Global finite-depth stability and nonreal zeros of admissible Fourier transforms

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Abstract

We prove a global finite-jet stability theorem for the differential iteration $\mathcal{T}f = (f')^2 - ff''$. A uniformly positive reference curvature and bounded curvature-normalized higher derivatives give explicit amplitude budgets for positive, uniformly log-concave iterates through any prescribed finite depth. Intensified hyperbolic-cosh mixtures realize the required jets on the whole half-line, with all spatial tails controlled. Combining this realization with a credited finite-atomic Fourier multiplier gives a simultaneous separation theorem. For every finite generalized-Laguerre cutoff, finite root-coordinate iteration depth and $0 < Y < H$, one fully admissible kernel has an order-one Fourier transform satisfying all retained global Laguerre inequalities and all polynomial moment orders throughout $0 \le y \le Y$, yet has nonreal zeros inside $|\operatorname{Im} z| < H$ and negative physical polynomial squares at a later parameter. Parameters depend on the prescribed finite depths. The construction concerns general admissible kernels.

1 Question, normalization and main result

Positivity tests for Fourier transforms occur at several levels: differential inequalities for the spatial kernel, generalized Laguerre inequalities for its transform, and polynomial moment forms of associated kernels. Infinite characterizations are classical [3]; the issue here is what a prescribed *finite* collection of strong tests can force.

For a real entire function F, define $L_n[F]$ by

$$|F(x+iy)|^2 = \sum_{n\ge 0} L_n[F](x) y^{2n}, \qquad L_n[F](x) = \sum_{j=0}^{2n} \frac{(-1)^{n+j} F^{(j)}(x) F^{(2n-j)}(x)}{j!(2n-j)!}. \tag{1}$$

Thus $L_0[F] = F^2$ and $\mathcal{L}_1[F] = (F')^2 - FF''$. If all $L_n[F](x)$ are nonnegative for every n, x, a nonzero F cannot have a nonreal zero: at a fixed real x, its first nonzero term in (1) is positive. This infinite condition must be distinguished from a finite cutoff.

We call a kernel ϕ *admissible* if it is smooth, positive and even, strictly decreasing on $(0,\infty)$, and for some $\varepsilon > 0$ each fixed derivative satisfies $\phi^{(j)}(s) = O(e^{-|s|^{2+\varepsilon}})$ as $|s| \to \infty$. Our examples have entire extensions and the stronger double-exponential real decay.

Put $F(z) = \int_{\mathbb{R}} \phi(s) e^{izs}\, ds$ and

$$\begin{aligned} H_y(s) &= \frac{\sinh(2ys)}{2ys}, \quad H_0 = 1, \\ J_y(t) &= \int_0^\infty s^2 H_y(s) \phi(s + t/2) \phi(s - t/2)\, ds, \\ g_y(x) &= \int_{\mathbb{R}} J_y(t) e^{ixt}\, dt, \qquad Q_y(P) = \frac{1}{2\pi} \int_{\mathbb{R}} P(x)^2 g_y(x)\, dx. \end{aligned} \tag{2}$$

For our kernels all these moments exist. The half-center normalization is

$$g_y(x) = \frac{\partial_y |F(x + iy)|^2}{8y} \quad (y > 0), \qquad g_0(x) = \frac{\mathcal{L}_1[F](x)}{4}. \tag{3}$$

Dividing the moment matrices by $J_y(0) > 0$ changes neither sign nor inertia.

For a positive function f of a real coordinate, write $\mathcal{T} f = (f')^2 - f f''$. This *continuous differential* iteration is different from the discrete coefficient operation $a_j \mapsto a_j^2 - a_{j-1} a_{j+1}$.

Theorem 1 (Simultaneous finite-hierarchy separation)**.** *For every integers $k, q \geq 1$ and reals $0 < Y < H$, there is an admissible entire kernel ϕ with uniformly negative ordinary logarithmic curvature such that:*

1. *With $S_0(u) = \phi(\sqrt{u})$ and $S_{r+1} = \mathcal{T} S_r$, every S_r is positive and $\log S_r$ is uniformly strictly concave on $[0, \infty)$ for $0 \leq r \leq q$.*

2. *F is real even entire of exact order one and $L_n[F](x) \geq 0$ for every real x and $1 \leq n \leq k$. For each retained n, $\int_{\mathbb{R}} P^2 L_n[F] > 0$ for every nonzero real polynomial P, of arbitrary finite degree.*

3. *$Q_y(P) > 0$ for every nonzero real polynomial P and every $0 \leq y \leq Y$.*

4. *All zeros of F lie in $|\operatorname{Im} z| < H$, but some are nonreal at a height b with $Y < b < H$. For some $y_* \in (Y, b)$ there are negative Q_{y_*} squares in both parities. The parameter y_* and the coefficients of those polynomials can be chosen rational.*

The kernel and its scales depend on k, q, Y, H.

The main mechanism is global finite-jet stability and its realization by intensified atomic cosh mixtures. The author's previous paper [1] proves a two-level shape construction and endpoint nonpropagation. Here any prescribed finite continuous depth is permitted, and the same kernel also meets a finite global generalized-Laguerre hierarchy and every moment degree over a whole initial parameter interval. The six-atom finite-cutoff factor is inherited from the independent construction in [2, §4.1]; its coefficient argument is included where it is used.

