilon(t)\,\mathrm{d}t}$, with alternating signs $\epsilon(t) \in \{-1, 1\}$, so their Fourier support is in $[-T, T]$. The total mass at level $2j$ is $2(DT)^{2j}/(2j)!$. Division by $2\cosh(TD)$ gives total mass one. The expansion converges in total variation, proving the assertion.

For the moving packet put

$$D = d + \varepsilon, \qquad A = \sqrt{D^2 - \varepsilon^2} = \sqrt{d^2 + 2d\varepsilon} \geq d, \tag{10}$$

and use the renormalized window

$$w_T(x) = \frac{\cos(T\sqrt{x^2 - D^2})}{\cosh(TA)}, \qquad V = \frac{\cosh(TD)}{\cosh(TA)} \geq 1. \tag{11}$$

Its Fourier measure has variation V, not one. Globally $|w_T| \leq V$; on $\operatorname{supp}\kappa$ it is at least one, and on $\operatorname{supp}\nu$ it is at most $\operatorname{sech}(TA)$ in absolute value. The normalization loss V must be retained in the approximation and full absolute norms.

2.3 Adaptive rational damping and a common Fourier span

Put $s = m + 1$ and, eventually along the declared sequence,

$$b = \sqrt{L/m} > 1, \qquad R_m(x) = (1 + x^2/b^2)^{-m}, \qquad \rho = (1 + \varepsilon^2/b^2)^{-m}.$$

By (7), $m = o(L)$ and $b \to \infty$. In particular

$$m \log b = \tfrac{m}{2} \log(L/m) = o(L), \qquad \log(1/\rho) \leq m\varepsilon^2/b^2 = o(L). \tag{12}$$

For $p_j = A_j/\sqrt{h_j}$, $0 \leq j \leq m$, root location and (6) bound its monomial coefficient sum by

$$A_L = (1 + \varepsilon)^m/\sqrt{h_m}.$$

Define $M = A_L(2b)^m$. Thus

$$\log M = -\tfrac{1}{2}\log h_m + o(L). \tag{13}$$

Here is a Fourier representation that pays the entire growing multiplier. The probability density $\lambda_b(t) = (b/2)e^{-b|t|}$ has characteristic function $r_b(x) = (1 + x^2/b^2)^{-1}$. Integration by parts makes $xr_b(x)$ the Fourier transform of $i\lambda_b'(t)\,\mathrm{d}t$, with convention $f(x) = \int e^{itx}\,\mathrm{d}\sigma(t)$. The density is absolutely continuous, so its first derivative has no atom at zero. Exactly,

$$\int e^{b|t|/2}\lambda_b(t)\,\mathrm{d}t = 2, \qquad \int e^{b|t|/2}|\lambda_b'(t)|\,\mathrm{d}t = 2b.$$

For $0 \le j \le m$, write $x^j R_m = (xr_b)^j r_b^{m-j}$. Convolution and $e^{a|u+v|} \le e^{a|u|}e^{a|v|}$ give weighted Fourier variation at most $(2b)^j 2^{m-j} \le (2b)^m$. Consequently every $p_j R_m$ has a Hermitian complex Fourier measure σ_j satisfying

$$\int e^{b|t|/2}\,\mathrm{d}|\sigma_j|(t) \le M. \tag{14}$$

Hermitian symmetry means $\sigma_j(-B) = \overline{\sigma_j(B)}$; its transform is real. In particular $|p_j R_m| \le M$ on the whole line. This is an elementary convolution construction, not an assumption about Fourier positivity of a polynomial multiplier.

Use D, A, w_T, V from (10)–(11). Let $J = R$ in the Gaussian class and $J = C_E(1 + R)$ in the exponential class, so $\|\nu\|_{\mathrm{TV}} \le J$. Choose

$$H = sM^2, \qquad T = \max\left\{1, \left\lceil A^{-1}\operatorname{arcosh}\max\{1, 2\sqrt{JH}/\rho\}\right\rceil\right\}. \tag{15}$$

For $f_j = p_j R_m w_T$ and EVERY coefficient vector $u \in \mathbb{R}^s$,

$$\mathcal{Q}_\mu\left(\sum_j u_j f_j\right) \le -\tfrac{3}{4}\rho^2\|u\|_2^2, \tag{16}$$

$$\left\|\sum_j u_j f_j\right\|_{L^2(|\mu|)} \le C_0\|u\|_2, \qquad C_0 = \sqrt{V^2 + \rho^2/4} \le 2V. \tag{17}$$

Indeed the background contribution is at most $JsM^2\operatorname{sech}^2(TA)\|u\|^2 \le \rho^2\|u\|^2/4$. On the negative support, $R_m \ge \rho$, $w_T \ge 1$, and the p_j are orthonormal; their full negative squared norm is at least $\rho^2\|u\|^2$ and at most $V^2\|u\|^2$. All mixed combinations are included.

Equations (12)–(13) give

$$\log(JH) = L + o(L), \quad AT = \tfrac{1}{2}L + o(L), \quad T = \frac{L}{2d} + o(L), \quad \log V = o(L).$$

For the last assertion, $\log V \le \log 2 + T(D - A)$ and $D - A = \varepsilon^2/(D + A) \to 0$.

2.4 A subleading Fourier extension

Let J_{abs} be a bound for the relevant absolute moment of the whole measure. For the Gaussian receiver at fixed $0 < a < c$ use

$$J_{\mathrm{abs}} = e^{a\varepsilon^2} + C_G R, \qquad \int e^{ax^2}\,\mathrm{d}|\mu| \le J_{\mathrm{abs}};$$

for the exponential receiver use $J_{\mathrm{abs}} = C_E(1+R)$ and its declared rate-c moment. In either case $\log J_{\mathrm{abs}} = \log R + O(1)$ and hence

$$\log(VM\sqrt{J_{\mathrm{abs}}}) = \tfrac{1}{2}L + o(L). \tag{18}$$

Set

$$\eta = \frac{\rho^2}{32C_0\sqrt{s}}, \qquad U = 3L/b = o(L), \qquad B = T + U = \frac{L}{2d} + o(L). \tag{19}$$

The Fourier measure τ_j of f_j is the convolution of σ_j with the Fourier measure of w_T, which has variation V and support $[-T,T]$. Therefore

$$|\tau_j|(\{|t| > T+U\}) \le VMe^{-bU/2}.$$

There is no factor $e^{bT/2}$ in this estimate: after convolution with a measure supported in $[-T,T]$, only the tail beyond U of the other measure is used. Restrict τ_j to $[-B,B]$ and let $\widetilde{f}_j$ be its Fourier transform. Hermitian symmetry is preserved and its variation is at most VM. Since $\mathrm{TV}(\mu) \le J_{\mathrm{abs}}$,

$$\|f_j - \widetilde{f}_j\|_{L^2(|\mu|)} \le \sqrt{J_{\mathrm{abs}}}VMe^{-3L/2} = e^{-L+o(L)} < \eta/3 \tag{20}$$

eventually. Here $\log(1/\eta) = o(L)$. We now approximate the entire bandlimited target; no growing polynomial multiplier is attached to the resulting approximating polynomial.

2.5 Gaussian return through Hermite polynomials

Fix $0 < a < c$ and $\theta > 0$. Write $h_n(u) = H_n(u)/\sqrt{2^n n!}$ for the normalized physicists' Hermite polynomial. The classical generating and Mehler formulas [6, §§18.12,18.18] give

$$e^{itx} = e^{-t^2/(4a)} \sum_{n\ge 0} H_n(\sqrt{a}x)\frac{(it/(2\sqrt{a}))^n}{n!}, \tag{21}$$

$$\sum_{n\ge 0} r^n h_n(u)^2 = (1-r^2)^{-1/2}\exp\left(\frac{2ru^2}{1+r}\right), \qquad 0 < r < 1. \tag{22}$$

Multiplying (22) by e^{-u^2}, retaining one term and taking $r = n/(n+1)$ proves

$$e^{-u^2/2}|h_n(u)| \le \sqrt{e}\,(n+1)^{1/4} \quad (n \ge 1);$$

the $n = 0$ bound is one. The absolute coefficient of h_n in (21) is the square root of a Poisson mass with mean $\lambda = t^2/(2a)$.

Put $\Lambda = B^2/(2a)$ and $N = \lceil (1+\theta)\Lambda \rceil$. For $|t| \le B$, the weighted tail after N is bounded uniformly by

$$C_\theta(N+2)^{1/4}e^{-c_\theta\Lambda}. \tag{23}$$

Here is an explicit reason for uniformity. The majorant terms $(n+1)^{1/4}e^{-\lambda/2}\lambda^{n/2}/\sqrt{n!}$ have consecutive ratio at most a fixed $q_\theta < 1$ for $n > N$ and large B, uniformly in $\lambda \le \Lambda$. Their first omitted term is maximized at $\lambda = \Lambda$, since $n > \Lambda$. The bound $n! \ge (n/e)^n$ gives logarithm at most

$$-\frac{\Lambda}{2}\left[(1+\theta)\log(1+\theta) - \theta\right] + O_\theta(\log(B+2)),$$

which proves (23). At $t = 0$ the omitted tail vanishes. These bounds also justify weighted absolute convergence and integration of the expansion.

Integrating the finite sum in (21) against the restricted Hermitian Fourier measure gives a real polynomial P_j of degree at most N. Its original absolute-L^2 error from $\widetilde{f_j}$ is at most

$$\sqrt{J_{\mathrm{abs}}} V M C_\theta (N+2)^{1/4} e^{-c_\theta \Lambda} < \eta/3$$

eventually. Indeed the positive logarithmic costs are $O(L)$, $\log(1/\eta) = o(L)$ and Λ has order L^2. Combining this with (20) and the whole-span sign payment below yields

$$\limsup \frac{D^G_{m+1}}{L^2} \le \frac{1+\theta}{8ad^2}. \tag{24}$$

Take the concentration limit first, then $\theta \downarrow 0$ and $a \uparrow c$ to obtain its sharp upper coefficient.

2.6 Critical exponential return without a growing multiplier

Put $h = \pi/c$ and $Q_n(x) = P_n^{(1/2)}(x/h; \pi/2)$, in the classical normalization [5, 6]. Their generating function is

$$\sum_{n\ge 0} Q_n(x) z^n = (1-iz)^{-1/2+ix/h}(1+iz)^{-1/2-ix/h}, \qquad |z| < 1. \tag{25}$$

Cauchy's estimate on $|z| = n/(n+1)$ gives

$$|Q_n(x)| \le e\sqrt{n+1}\, e^{c|x|/2}, \qquad n \ge 0.$$

For $n \ge 1$ use $|\operatorname{Re} \arctan z| \le \arctan |z| < \pi/4$; $Q_0 = 1$ handles zero. Substitution $z = i \tanh(ht/2)$ in (25) gives

$$e^{itx} = \operatorname{sech}(ht/2) \sum_{n\ge 0} Q_n(x) [i \tanh(ht/2)]^n.$$

Integrating against the restricted measure of $\widetilde{f_j}$ therefore gives coefficients of magnitude at most VMr^n, where $r = \tanh(hB/2)$ and $\alpha = (1-r)^{-1} = (e^{hB}+1)/2$. The degree-$N$ error in the original absolute norm is at most

$$F(N+2) r^{N+1} \alpha^2, \qquad F = eVM\sqrt{J_{\mathrm{abs}}}.$$

The estimate follows by majorizing $\sqrt{n+1}$ by $n+1$ and summing the geometric first-moment tail; weighted absolute convergence also justifies integration. Choose the explicit cutoff $N = \lceil \alpha L^2 \rceil$. Since $N + 2 \le \alpha(L^2+3)$, $\log r \le -1/\alpha$, $\log F = L/2 + o(L)$ and $\log \alpha = hB + O(1) = O(L)$, the error logarithm is at most $-L^2 + O(L) < \log(\eta/3)$ eventually. The integrated finite sum is real and satisfies

$$\limsup \frac{\log D^E_{m+1}}{L} \le \frac{\pi}{2cd}, \tag{26}$$

because $\log N = hB + O(\log L) = \pi L/(2cd) + o(L)$.

2.7 The complete Gram sign and rational output

In either class the per-target approximation errors above total less than $2\eta/3$. Optional coefficient rounding can consume the remaining $\eta/3$ at unchanged degree. The full absolute moment budget pays it: in the Gaussian case $\int |x|^{2j}\, \mathrm{d}|\mu| \le J_{\mathrm{abs}} j!/a^j$; in the exponential case it is at most $J_{\mathrm{abs}}(2j)!/c^{2j}$. This is an existence statement; certified effective output additionally requires computable class data and the negative moments defining the p_j. The construction uses only the stated input data.