Curvature doubling, logarithmic corrections and normalized telescoping have methodological precedents in discrete iteration work [5, 6]. Those general ideas are classical. The distinction here is the global real-jet norm, the paid unbounded-tail realization and the simultaneous admissible Fourier construction. Classical discrete iterated Turán results [7] do not identify that coefficient iteration with $\mathcal{T}$ applied to the root kernel.

The finite quantifiers are essential. We construct one example for each prescribed finite pair (k, q), not one fixed example for every depth. Theorem 1 concerns the general admissible class.

2 Global stability at any prescribed finite depth

For a twice differentiable real function f write

$$\mathcal{L}_1 f = f'^2 - f f''.$$

When $f > 0$ and $g = \log f$, the identities

$$\mathcal{L}_1 f = -f^2 g'', \qquad \log(\mathcal{L}_1 f) = 2g + \log(-g'') \quad (g'' < 0) \tag{4}$$

exhibit the doubling of the leading logarithm and its logarithmic correction. This amplification strategy has classical discrete analogues; see [5, 6]. Our use of it requires a uniform finite-jet estimate on the entire half-line and will subsequently be realized by admissible Fourier kernels. It concerns iteration of functions in one real coordinate, rather than iteration of their coefficient sequences.

Theorem 2 (Curvature-normalized finite-depth stability)**.** *Let $q \geq 1$ be an integer, $K = 2q + 2$, and let h, g_0 be real C^K functions on $[0, \infty)$. Suppose, everywhere on that half-line,*

$$h'' \geq a > 0, \qquad |h^{(i+2)}| \leq R_i h'' \quad (1 \leq i \leq K - 2) \tag{5}$$

for finite constants $R_i \geq 0$. Define

$$R = \max\{1, \max_{1 \leq i \leq K-2} (2R_i + 1)\}, \qquad C_j = j^j (j-1)! R^j \quad (2 \leq j \leq K), \tag{6}$$

and choose

$$A \geq \max\{1,\ 4 \max_{2 \leq j \leq K} C_j / a\}. \tag{7}$$

Assume the global initial errors obey

$$|g_0^{(j)}(v) + A h^{(j)}(v)| \leq \frac{A}{4} h''(v) \quad (2 \leq j \leq K,\ v \geq 0). \tag{8}$$

Starting from $f_0 = e^{g_0}$, the recursion $f_{r+1} = \mathcal{L}_1 f_r$ is well defined through $r = q$. All its functions f_r, $0 \leq r \leq q$, are strictly positive, belong to C^{K-2r}, and satisfy

$$(\log f_r)''(v) \leq -2^{r-1} A h''(v) \leq -2^{r-1} A a < 0 \qquad (v \geq 0). \tag{9}$$

At the endpoint derivatives mean the continuous right derivatives. The constants and the initial function may depend on q.

Proof. We track the derivatives that remain after each iteration. Put $g_r = \log f_r$. For $0 \leq r \leq q$ and $2 \leq j \leq K - 2r$ we prove the sharper estimate

$$|g_r^{(j)} + 2^r A h^{(j)}| \leq 2^r \left\{ \frac{A}{4} h'' + (1 - 2^{-r}) C_j \right\} \leq \frac{1}{2}\, 2^r A h''. \tag{10}$$

The last inequality follows from $C_j \leq Aa/4 \leq Ah''/4$. The first inequality at $r = 0$ is exactly (8). In particular it gives $g_0'' < 0$.

Suppose the estimate has been established at a level $r < q$. The case $j = 2$ gives

$$-g_r'' \geq \tfrac{1}{2}\, 2^r A h'' > 0.$$

Thus $f_{r+1} = -f_r^2 g_r''$ is positive, is C^{K-2r-2}, and (4) applies. For every available $i \geq 1$,

$$\left| \frac{g_r^{(i+2)}}{g_r''} \right| \leq \frac{2^r A(R_i + 1/2)h''}{(1/2)2^r A h''} = 2R_i + 1 \leq R. \tag{11}$$

No positivity of $h^{(i+2)}$ has been used.

For $2 \leq j \leq K - 2r - 2$, the logarithmic derivative partition formula is

$$[\log(-g_r'')]^{(j)} = \sum_{\pi \in \Pi_j} (-1)^{|\pi|-1}(|\pi| - 1)! \prod_{B \in \pi} \frac{g_r^{(|B|+2)}}{g_r''}, \tag{12}$$

where Π_j denotes the set of partitions of $\{1, \ldots, j\}$. It follows, for example, by taking the jth derivative of the composition of log with the positive function $-g_r''$. There are at most j^j partitions; their factorial coefficients are at most $(j-1)!$. Each product contains at most j factors, all bounded by R, and $R \geq 1$. Hence

$$\left|[\log(-g_r'')]^{(j)}\right| \leq C_j. \tag{13}$$

Every derivative used here is available: its order is at most $j + 2 \leq K - 2r$.