For every unit coefficient vector the total absolute-norm error is at most $\Delta = \sqrt{s}\eta = \rho^2/(32C_0)$. Its original square-form perturbation is at most

$$\Delta(2C_0 + \Delta) \le \rho^2/16 + \rho^4/(1024C_0^2) < \rho^2/8.$$

Thus the strict negative margin in (16) survives on the entire s-dimensional span. In particular the polynomials are independent. This completes both upper bounds without an extra restriction on the requested index.

2.8 Classical Laguerre estimates for the lower bound

We use the ordinary Laguerre normalization

$$L_n(u) = \sum_{j=0}^{n} \binom{n}{j} \frac{(-u)^j}{j!}, \qquad \int_0^\infty L_n L_j e^{-u}\,\mathrm{d}u = \delta_{nj}, \qquad |L_n(u)| \le e^{u/2} \quad (u \ge 0). \tag{27}$$

The envelope is the classical Laguerre inequality [6, Eq. 18.14.8]. For each fixed $a > 0$ we will use

$$\log L_n(-a) = 2\sqrt{an} + O_a(\log(n+2)), \tag{28}$$

$$\log \sum_{j=0}^{n} L_j(-a)^2 = 4\sqrt{an} + O_a(\log(n+2)). \tag{29}$$

Here the required estimates have a short direct proof. Positivity of coefficients gives $L_n(-a) \le \sum_{j\ge0}(na)^j/(j!)^2 \le e^{2\sqrt{na}}$; the last inequality follows by comparing even coefficients of the exponential. For a lower bound retain $j = \lfloor\sqrt{na}\rfloor$. For large n, $j \le n/2$ and $\binom{n}{j} \ge n^j e^{-j^2/n}/j!$, using $\log(1-u) \ge -2u$ for $0 \le u \le 1/2$. Stirling's logarithmic formula gives (28). The sum in (29) lies between its last term squared and $(n+1)e^{4\sqrt{an}}$.

2.9 A fixed Gaussian lower law and a moving codimension

Choose a small fixed $\varepsilon_0 > 0$ with $D = d + \varepsilon_0$ and $e^{cD^2} < C_G$. Fix $\sigma^2 < 1/c$ and let

$$X = \pm\sqrt{D^2 + \sigma^2 U}, \qquad U \text{ with density } e^{-u}\ (u \ge 0)$$

define τ_*. Its Gaussian moment is $E_* = e^{cD^2}/(1 - c\sigma^2)$. Let $\tau_D = (\delta_D + \delta_{-D})/2$ and choose fixed $\eta > 0$ so

$$\tau = (1-\eta)\tau_D + \eta\tau_*, \qquad \int e^{cx^2}\,\mathrm{d}\tau \le C_G.$$

For example one may take $\eta = \min\{1, (C_G - e^{cD^2})/[2(E_* - e^{cD^2})]\}$. The same positive law is fixed before m, κ or R vary. The remainder $R\tau$ belongs to (2) whenever $\varepsilon \le \varepsilon_0$.

Let A_m be the monic κ-orthogonal factor. On reference support each of its root factors has magnitude at least d, so $|A_m| \ge d^m$. For $N \ge m$, the space

$$\{A_m Q : \deg Q \le N - m\} \subset \mathcal{P}_N$$

has codimension exactly m. On that whole space the positive contribution of $R\tau$ is at least $R\eta d^{2m}\|Q\|_{\tau_*}^2$. The complete negative contribution is bounded, by Lemma 2.2, by

$$h_m(1 + \varepsilon_0^2/D^2)K_{\lfloor(N-m)/2\rfloor}\|Q\|_{\tau_*}^2, \qquad K_l = \sum_{j=0}^{l} L_j(-D^2/\sigma^2)^2.$$

Indeed $\int A_m^2 \, \mathrm{d}\kappa = h_m$, and the lemma bounds the supremum of Q^2 on the whole negative interval. The original form is therefore nonnegative on that codimension-m space if

$$(1 + \varepsilon_0^2/D^2) K_{\lfloor (N-m)/2 \rfloor} \le \eta \exp(L + 2m \log d). \tag{30}$$

This excludes $m + 1$ negative directions. It retains the entire form, not only separate tests.

Fix $0 < \theta < 1$ and choose

$$N = m + 2 \left\lfloor \frac{(1-\theta)\sigma^2 L^2}{16 D^2} \right\rfloor .$$

By (29), the left-hand logarithm in (30) is $\sqrt{1-\theta}\, L + O(\log L)$. The right-hand logarithm is $L + 2m \log d + O(1) = L + o(L)$ by (7), also when $d < 1$. The strict deficit therefore pays every reference constant. Consequently $n_{m+1}(-\kappa + R\tau) > N$. Take the sequence limit first, then $\theta \downarrow 0$, $\sigma^2 \uparrow 1/c$ and $\varepsilon_0 \downarrow 0$. The resulting lower coefficient is $1/(8cd^2)$, proving (4). No endpoint law with an infinite Gaussian moment was used.

2.10 A fixed exponential lower law and growing codimension

Put $\alpha = \pi/(2cd)$. We state the classical continuous dual Hahn input with its probability normalization [6, Section 18.25], [5]. For $a_0 > 0$, the law on $u > 0$ with parameters $(a_0, 1/2, 1/2)$ has density

$$\frac{2u \tanh(\pi u) |\Gamma(a_0 + iu)|^2}{\Gamma(a_0 + 1/2)^2}. \tag{31}$$

In the standard normalization it is

$$S_j(u^2) = (a_0 + 1/2)_j^2 \, {}_3F_2\left(\begin{matrix} -j, \ a_0 + iu, \ a_0 - iu \\ a_0 + 1/2, \ a_0 + 1/2 \end{matrix} ; 1 \right).$$

At $u^2 = -a_0^2$ one numerator parameter is zero, so its terminating series is one. The classical orthogonality norm and this value are

$$S_j(-a_0^2) = (a_0 + 1/2)_j^2, \qquad \|S_j\|^2 = (j!)^2 (a_0 + 1/2)_j^2.$$

For the even lift $X = \pm\sigma\sqrt{u^2 + a_0^2}$ the squared-variable origin kernel is therefore

$$K_l = \sum_{j=0}^{l} \left[\frac{(a_0 + 1/2)_j}{j!} \right]^2 \sim \frac{l^{2a_0}}{2a_0 \Gamma(a_0 + 1/2)^2} \le K_0 (l+1)^{2a_0}. \tag{32}$$

The last bound holds with fixed K_0 for every $l \ge 0$. The asymptotic follows by the gamma-ratio estimate and summing $j^{2a_0 - 1}$. Stirling's formula on a vertical line makes (31) of order $u^{2a_0} e^{-\pi u}$ at infinity; the lifted law has finite rate-c exponential moment when $c\sigma < \pi$.

Fix $0 < \beta < \alpha$. Choose a small fixed $\varepsilon_0 > 0$ with $D_0 = d + \varepsilon_0$, $e^{cD_0} < C_E$ and $cD_0/\pi < 1/(2\beta)$. Choose

$$cD_0/\pi < a_0 < 1/(2\beta), \qquad \sigma = D_0/a_0.$$

The corresponding lift τ_* is supported on $|x| \ge D_0$ and has a finite exponential moment E_*. Mix once with the fair endpoint law τ_{D_0} to obtain $\tau \ge \eta_0 \tau_*$ with moment at most C_E, where

$$\eta_0 = \min\left\{ 1, \frac{C_E - e^{cD_0}}{2(E_* - e^{cD_0})} \right\} > 0.$$

The law is fixed before m, κ and R vary. For every $\varepsilon \le \varepsilon_0$, $R\tau$ meets the support condition, and $-\kappa + R\tau$ meets the SAME full budget: the background contributes at most $C_E R$ and the negative probability at most $e^{c\varepsilon_0} < C_E$.

Use again the monic factor A_m. On the codimension-m space $P = A_m Q$ with $\deg Q \le N-m$, the positive contribution is at least $R\eta_0 d^{2m}\|Q\|_{\tau_*}^2$. By Lemma 2.2, the complete negative contribution is at most

$$h_m(1+\varepsilon_0^2/D_0^2)K_{\lfloor (N-m)/2\rfloor}\|Q\|_{\tau_*}^2.$$

The whole subspace is therefore nonnegative if

$$(1+\varepsilon_0^2/D_0^2)K_{\lfloor (N-m)/2\rfloor} \le \eta_0 \exp(L + 2m\log d). \tag{33}$$

For $N < m$ ambient dimension already excludes $m+1$ negative directions. Take $N = \lfloor e^{\beta L}\rfloor$. Eventually $N \ge m$ since $m = o(L)$. The logarithm of the left side of (33) is at most $2a_0\beta L + O(1)$ by (32); the right-hand logarithm is $L + o(L)$. Since $2a_0\beta < 1$, the condition eventually holds and $n_{m+1}(-\kappa + R\tau) > N$. Every allowed concentration sequence eventually enters each fixed reference interval. Taking its limit first, then $\beta \uparrow \alpha$, gives $\liminf \log D_{m+1}^E/L \ge \alpha$. Together with (26) this proves (5) and completes Theorem 2.1.

2.11 A slow continuous concentration example and boundaries

For uniform probability on $[-\varepsilon,\varepsilon]$, the standard Legendre polynomial has leading coefficient $\ell_m = 2^{-m}\binom{2m}{m}$ and probability squared norm $1/(2m+1)$. Consequently

$$h_m = \frac{\varepsilon^{2m}}{(2m+1)\ell_m^2}.$$

Take $R = 1$, $m \to \infty$ and $\varepsilon = 1/\log(m+2)$. Stirling's formula gives

$$L = 2m\log\log(m+2) + O(m).$$

Here $m\log(m+2)/L \to \infty$, so the earlier sufficient index-growth restriction does not hold. The present theorem still yields

$$D_{m+1}^G \sim \frac{m^2[\log\log(m+2)]^2}{2cd^2}, \qquad \log D_{m+1}^E \sim \frac{\pi m\log\log(m+2)}{cd}.$$

The negative law is continuous and the requested index is prescribed; this is not merely a fixed-atom or fixed-index consequence.

The support must still collapse and the gap is from its entire containing interval. An objective-preserving measure representation and its hypotheses are prerequisites for a spectral application. Signed-band index laws for a fixed supplied density remain distinct from this arbitrary shrinking-probability joint minimax statement.

3 A derivative-free inverse-scale law for local concentration

We next consider a scalar negative atom without a support gap. Fix $q, c, C, L, r_0 > 0$ and an increasing tail function H. For $R \ge 1$, let $\mathcal{A}_{H,q}(R)$ consist of real signed measures

$$\mu = -\delta_0 + \nu, \qquad \mathrm{TV}(\nu) \le R, \quad \int_{\mathbb{R}} e^{cH(|x|)}\,\mathrm{d}|\nu|(x) \le CR, \quad |\nu|([-r,r]) \le LRr^q \quad (0 < r \le r_0). \tag{34}$$

The local condition forces $\nu(\{0\}) = 0$. Write $D_{H,q}(R) = \sup_{\mu \in \mathcal{A}_{H,q}(R)} n_1(\mu)$, where n_1 is the first ordinary polynomial degree producing a strictly negative square integral, with value infinity if none exists. The upper backgrounds may be signed, singular or atomic away from zero. No lower mass bound, density, doubling condition on ν, or hard gap is assumed.

Theorem 3.1 (Inverse tail-scale law)**.** *Let $H : [0, \infty) \to [0, \infty)$ be continuous, strictly increasing and unbounded, with $H(0) = 0$. Suppose that, for one fixed $\rho > 1$,*

$$\frac{H(y)}{H(x)} \geq \left(\frac{y}{x}\right)^{\rho} \qquad (0 < x \leq y). \tag{35}$$

Set $A(t) = H^{-1}(t)$ and $F(t) = t/A(t)$ for $t > 0$. Then F is a continuous increasing bijection of $(0, \infty)$, and

$$D_{H,q}(R) = \Theta\Big(F^{-1}(R^{1/q})\Big) \qquad (R \to \infty). \tag{36}$$

The constants may depend on all fixed data, including H and ρ. A fixed positive reference density, multiplied by a fixed scalar times R, supplies the lower bound. The upper constructs a polynomial on the original signed measure with value one at the negative atom and a fixed strict sign margin.

There is no differentiability, curvature, convexity or upper-growth assumption on H. Condition (35) is nevertheless a substantive uniform superlinearity condition; arbitrary convex tails with long nearly linear stretches are not included. The theorem concerns a minimax order, not a leading coefficient, a rate for every individual background, or computational complexity.