Differentiate (4) j times, double the preceding induction error, and add (13). The resulting bound is

$$\begin{aligned} |g_{r+1}^{(j)} + 2^{r+1} A h^{(j)}| &\leq 2^{r+1} \left\{ \frac{A}{4} h'' + (1 - 2^{-r}) C_j \right\} + C_j \\ &= 2^{r+1} \left\{ \frac{A}{4} h'' + (1 - 2^{-r-1}) C_j \right\}. \end{aligned}$$

This proves (10) at $r+1$. At $r = q$, two controlled derivatives remain because $K - 2q = 2$. Its second-derivative case proves (9). All identities and bounds extend to $v = 0$ by the assumed regularity. The hypotheses are global, so no compactness argument or unstated tail extension is needed. □

Corollary 3 (Powering and an open global jet neighborhood)**.** *For a potential h satisfying* (5)*, every amplitude satisfying* (7) *makes e^{-Ah} have positive, uniformly logarithmically concave iterates through depth q. The same holds for every initial logarithm satisfying* (8)*.*

Proof. For the exact power take $g_0 = -Ah$, whose initial errors vanish. The general assertion is Theorem 2. □

The theorem allows vanishing and sign-changing higher derivatives. For example, $h(v) = v^2$ satisfies the hypotheses with all $R_i = 0$. So does $h(v) = v^2 + \eta \sin v$ for $0 < \eta < 2$: its curvature is at least $2 - \eta$, and every derivative of order at least three has absolute value at most η. These examples clarify the role of curvature normalization in (8).

For the admissible realization below, the relevant potential is

$$h(v) = \cosh \sqrt{v} = \sum_{n \geq 0} \frac{v^n}{(2n)!}. \tag{14}$$

It is real analytic at the endpoint. Its positive series gives

$$h'' \geq \frac{1}{12}, \qquad 0 < h^{(j)} \leq \frac{12 j!}{(2j)!} h'' \leq h'' \quad (j \geq 2). \tag{15}$$

To verify the comparison, the coefficient of v^n in $h^{(j)}$ is $(n+j)!/[n!(2n+2j)!]$. For $j \geq 2$ its ratio to the coefficient in h'' is

$$\prod_{k=3}^{j} \frac{1}{2(2n+2k-1)} \leq \prod_{k=3}^{j} \frac{1}{2(2k-1)} = \frac{12j!}{(2j)!},$$

with the empty product equal to one. Thus one may use $R_i = 12(i+2)!/(2i+4)!$ in Theorem 2. In particular, the stronger relative initial error $|g_0^{(j)} + Ah^{(j)}| \leq Ah^{(j)}/4$ implies (8). The construction below pays this estimate on the entire half-line.

The dependence on prescribed finite depth must be retained. There are simple infinite-depth examples in other settings: for instance $f_0(v) = e^{-Ae^v}$ has

$$f_r(v) = A^{2^r-1}2^{2^r-r-1}\exp\{(2^r-1)v - 2^rAe^v\} \qquad (r \geq 0).$$

Direct substitution in (4) verifies this formula. The example illustrates the iteration in its stated setting; the admissible Fourier realization in Theorem 2 is constructed at each prescribed finite depth.

3 Intensity-aware atomic lifts at arbitrary finite depth

The finite-atomic construction in [1] imposed two logarithmic shape levels at a fixed finite derivative budget. We now allow an arbitrary prescribed finite depth. The additional intensity parameter supplies the margin needed to pay every logarithmic-curvature correction in that finite hierarchy. All estimates below are global on the real half-line; no compact perturbation argument is used to leave the tails untreated.

Let

$$\mu = \sum_{j=1}^{n} w_j \delta_{x_j}$$

be a nonzero positive even finite atomic measure. Combine coincident atoms, so the locations are distinct and every $w_j > 0$. Set

$$D = \max_j x_j - \min_j x_j, \qquad Q = \frac{\max_j w_j}{\min_j w_j}, \qquad T = \binom{n}{2} Q.$$

For $A, L > 0$ define

$$\phi_{A,L}(s) = \sum_{j=1}^{n} w_j \exp\left[-A\cosh\left(\frac{s-x_j}{L}\right)\right], \qquad \Delta = \frac{D}{L}. \tag{16}$$

The intensity is A; L controls the width relative to the atom span. Neither parameter changes the atomic Fourier multiplier.

For a fixed integer $q \geq 1$, put

$$\begin{aligned} K &= 2q+2, \qquad m = 2K, \qquad \alpha_j = \frac{j!}{(2j)!}, \qquad C_j = j^j(j-1)! \quad (j \geq 2), \\ M_r &= 2^r\{2(r-1)^{r-1} + 2^r\} \quad (2 \leq r \leq m), \qquad B_m = m^{2m}(m-1)!M_m. \end{aligned} \tag{17}$$

These are sufficient budgets, not optimized constants.

Theorem 4 (Atomic lifting at every prescribed finite depth)**.** *Choose*

$$A \geq \max\left\{1,\ 4 \max_{2\leq j\leq K-2} \frac{C_j}{\alpha_j}\right\}. \tag{18}$$

For $n \geq 2$, choose

$$L \geq \max\{1,\ 16D,\ AD,\ 16B_mTD,\ 10AD(T+1)\}. \tag{19}$$

Then $\phi_{A,L}$ is positive, even and entire, strictly decreasing on $(0,\infty)$, and

$$(\log \phi_{A,L})''(s) < -\frac{9A}{10L^2} \qquad (s \in \mathbb{R}). \tag{20}$$

For every fixed derivative order $k \geq 0$ and every $\varepsilon > 0$,

$$\phi_{A,L}^{(k)}(s) = O\big(e^{-|s|^{2+\varepsilon}}\big) \qquad (|s| \longrightarrow \infty).$$

In particular the kernel meets the full super-Gaussian admissibility condition.