For $H(x) = x^p$, $p > 1$, it gives

$$D_{p,q}(R) = \Theta\Big(R^{p/[q(p-1)]}\Big), \qquad p > 1, \quad q > 0. \tag{37}$$

Thus the lower argument includes subquadratic powers as well as the usual smooth-curvature cases. Classical fast-decreasing polynomials, compact Jacobi kernels and weighted polynomial comparison underlie the proof. In particular, the multiplier existence used below belongs to the established theory of [7, 8]; we give an explicit construction to specify its degree and hypotheses. The recovery formulation continues the original-measure framework of [1, 2, 3].

3.1 The local target and the upper inverse-scale bound

Choose an integer $\ell \geq 1$ with $\beta = 2\ell > q$ and put

$$B = r_0^{-q} + \frac{\beta L}{\beta - q}, \qquad T = \max\{1, r_0^{-1}, (16BR)^{1/q}\}, \qquad W = \ell T. \tag{38}$$

The real function $f_T(x) = \operatorname{sinc}(Tx)^{\ell}$, with its value at zero defined as one, is the Fourier transform of a symmetric probability π_T on $[-W, W]$: take the convolution of ℓ uniform probabilities on $[-T, T]$. Its squared modulus is at most $\min\{1, (T|x|)^{-\beta}\}$. For completeness, layer cake split at $u_0 = (Tr_0)^{-\beta} \leq 1$ gives

$$\int f_T^2 \, \mathrm{d}|\nu| \leq Ru_0 + LRT^{-q} \int_{u_0}^{1} u^{-q/\beta} \, \mathrm{d}u \leq BRT^{-q} \leq \frac{1}{16}. \tag{39}$$

The first term uses $(Tr_0)^{-\beta} \leq T^{-q} r_0^{-q}$. The closed-interval local condition suffices; changing boundary inequalities at radii affects no layer integral.

The scaling condition implies, for $v \geq u > 0$,

$$\frac{A(v)}{A(u)} \leq (v/u)^{1/\rho}, \qquad (v/u)^{1-1/\rho} \leq \frac{F(v)}{F(u)} \leq v/u. \tag{40}$$

It follows that F is strictly increasing, tends to zero at zero and to infinity at infinity. For $\delta = 1-1/\rho$ and $K \geq 1$,

$$KF^{-1}(t) \leq F^{-1}(Kt) \leq K^{1/\delta}F^{-1}(t). \tag{41}$$

The corresponding inequalities for factors below one follow by reciprocation. These facts pay fixed constants inside all inverse scales.

For every integer $j \geq 1$,

$$\sup_{x\geq 0} x^j e^{-cH(x)/2} \leq K_c^j A(j)^j, \qquad K_c = \max\{1, (2/(c\rho e))^{1/\rho}\}. \tag{42}$$

Indeed, for $x \leq A(j)$ discard the exponential. For $x \geq A(j)$, put $u = H(x) \geq j$. Then $x/A(j) \leq (u/j)^{1/\rho}$, and the unrestricted maximum of $(u/j)^{j/\rho}e^{-cu/2}$ is $(2/(c\rho e))^{j/\rho}$.

Define a real polynomial by integrating a finite Taylor expansion:

$$P_N(x) = \int_{-W}^{W} \sum_{j=0}^{N} \frac{(itx)^j}{j!} \, \mathrm{d}\pi_T(t).$$

Symmetry removes odd coefficients, and $P_N(0) = 1$ exactly. Using $j! \geq (j/e)^j$ and (42), the jth weighted-uniform remainder term is bounded by $[eK_cW/F(j)]^j$. Choose an integer N at least

$$1, \qquad F^{-1}(2eK_cW), \qquad \log_4 \max\{1, 16CR\}. \tag{43}$$

Every term beyond N is at most 2^{-j}, so absolute weighted convergence justifies the Fourier integration and gives

$$\sup_x e^{-cH(|x|)/2}|P_N(x) - f_T(x)| \leq 2^{-N}, \qquad \|P_N - f_T\|^2_{L^2(|\nu|)} \leq CR4^{-N} \leq \frac{1}{16}.$$

Together with (39), Minkowski yields $\|P_N\|_{L^2(|\nu|)} \leq 1/2$, and hence

$$\int P_N^2 \, \mathrm{d}\mu \leq -\frac{3}{4}. \tag{44}$$

This is an original-measure sign estimate, not a compact restriction or a convolution.

Since $W = O(R^{1/q})$, (41) pays the second cutoff in (43) by a fixed multiple of $F^{-1}(R^{1/q})$. The logarithmic cutoff is smaller in order: $A(t) \geq A(1) > 0$ for $t \geq 1$, so $F^{-1}(u) \geq A(1)u$ for sufficiently large u. Integer ceilings are harmless. This proves the upper in (36).

For certified computable H and inverse-scale bounds, rational coefficient rounding at the same degree is also possible without original moments. The absolute measure has moment budget at most $1 + CR$, and a coefficient error $\sum_j \epsilon_j x^j$ has norm at most

$$\sqrt{1+CR}\left[|\epsilon_0| + \sum_{j\geq 1} |\epsilon_j| K_c^j A(j)^j\right].$$

A sufficiently small value preserves a fixed part of the margin in (44). The existence theorem applies to every tail function satisfying its stated hypotheses.

3.2 A degree-paid dyadic multiplier

We first give the construction with parameters that can also be used for a nearcritical tail.

Lemma 3.2 (Compact dyadic multiplier)**.** *Let $Q:[0,\infty)\to[0,\infty)$ be increasing and continuous, with $Q(0)=0$. Fix $A>0$, an integer $J\ge 1$, and $d_j=A2^{-j}$, $0\le j\le J$. Put*

$$m_j=\left\lceil \frac{AQ(d_{j-1})}{4d_j}\right\rceil,\qquad C_j(x)=\frac{T_{m_j}((A^2+d_j^2-2x^2)/(A^2-d_j^2))}{T_{m_j}((A^2+d_j^2)/(A^2-d_j^2))},\qquad S(x)=\prod_{j=1}^{J}C_j(x).$$

If $m_j=0$, take that factor as one. Then $S(0)=1$,

$$\deg S\le\sum_{j=1}^{J}2^{j-1}Q(A2^{-(j-1)})+2J, \tag{45}$$

$$|S(x)|\le\max\{2,e^{Q(d_J)/2}\}e^{-Q(|x|)/2}\qquad(|x|\le A). \tag{46}$$

Proof. Every denominator is positive. For $|x|\le A$, the numerator argument lies between -1 and the positive denominator argument, so $|C_j(x)|\le 1$. For $d_j\le|x|\le A$ its argument is in $[-1,1]$, whereas

$$\operatorname{arcosh}\frac{A^2+d_j^2}{A^2-d_j^2}=\log\frac{A+d_j}{A-d_j}\ge 2d_j/A.$$

The Chebyshev identity therefore gives $|C_j(x)|\le 2e^{-2m_jd_j/A}\le 2e^{-Q(d_{j-1})/2}$. For $d_j\le|x|\le d_{j-1}$ this and the bounds one for the other factors imply $|S(x)|\le 2e^{-Q(|x|)/2}$. On the remaining central interval, $|S|\le 1$ and $Q(|x|)\le Q(d_J)$. These observations prove (46). Finally, $\deg S\le 2\sum_j m_j$ and the integer ceilings cost at most $2J$, which gives (45). □

The lemma only bounds the polynomial on a finite interval. The bound holds on the stated interval. Such compact fast-decreasing multipliers and their degree characterization are classical [7, 8]; the explicit factors make the constants and the comparison direction below transparent.