In the normalized square coordinate define

$$f_0(v) = \phi_{A,L}(L\sqrt{v}), \qquad f_{r+1}(v) = f_r'(v)^2 - f_r(v)f_r''(v) \quad (0 \leq r < q). \tag{21}$$

Each f_r is positive for $v \geq 0$, and, with $g_r = \log f_r$,

$$g_r''(v) \leq -\frac{2^{r-1}A}{12} \qquad (v \geq 0,\ 0 \leq r \leq q). \tag{22}$$

All values at $v = 0$ are the removable real-analytic values. For a single even atom, $x_1 = 0$ and $D = T = 0$; the same conclusions hold for every $L > 0$ at an intensity satisfying (18)*.*

The theorem asserts one construction for each prescribed finite q. Intensity and width may depend on depth. The Fourier consequences of retaining the multiplier are treated separately.

3.1 Global centered moments with intensity

Write $t = s/L$, $d_j = x_j/L$ and

$$U_j(t) = A\cosh(t - d_j), \qquad V_j(t) = A\sinh(t - d_j), \qquad r_j(t) = \frac{w_j e^{-U_j(t)}}{\sum_i w_i e^{-U_i(t)}}.$$

Expectations here use these actual t-dependent probabilities. Put $\ell(t) = \log \phi_{A,L}(Lt)$. Exact differentiation gives

$$\ell'' = -\mathbb{E}U + \operatorname{Var} V, \qquad \operatorname{Var} V = \sum_{i<j} r_i r_j (V_i - V_j)^2. \tag{23}$$

For $U_{\min} = \min_j U_j \geq A$ and $u_{ij} = |U_i - U_j|$,

$$U_j \leq e^{\Delta} U_{\min}, \quad u_{ij} \leq (e^{\Delta} - 1)U_{\min}, \quad r_i r_j \leq Q e^{-u_{ij}}. \tag{24}$$

Indeed $\cosh(a+b) \leq e^{|b|}\cosh a$. For the last inequality, if $U_i \geq U_j$, bound the squared normalization denominator below by $w_j^2 e^{-2U_j}$; the opposite order is identical. The hyperbolic pair identity is

$$(V_i - V_j)^2 = (U_i - U_j)^2 + 4A^2 \sinh^2((d_i - d_j)/2).$$

Since $u^2e^{-u} \le u/e$ and $\sum_{i<j} r_i r_j \le 1/2$,

$$\operatorname{Var} V \le \mathcal{C}\mathbb{E}U, \qquad \mathcal{C} = \frac{T}{e}(e^{\Delta}-1) + 2A\sinh^2(\Delta/2). \tag{25}$$

The factor A in the second term is important: the constant offset $2A^2\sinh^2(\Delta/2)$ is absorbed using $\mathbb{E}U \ge A$.

Under (19), $\Delta \le 1/[10A(T+1)]$ and $\Delta \le 1/16$. The elementary estimates $e^{\Delta}-1 \le 2\Delta$ and $2\sinh^2(\Delta/2) \le \Delta^2$ therefore imply

$$\mathcal{C} < T\Delta + A\Delta^2 \le \frac{T}{10(T+1)} + \frac{1}{100(T+1)^2} < \frac{1}{10}.$$

Equations (23)–(25) give (20).

Lemma 5 (Uniform even logarithmic jets)**.** *Suppose $A \ge 1$, $0 \le \Delta \le 1/4$ and $A\Delta \le 1$. For every even integer k with $2 \le k \le m$,*

$$|\ell^{(k)}(t) + \mathbb{E}U(t)| \le B_m T\Delta\, \mathbb{E}U(t) \qquad (t \in \mathbb{R}). \tag{26}$$

Proof. For any component-index random variable X and integer $r \ge 2$, conditional Jensen yields

$$\mathbb{E}|X - \mathbb{E}X|^r \le \sum_{i,j} r_i r_j |X_i - X_j|^r.$$

Set $b = 2A\sinh(\Delta/2) \le 2A\Delta$. The pair identity gives $|V_i - V_j| \le u_{ij} + b$. The intensity restriction pays the offset at every required order:

$$b^r \le 2^r(A\Delta)^r \le 2^r A\Delta \le 2^r\Delta\mathbb{E}U.$$

Also

$$u^r e^{-u} \le \left(\frac{r-1}{e}\right)^{r-1} u \le (r-1)^{r-1}u.$$

Use (24), the ordered-pair factor two and $(u+b)^r \le 2^{r-1}(u^r + b^r)$. For $2 \le r \le m$ this gives

$$\begin{aligned}\mathbb{E}|V - \mathbb{E}V|^r &\le 2^r T\{(r-1)^{r-1}(e^{\Delta}-1)U_{\min} + b^r\} \\ &\le M_r T\Delta\mathbb{E}U.\end{aligned}$$

The U-centered moment is at most $4T(r-1)^{r-1}\Delta\mathbb{E}U$, and is bounded by the same expression. These estimates hold on the entire real line.

To control mixed cumulants without creating a power of $\mathbb{E}U$, let $X_1, \dots, X_r$ each be a signed U or V, put $Y_i = X_i - \mathbb{E}X_i$ and $H_r = M_r T\Delta\mathbb{E}U$. For a block C of a set partition of $\{1, \dots, r\}$, generalized Hölder, with the constant function as an additional factor when $|C| < r$, gives

$$\left|\mathbb{E}\prod_{i\in C} Y_i\right| \le \prod_{i\in C}(\mathbb{E}|Y_i|^r)^{1/r} \le H_r^{|C|/r}.$$

The product over the blocks is at most H_r, even when $H_r < 1$. Cumulants of order at least two are unchanged by centering. Their partition formula, with coefficients $(|\pi|-1)!$, hence yields

$$|\operatorname{cum}_r(X_1, \dots, X_r)| \le r^r(r-1)!M_r T\Delta\mathbb{E}U. \tag{27}$$

Singleton blocks vanish; retaining them in this upper count only enlarges the bound.

Let $a_j(t) = -U_j(t)$. At a fixed real t, the local finite-jet identity

$$\ell(t+z) - \ell(t) = \log \mathbb{E}_t \exp\left(\sum_{d\geq 1} a^{(d)}(t)\frac{z^d}{d!}\right)$$

implies the exact derivative formula

$$\ell^{(k)}(t) = \sum_{\pi\in\Pi_k} \mathrm{cum}_{|\pi|}\big(a^{(|C_1|)}, \dots, a^{(|C_{|\pi|}|)}\big),$$

where Π_k denotes labeled set partitions and cum_1 is expectation. Thus all derivatives of the tilted probabilities are included. The component derivatives alternate between $-U$ and $-V$. For even k, the unique one-block term is $-\mathbb{E}U$. Every other term is bounded by (27). For $r \leq k \leq m$, use $r^r \leq m^m$, $(r-1)! \leq (m-1)!$, $M_r \leq M_m$ and $|\Pi_k| \leq m^m$. This proves (26) with the complete budget B_m. Only local analyticity of the real logarithm is used; a uniformly zero-free complex strip for that logarithm is unnecessary. □

3.2 The positive change of coordinate

For every smooth even function ℓ, $v = t^2 > 0$ and $j \geq 1$,

$$\frac{d^j}{dv^j}\ell(\sqrt{v}) = \frac{1}{2^{2j-1}(j-1)!}\int_0^1 (1-z^2)^{j-1}\ell^{(2j)}(z\sqrt{v})\,dz. \tag{28}$$

For $j = 1$ this follows from $\ell'(0) = 0$ and the fundamental theorem of calculus. Differentiating in $v = t^2$ and integrating by parts using $[(1-z^2)^j]' = -2jz(1-z^2)^{j-1}$ multiplies the coefficient by $1/(4j)$. The boundary at one vanishes by the weight, and that at zero by evenness. This proves the induction. In the real-analytic case its removable endpoint value is

$$\frac{d^j}{dv^j}\ell(\sqrt{v})\bigg|_{v=0} = \frac{j!}{(2j)!}\ell^{(2j)}(0).$$

Put $h(v) = \cosh\sqrt{v}$. Its series gives

$$h^{(j)}(v) = \alpha_j\, {}_0F_1(;j+1/2;v/4), \qquad {}_0F_1(;b;z) = \sum_{d\geq 0}\frac{z^d}{d!(b)_d}, \tag{29}$$

where $(b)_d = b(b+1)\cdots(b+d-1)$ and $(b)_0 = 1$. All coefficients are positive. For $j \geq 2$ and $v \geq 0$,

$$h^{(j)}(v) \geq \alpha_j, \qquad \frac{h^{(j)}(v)}{h''(v)} \leq 12\alpha_j. \tag{30}$$

Indeed the hypergeometric factor in (29) decreases coefficientwise as its second parameter increases; $\alpha_2 = 1/12$.

Evenness centers the extreme atoms, so $|d_j| \leq \Delta/2$ and

$$e^{-\Delta/2}A\cosh t \leq \mathbb{E}U(t) \leq e^{\Delta/2}A\cosh t.$$

The width budget gives $\delta = B_m T\Delta \leq 1/16$ and $\Delta \leq 1/16$. Apply (28) to the even jets of Lemma 5. Its kernels are nonnegative, so for $g_0(v) = \log\phi_{A,L}(L\sqrt{v})$ and $2 \leq j \leq K$,

$$(1-\delta)e^{-\Delta/2}Ah^{(j)}(v) \leq -g_0^{(j)}(v) \leq (1+\delta)e^{\Delta/2}Ah^{(j)}(v).$$

Here $e^{\Delta/2} \le 1+\Delta$, and $e^{-\Delta/2} \ge 1-\Delta/2$. Since $(17/16)^2 < 5/4$ and $(31/32)(15/16) > 3/4$, we obtain the global initial comparison

$$|g_0^{(j)} + Ah^{(j)}| \le \frac{A}{4} h^{(j)} \qquad (2 \le j \le K). \tag{31}$$

The endpoint $v = 0$ follows by continuity. This compares logarithmic derivatives; an undifferentiated relative comparison of the mixture with the centered base would fail in the far tails.