Apply the lemma to $Q=aH$, $a>0$, with $A=(aH)^{-1}(n)$, $J=\lceil\log_2 n\rceil$ and integer $n\ge 2$. The lower scaling in (35) gives

$$Q(A2^{-j})\le n2^{-\rho j},\qquad Q(d_J)\le n^{1-\rho}\le 1.$$

Consequently a polynomial S_n satisfies

$$S_n(0)=1,\quad |S_n(x)|\le 2e^{-aH(|x|)/2}\quad(|x|\le A),\quad \deg S_n\le K_\rho n,\qquad K_\rho=2+\frac{1}{1-2^{1-\rho}}. \tag{47}$$

Indeed the main degree sum is a convergent geometric sum, and $J\le n$. No second derivative of H occurs.

3.3 The compact power kernel, with normalization

Lemma 3.3 (Compact singular-power evaluation)**.** *For each fixed $q>0$ there is $c_q>0$ such that, for every $A>0$, integer $M\ge 0$, and real polynomial $U\in\mathcal{P}_M$,*

$$\int_{-A}^{A}U(x)^2|x|^{q-1}\,\mathrm{d}x\ge c_q\left(\frac{A}{M+1}\right)^q|U(0)|^2. \tag{48}$$

Proof. Put $\alpha = q/2 > 0$. For the unnormalized weight $v^{\alpha-1}$ on $[0,1]$, the shifted Jacobi polynomials can be written as

$$R_j(v) = \frac{v^{1-\alpha}}{j!}\left(\frac{d}{dv}\right)^j [v^{j+\alpha-1}(v-1)^j].$$

They have degree j, value $R_j(0) = (-1)^j(\alpha)_j/j!$, and squared norm $1/(2j+\alpha)$. These classical identities [4, 6] follow directly here: j integrations by parts annihilate every lower-degree polynomial; all boundary terms vanish, since at zero the remaining exponent is at least $\alpha > 0$, and the factor at one has order j. The leading coefficient is $(j+\alpha)_j/j!$, so the squared norm is $[(j+\alpha)_j/j!]\mathrm{B}(j+\alpha, j+1) = 1/(2j+\alpha)$. Thus the degree-$l$ kernel at zero is exactly

$$\sum_{j=0}^{l}(2j+\alpha)\left[\frac{(\alpha)_j}{j!}\right]^2 = \frac{(l+\alpha)^2}{\alpha}\left[\frac{(\alpha)_l}{l!}\right]^2. \tag{49}$$

The sum telescopes using the ratio of consecutive Pochhammer terms. The fixed-parameter gamma-ratio estimate bounds (49) by $C_q(l+1)^q$ for every $l \geq 0$.

Split U into its even and odd parts. The odd part vanishes at zero and adds a nonnegative norm. For the even part use $v = x^2/A^2$; its norm acquires the factor A^q and its degree in v is at most $l = \lfloor M/2 \rfloor$. The original compact kernel is therefore A^{-q} times (49). Cauchy–Schwarz in that orthogonal basis proves (48). No value of the possibly singular weight at zero has been divided by. □

3.4 A fixed full-line lower example

Choose $a \geq \max\{1, 2c\}$ and define

$$Z_a = \int_{\mathbb{R}} |x|^{q-1} e^{-aH(|x|)}\,\mathrm{d}x, \qquad \mathrm{d}\tau_a(x) = Z_a^{-1}|x|^{q-1}e^{-aH(|x|)}\,\mathrm{d}x. \tag{50}$$

The probability exists and has all moments: (35) gives superlinear power growth at infinity, and $q > 0$ gives integrability at zero. For $\deg P \leq n$, multiply by S_n from (47) and restrict to its comparison interval $[-A, A]$, where $A = H^{-1}(n/a)$. The direction of the comparison is

$$\int P^2\,\mathrm{d}\tau_a \geq \frac{1}{4Z_a}\int_{-A}^{A}(PS_n)^2|x|^{q-1}\,\mathrm{d}x.$$

Lemma 3.3, $\deg(PS_n) \leq (1+K_\rho)n$, and $S_n(0) = 1$ give a fixed $b > 0$ with

$$\int P^2\,\mathrm{d}\tau_a \geq b\left(\frac{A(n)}{n}\right)^q |P(0)|^2 = bF(n)^{-q}|P(0)|^2, \qquad n \geq 2. \tag{51}$$

Here $H^{-1}(n/a) \geq a^{-1/\rho}A(n)$ by (35); all fixed degree and normalization factors have been absorbed into b. This is an estimate on every degree-n polynomial, not a test of selected squares.

The fixed law obeys, for every $r > 0$,

$$\tau_a([-r,r]) \leq \frac{2r^q}{qZ_a}, \qquad E_a := \int e^{cH(|x|)}\,\mathrm{d}\tau_a = \frac{Z_{a-c}}{Z_a} < \infty.$$

Choose one fixed

$$0 < \eta \leq \min\{1, C/E_a, LqZ_a/2\}. \tag{52}$$

Then $\nu_R = \eta R\tau_a$ satisfies all budgets in (34) for every $R \geq 1$ and every prescribed r_0. This permits arbitrarily small positive C, L because the remainder mass is at most R, rather than artificially fixed to equal R.

For $\mu_R = -\delta_0 + \eta R\tau_a$, the whole degree-$n$ form is nonnegative if $F(n)^q \le b\eta R$. Take, for large R, $n = \lfloor F^{-1}((b\eta R/2)^{1/q})\rfloor \ge 2$. Then $n_1(\mu_R) > n$. The inverse comparisons (40)–(41) pay the fixed argument and the floor, giving the lower in (36). The probability and its scalar η were fixed before R varied. Equality in the nonnegativity condition never counts as a strictly negative square.

Remark 3.4 (Nonpower tails). The hypothesis is not restricted to regularly varying powers. For $H(x) = e^{x^2} - 1$ it holds with $\rho = 2$, as is verified by $xH'(x)/H(x) \ge 2$ for $x > 0$. Now $A(t) = \sqrt{\log(1+t)}$ and $F(t) = t/\sqrt{\log(1+t)}$; Theorem 3.1 gives $D_{H,q}(R) = \Theta(R^{1/q}\sqrt{\log(R+2)})$. This uses the declared double-exponential moment budget and does not follow from a Gaussian moment assumption alone. The theorem assumes (35).

4 Logarithmically strengthened critical tails

Uniform power superlinearity does not include

$$H_\beta(x) = x[\log(e+x)]^\beta, \qquad \beta \ge 0.$$

These tails have a different scale. Keep exactly the class (34) with $H = H_\beta$ and fixed positive q, c, C, L, r_0, and write $D_{\beta,q}(R) = D_{H_\beta,q}(R)$.

Theorem 4.1 (Nearcritical local-concentration law)**.** *For every fixed $\beta \ge 0$ and $q, c, C, L, r_0 > 0$,*

$$\log D_{\beta,q}(R) = \Theta\Big(R^{1/[q(\beta+1)]}\Big) \qquad (R \to \infty). \tag{53}$$

The upper permits signed and singular remainders. A single fixed atomless positive reference density, multiplied by a fixed scalar times R, supplies the lower. The conclusion is not uniform as β varies and does not determine a sharp leading coefficient.

For $\beta = 0$ this is the critical exponential order. For $\beta > 0$, neither a substitution into a fixed-rate exponential estimate nor discarding the enlarged exponential moment would prove the upper. We retain that moment explicitly and optimize its rate. The nearcritical model Christoffel scale for $q = 1$ is classical [9].

4.1 A uniform critical expansion at a variable moment rate

Use f_T, T and $W = \ell T$ from (38)–(39). Thus $T \ge 1$, $T \asymp R^{1/q}$ for large R, $f_T(0) = 1$, and $\|f_T\|_{L^2(|\nu|)} \le 1/4$. First suppose $\beta > 0$. For a rate parameter $a \ge 1$, put

$$M(a) = \sup_{x\ge 0}\{ax - cx[\log(e+x)]^\beta\}.$$

It is finite and nonnegative. If $\log(e+x) \ge (2a/c)^{1/\beta}$, the expression is at most $-ax$; otherwise $x \le \exp((2a/c)^{1/\beta})$. Consequently

$$0 \le M(a) \le \overline{M}(a) := a\exp((2a/c)^{1/\beta}), \qquad \int e^{a|x|}\,\mathrm{d}|\mu| \le \mathcal{J}_a := 1 + CRe^{\overline{M}(a)}. \tag{54}$$

The unit negative atom is included. In particular $\log \mathcal{J}_a \le \log(1+CR) + \overline{M}(a)$. We do not replace $\mathcal{J}_a$ by a fixed multiple of R.

Set $h = \pi/a$ and use classical Meixner–Pollaczek polynomials $Q_n(x) = P_n^{(1/2)}(x/h;\pi/2)$. Their generating function [5, 6] is

$$\sum_{n\ge 0} Q_n(x)z^n = (1+z^2)^{-1/2}\exp\left(\frac{2x}{h}\arctan z\right), \qquad |z| < 1.$$

For $|z| = r < 1$, $|\operatorname{Re} \arctan z| \le \arctan r$; this follows, for example, from the argument range of the associated Cayley ratio. Cauchy's estimate with $r = n/(n+1)$ gives

$$|Q_n(x)| \le e\sqrt{n+1}\, e^{a|x|/2}, \qquad n \ge 0. \tag{55}$$

Indeed $r^{-n} \le e$ and $(1-r^2)^{-1/2} \le \sqrt{n+1}$ for $n \ge 1$, while $2 \arctan r < \pi/2$ and $h = \pi/a$. For $n = 0$ use $Q_0 = 1$. The numerical constant in (55) is independent of the possibly growing rate a.

Substitute $z = i \tanh(ht/2)$ into the generating function to obtain

$$e^{itx} = \operatorname{sech}(ht/2) \sum_{n \ge 0} Q_n(x) [i \tanh(ht/2)]^n.$$

Let

$$r = \tanh(hW/2), \qquad \delta = 1 - r, \qquad b = \delta^{-1} = (e^{hW} + 1)/2.$$

Integrating this expansion against the symmetric Fourier probability of f_T gives real coefficients of magnitude at most r^n. Absolute convergence in the weighted uniform norm from (55) justifies the integration. The degree-N truncation U_N satisfies

$$\|U_N - f_T\|_{L^2(|\mu|)} \le e\sqrt{\mathcal{J}_a}\, (N+2) r^{N+1} \delta^{-2}. \tag{56}$$

Here we used $\sum_{n>N} \sqrt{n+1}\, r^n \le \sum_{n>N} (n+1) r^n \le (N+2) r^{N+1}/(1-r)^2$. No density or moment oracle for ν enters this estimate.

There is an explicit, noncircular cutoff. Put

$$F_a = e\sqrt{\mathcal{J}_a}, \qquad Z = 4 + 2\log(8 F_a b^3), \qquad N = \lceil bZ \rceil. \tag{57}$$

Since $b \ge 1$ and $\mathcal{J}_a \ge 1$, $Z \ge 4$. Then $N + 2 \le b(Z+3)$ and $r^{N+1} \le e^{-\delta(N+1)} \le e^{-Z}$. The elementary inequality $Z + 3 \le e^{Z/2}$ holds for $Z \ge 4$, as is checked at four and by differentiating. Thus (56) is at most

$$F_a b^3 (Z+3) e^{-Z} \le F_a b^3 e^{-Z/2} = e^{-2}/8 < 1/8. \tag{58}$$

The full moment, however large, has been paid by the declared Z.