3.3 Propagation through the finite hierarchy

Proof of Theorem 4. The ordinary curvature has already been proved. Positivity and the entire extension follow from the finite sum in (16). Evenness gives $(\log \phi_{A,L})'(0) = 0$; strictly negative curvature implies strict decrease on $(0, \infty)$. Every fixed derivative is a finite sum of a polynomial in $e^{s/L}, e^{-s/L}$ times a shifted cosh exponential. Its double-exponential tails prove the stated derivative decay, for each fixed order and each fixed $\varepsilon > 0$. This also pays all finite real differentiations used above.

We prove inductively that, whenever $0 \le r \le q$ and $2 \le j \le K - 2r$,

$$|g_r^{(j)} + 2^r A h^{(j)}| \le 2^r \left\{ \frac{A}{4} h^{(j)} + (1 - 2^{-r}) C_j \right\}. \tag{32}$$

At $r = 0$ this is (31). At every positive level all available derivative orders are at most $K - 2$. The intensity bound and (30) show that the error in (32) is at most $2^{r-1} A h^{(j)}$. Hence

$$\frac{1}{2}\, 2^r A h^{(j)} \le -g_r^{(j)} \le \frac{3}{2}\, 2^r A h^{(j)}. \tag{33}$$

For $r = 0$, the stronger factors $3/4$ and $5/4$ from (31) imply (33) at every available order, including $K - 1$ and K. Thus no intensity bound involving C_{K-1} or C_K is required.

Suppose $r < q$. In particular $g_r'' < 0$, so $f_{r+1} = -f_r^2 g_r'' > 0$ and

$$g_{r+1} = 2g_r + \log(-g_r''). \tag{34}$$

For $1 \le i \le K - 2r - 2$, equations (30) and (33) give

$$\left| \frac{g_r^{(i+2)}}{g_r''} \right| \le 3 \frac{h^{(i+2)}}{h''} \le 36 \frac{(i+2)!}{(2i+4)!} \le 1.$$

The last sequence starts at $3/10$ for $i = 1$ and decreases. For $a = -g_r'' > 0$, the exact logarithmic derivative formula is

$$(\log a)^{(j)} = \sum_{\pi \in \Pi_j} (-1)^{|\pi|-1} (|\pi| - 1)! \prod_{C \in \pi} \frac{a^{(|C|)}}{a}.$$

Every factor is bounded in modulus by one. Consequently

$$|[\log(-g_r'')]^{(j)}| \le \sum_{\pi \in \Pi_j} (|\pi| - 1)! \le j^j (j-1)! = C_j \quad (2 \le j \le K - 2r - 2). \tag{35}$$

Using (34), the new error is at most twice the preceding error plus C_j. This is precisely (32) at $r + 1$, because $2(2^r - 1)C_j + C_j = (2^{r+1} - 1)C_j$. The induction reaches $r = q$ with derivative order two still available. Equation (33) and $h'' \ge 1/12$ give (22).

For one atom, evenness forces its location to be zero. All centered moments vanish and g_0 is a constant minus Ah; thus (31) holds with zero error for every $L > 0$. The same induction proves the hierarchy, while the spatial curvature is directly at most $-A/L^2$. □

The functions in (21) are real-analytic at zero: the original kernel is even and entire, so f_0 is entire in v, and the polynomial differential iteration preserves that property. Their real logarithms are legitimate on the entire closed half-line by the positivity just proved.

In the unnormalized square coordinate $u = s^2$, let $\widetilde{f}_0(u) = \phi_{A,L}(\sqrt{u})$ and apply the same iteration using u derivatives. Then

$$\widetilde{f}_r(u) = L^{-4(2^r-1)} f_r(u/L^2).$$

This follows inductively from the two derivatives in each Laguerre operation. Thus every positivity statement is unchanged and

$$(\log \widetilde{f}_r)''(u) \le -\frac{2^{r-1}A}{12L^4}.$$

The construction pays finitely many logarithmic jets through order $m = 4q+4$ in the original scaled spatial variable. Its new ingredient is that a large leading intensity makes the logarithmic-curvature corrections uniformly bounded at each finite step, while the negative leading derivatives grow as $2^r A$. It does not presume an infinite-depth property of the unamplified base.

4 A positive carrier and a whole initial parameter interval

The finite-cutoff multiplier below is based on the six-atom construction in [2, §4.1]. A larger carrier also pays the whole physical interval.

Lemma 6. *Fix $k \ge 1$ and $0 < Y < H$. Put*

$$c = 17/16, \quad \beta_0 = \operatorname{arcosh} c, \quad b = (Y+H)/2, \quad \lambda = \beta_0/b, \quad V = \lambda Y,$$

and define

$$d = c - \cosh V > 0, \qquad C = \frac{2c \sinh V}{V d^2}, \qquad M \in \mathbb{N}, \quad M > \max\{4k, CV, \sqrt{C}, 2\}. \tag{36}$$

Then

$$p(z) = \frac{[c + \cos(\lambda z)]\cos(M\lambda z)}{c+1} \tag{37}$$

is the Fourier transform of a positive even six-atom probability measure. It has $L_n[p](x) \ge 0$ for all real x and $1 \le n \le k$, and

$$\frac{\partial_y \log |p(x+iy)|^2}{y} \ge \lambda^2 \epsilon > 0 \quad (0 < y \le Y), \qquad \epsilon = \frac{2M^2}{1+MV} - C. \tag{38}$$

Its nonreal zeros have heights exactly $\pm b$; its other zeros are real.

Proof. The atoms are $\pm M\lambda$ with individual weights $c/[2(c+1)]$ and $\pm(M-1)\lambda, \pm(M+1)\lambda$ with individual weights $1/[4(c+1)]$. They are distinct, their weights sum to one, and their span and pair budget are

$$D = 2\lambda(M+1), \qquad T = \binom{6}{2}\frac{w_{\max}}{w_{\min}} = 255/8. \tag{39}$$

The zeros of $c + \cos z$ are $(2j+1)\pi \pm i\beta_0$.

Before the λ rescaling let $v = \cos x$. The coefficients of y^{2n} in $|c + \cos(x+iy)|^2$ are

$$h_0 = (c+v)^2, \qquad h_n = \frac{2cv + 2^{2n-1}}{(2n)!} \quad (n \ge 1).$$

Thus $h_0 \geq 1/256$, $h_1 \geq -1/16$ and $h_n \geq 0$ for $n \geq 2$. For $|\cos(M(x+iy))|^2$ the coefficients are $Q_0 = \cos^2(Mx) \in [0,1]$ and $Q_n = (2M)^{2n}/[2(2n)!]$ for $n \geq 1$. At $n = 1$ the product coefficient is at least $M^2/256 - 1/16 > 0$. For $2 \leq n \leq k$ its only potentially negative summand is $h_1 Q_{n-1}$, and it is paid by $h_0 Q_n$, since

$$\frac{1}{16}\frac{Q_n}{Q_{n-1}} = \frac{M^2}{8n(2n-1)} > 1.$$

Normalization and dilation preserve these signs.

Write $Q(z) = c + \cos z$. For $0 < y \leq V$, $|Q(x+iy)| \geq c - \cosh y \geq d$. Direct differentiation gives

$$\partial_y \log |Q(x+iy)|^2 = \frac{2\sinh y(c\cos x + \cosh y)}{|Q(x+iy)|^2} \geq -Cy.$$

Here $\sinh y/y \leq \sinh V/V$ and the negative part of the numerator's parenthesis is at most c. Also

$$\partial_y \log |\cos(M(x+iy))|^2 = \frac{2M\sinh(2My)}{\cosh(2My) + \cos(2Mx)} \geq 2M\tanh(My) \geq \frac{2M^2 y}{1+My}.$$

The last inequality follows from $e^{2u} \geq 1 + 2u$ for $u \geq 0$. Our strict choices give $M^2 > C$ and $M^2 > CMV$, hence $2M^2 > C(1+MV)$ and $\epsilon > 0$. Rescaling $z \mapsto \lambda z$ gives (38), including its factor λ^2. □

5 Fourier inheritance and completion of the construction

Lemma 7. *For $A, L > 0$,*

$$B(z) = \int_{\mathbb{R}} e^{-A\cosh(s/L)} e^{izs}\, ds = 2LK_{iLz}(A)$$

is real even entire of exact order one, with only real zeros. Consequently $L_n[B](x) \geq 0$ for every n and every real x, and $\partial_y \log |B(x+iy)|^2 \geq 0$ for $y > 0$.

Proof. The integral is the standard modified Bessel representation; its real-zero property is classical [4]. For completeness, if $K_{iLz}(A) = 0$, then $w(t) = K_{iLz}(Ae^t)$, $t \geq 0$, satisfies

$$-w'' + A^2 e^{2t} w = L^2 z^2 w, \qquad w(0) = 0.$$

The large-argument Bessel asymptotic pays w, w' at infinity and ensures $w \not\equiv 0$. Multiplying by $\overline{w}$ and integrating gives

$$L^2 z^2 \int_0^\infty |w|^2 dt = \int_0^\infty \big(|w'|^2 + A^2 e^{2t}|w|^2\big) dt > 0.$$

Thus z is real. The same integral bounds used below show exact order one. Evenness and Hadamard factorization yield

$$B(z) = B(0)\prod_{\rho>0}(1 - z^2/\rho^2), \qquad \sum_{\rho>0}\rho^{-2} < \infty,$$

with multiplicities. Each factor's squared modulus at $x+iy$ is a polynomial in y^2 with nonnegative coefficients. Locally uniform convergence passes each coefficient. For $y > 0$ each real zero contributes $2y/[(x-\rho)^2 + y^2]$ to the vertical logarithmic derivative, which proves the last assertion by locally normally convergent paired sums. □

We record the needed transform and polynomial-density details to make clear which infinite-order statements are being used.

Lemma 8. *For any finite positive even cosh mixture $\phi(s) = \sum w_j e^{-A\cosh((s-x_j)/L)}$, its transform is $F = Bp$, where $p(z) = \sum w_j e^{ix_j z}$. It is real even entire of exact order one. Formula* (3) *holds; each fixed-y density g_y is real even, continuous and exponentially decreasing in $|x|$. If g_y has a negative open interval, it has a negative polynomial square in each parity.*

Proof. Translation gives the factorization. Since $\cosh(u/L) \geq e^{|u|/L}/2$, bounding the integral on $|z| \leq R$ and substituting $v = e^{|u|/L}$ gives $\log\max|F| = O(R\log(R+2))$; shifts contribute only $O(R)$. Conversely restrict the positive integral $F(iR)$ to a unit interval near $s = -L\log R$. One summand there is at least $w_j e^{-c_j R}$, and $e^{-Rs} \geq e^{LR\log R}$ after choosing the interval on the negative side. Hence $\log F(iR) \geq LR\log R - O(R)$ and the order is exactly one. The same argument applies to B.