Because $|\mu|$ has an atom of mass one at zero, $|U_N(0) - 1| < 1/8$. Also $\|U_N\|_{L^2(|\nu|)} < 1/4 + 1/8 = 3/8$. Therefore

$$\int U_N^2 \, \mathrm{d}\mu \le -(7/8)^2 + (3/8)^2 = -5/8.$$

Normalize by the nonzero value $U_N(0)$ to get a polynomial P with $P(0) = 1$ and

$$\int P^2 \, \mathrm{d}\mu \le -1 + (3/7)^2 = -40/49. \tag{59}$$

For certified computable inputs, rational coefficient rounding preserves a fixed part of this margin at the same degree. It can be paid by $\sqrt{\mathcal{J}_a} \sum_j |\epsilon_j| \sqrt{(2j)!}/a^j$; the constant coefficient may be kept exactly one.

Now choose

$$a = T^{\beta/(\beta+1)}. \tag{60}$$

The Fourier cost satisfies $\log b \le hW = \pi \ell T/a = O(T^{1/(\beta+1)})$. The definitions of Z and $\mathcal{J}_a$ give

$$Z \le K[1 + \log(1 + CR) + \overline{M}(a) + T/a].$$

In taking its logarithm, every term remains controlled:

$$\log(1+\overline{M}(a)) \le K_{\beta,c}a^{1/\beta} + \log(1+a) + K,$$

and hence

$$\log Z \le K'[T^{1/(\beta+1)} + \log(e+T) + \log\log(e+R)] = O(T^{1/(\beta+1)})$$

for large R. Consequently (57) gives

$$\log N \le \log b + \log(Z+1) + K = O(R^{1/[q(\beta+1)]}). \tag{61}$$

This proves the upper for $\beta > 0$. For $\beta = 0$, choose any fixed $0 < a < c$ and use $\mathcal{J}_a = 1 + CR$ in the same uniform expansion and cutoff; then $\log N = O(T + \log\log(e+R)) = O(R^{1/q})$. The auxiliary assumption $a \ge 1$ was only used in the convenient bound on $M(a)$, not in (55) or (57).

4.2 A fixed lower law at the nearcritical scale

Fix $a > c$ and put $Q(x) = aH_\beta(x)$ for $x \ge 0$. For integer $n \ge 2$, use Lemma 3.2 with

$$A_n = \frac{n}{[\log(e+n)]^{\beta+1}}, \qquad J = \lceil \log_2 n \rceil, \qquad d_j = A_n 2^{-j}.$$

The degree sum in (45) is now

$$\begin{aligned}\deg S_n &\le aA_n \sum_{j=1}^{J} [\log(e+d_{j-1})]^\beta + 2J \\ &\le \frac{anJ}{\log(e+n)} + 2J \le (3a+2)n.\end{aligned}$$

We used $A_n \le n$, $\beta \ge 0$, $J \le n$ and $J/\log(e+n) \le 3$ for $n \ge 2$. The innermost radius is $d_J \le A_n/n \le 1$, so $Q(d_J) \le a[\log(e+1)]^\beta$. Thus (46) gives a fixed B_β with

$$S_n(0) = 1, \qquad |S_n(x)| \le B_\beta e^{-aH_\beta(|x|)/2} \quad (|x| \le A_n), \qquad \deg S_n \le (3a+2)n. \tag{62}$$

This calculation does not assume (35); the logarithmic number of dyadic scales is included in the choice of A_n.

Define the probability

$$\mathrm{d}\tau_a(x) = Z_a^{-1}|x|^{q-1}e^{-aH_\beta(|x|)}\,\mathrm{d}x, \qquad Z_a = \int_{\mathbb{R}} |x|^{q-1}e^{-aH_\beta(|x|)}\,\mathrm{d}x.$$

It has all moments since $H_\beta(x) \ge x$. For every $\deg P \le n$, compare its original norm with the compact norm of PS_n, exactly as before:

$$\begin{aligned}\int P^2\,\mathrm{d}\tau_a &\ge \frac{1}{B_\beta^2 Z_a}\int_{-A_n}^{A_n} (PS_n)^2|x|^{q-1}\,\mathrm{d}x \\ &\ge b_0[\log(e+n)]^{-q(\beta+1)}|P(0)|^2.\end{aligned} \tag{63}$$

Lemma 3.3 pays the degree at most $(3a+3)n$, and $b_0 > 0$ is fixed. There is no unsupported full-line bound on S_n, no curvature estimate, and no division by a singular weight at the origin.

The exact local and tail budgets are again $\tau_a([-r,r]) \le 2r^q/(qZ_a)$ and $\int e^{cH_\beta}\,\mathrm{d}\tau_a = Z_{a-c}/Z_a < \infty$. Fix $\eta > 0$ by (52) with these normalizations. Then $-\delta_0 + \eta R\tau_a$ belongs to the same class for every $R \ge 1$. Equation (63) makes its whole degree-n square form nonnegative whenever

$$[\log(e+n)]^{q(\beta+1)} \le b_0\eta R.$$

For large R, put $z = (b_0\eta R/2)^{1/[q(\beta+1)]}$ and $n = \lfloor e^z \rfloor - 3 \ge 2$. Since $e < 3$, $e + n \le e^z$, and the condition holds. Thus $n_1(-\delta_0 + \eta R\tau_a) > n$, proving the lower in (53). The density and η were fixed before R varied, and the argument covers every real $q > 0$, including an integrable cusp at zero.

The classical nearcritical origin-kernel scale for a nonsingular reference ($q = 1$) is described in [9] through the integral of $Q'(t)/t$. Fast-decreasing polynomial theory also anticipates the compact multipliers [7, 8]. The assertions here retain the original signed-measure class, arbitrary local exponent, explicit moment payment at a moving exponential rate, and fixed-law lower examples. The proof uses the classical Christoffel transition and orthogonal-polynomial identity. The nearcritical tail conclusion treats each fixed $\beta \ge 0$ in the stated class.

5 Exact finite controls and remaining questions

The accompanying standard-library program `verify_finite_controls.py` reconstructs uniform-packet moments with exact rational arithmetic. It verifies the monic squared norms, mixed orthogonality, finite evaluation-kernel bounds and a compact rising-factorial kernel identity. Its explicit checks remain active under optimized Python. These finite controls corroborate algebra and normalization.

The packet theorem permits every joint index and contrast sequence under support concentration. Noncollapsing negative support and varying gap or tail-rate parameters require additional analysis. The gap is from the containing interval; backgrounds in holes of disconnected negative support require a different argument. The nearcritical upper proved here covers $\beta \ge 0$ and does not extend to negative logarithmic powers by formal substitution, since a positive exponential moment may then be unavailable. Leading gapless minimax constants and wider tail-scale classes remain separate questions.

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