Define $C_y(t) = \int_0^\infty \cosh(2ys)\phi(s+t/2)\phi(s-t/2)ds$. Using $a = s+t/2, b = s-t/2$, with absolute Jacobian one, and the symmetry under $(a,b) \mapsto (-a,-b)$, gives $\widehat{C}_y(x) = |F(x+iy)|^2/2$. Since $\partial_y C_y = 4yJ_y$, (3) follows. Expanding at $y = 0$ gives its continuous endpoint $\mathcal{L}_1[F]/4$. Double-exponential decay pays differentiation, Fubini and every polynomial moment.

For exponential Fourier decay shift the s contour by $\pm i\pi L/4$ with the sign of x. The real part of each shifted cosh is $\cosh((s-x_j)/L)/\sqrt{2}$. This pays the complete shifted integral and the polynomial factors for any fixed derivatives of F, giving $|F^{(j)}(x+iy)| \leq C_{j,y}e^{-\pi L|x|/4}$ for fixed j, y. Vertical sides vanish by the same estimate. The physical identity therefore gives an exponential bound for g_y, including the endpoint.

Put $d\nu = |g_y(x)|dx$. Polynomials are dense in $L^2(\nu)$: if h is orthogonal to all polynomials, Cauchy–Schwarz gives a finite signed measure $h\,d\nu$ with an exponential moment at a smaller rate. Its Fourier transform is analytic in a strip and all derivatives at zero vanish, so it and then the measure are zero. For a real even measure, even and odd projection show density in each parity subspace. Take a nonzero compactly supported function in a negative interval and its reflection, of either parity. Its signed squared integral is negative. Polynomial approximation retains the strict sign since

$$\left|\int \operatorname{sign}(g_y)(P^2 - f^2)d\nu\right| \leq \|P-f\|_{L^2(\nu)}\big(\|P\|_{L^2(\nu)} + \|f\|_{L^2(\nu)}\big).$$

This supplies both parities at finite degrees, without a numerical sample. □

Proof of Theorem 1. Choose the carrier by Lemma 6, with data (39), and choose A, L from Theorem 4 at depth q. The resulting ϕ supplies part 1, including its u-coordinate curvature floor. Its Fourier transform is $F = Bp$. Lemma 7 and coefficient multiplication in (1) give

$$L_n[F] = \sum_{j=0}^{n} L_j[B]L_{n-j}[p] \geq 0 \quad (n \leq k).$$

At $x = 0$, every nonzero summand from even derivatives in the formula for $L_n[F](0)$ is positive, since the even moments of ϕ are positive. Thus $L_n[F](0) > 0$ for every n. Exponential decay gives finite moments, and every nonzero polynomial is nonzero off a finite set. This proves the strict Gram assertion in part 2.

For $0 < y \leq Y$, neither B nor p has a zero and

$$g_y(x) = \frac{|F(x+iy)|^2}{8y}\partial_y \log|F(x+iy)|^2 \geq \frac{\lambda^2\epsilon}{8}|F(x+iy)|^2 > 0.$$

At $y = 0$, continuity gives $g_0 \geq 0$, while $F(0) > 0, F'(0) = 0, F''(0) < 0$ give $g_0(0) > 0$. This proves part 3 at every degree and every parameter in the closed interval.

All zeros added by B are real; p has nonreal heights exactly $\pm b$, so all zeros lie in the required strip. At $z_0 = \pi/\lambda + ib$ let the zero multiplicity of F be m. Writing $F(z) = (z - z_0)^m U(z)$ with $U(z_0) \neq 0$ shows

$$\partial_y |F(\operatorname{Re} z_0 + iy)|^2 = 2m(y - b)^{2m-1} |U(z_0)|^2 + O((y - b)^{2m}) < 0$$

just below b. Choose such a parameter above Y; continuity gives an open negative band and Lemma 8 supplies negative squares in both parities. Strictness persists after rational approximation of the finitely many coefficients and then a small change to a rational parameter $y_* \in (Y, b)$. This completes part 4. □

6 Scope of the separation

The same example meets all stated finite diagnostics simultaneously. Increasing a cutoff or depth changes its parameters; no passage to an infinite hierarchy for one fixed kernel is justified. The continuous iteration is a condition on $\phi(\sqrt{u})$, the L_n are conditions on its Fourier transform, and the physical moment degree is a third index. Keeping these indices separate is necessary to interpret the conclusion.

The abstract stability theorem allows higher potential derivatives to vanish or change sign. The cosh realization additionally retains an explicit Fourier factor with prescribed nonreal zeros. It is this realization and the complete global estimates that give the admissible separation theorem. The separation theorem concerns the constructed admissible kernels.

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References

[1] S. Schubr, *Admissible kernel lifts with positive endpoint forms and nonreal Fourier zeros*, Luminary Research Archive, edition 1, October 24, 2026, https://research.luminaryoddities.com/admissible-kernel-lifts-positive-endpoint-forms-nonreal-fourier-zeros/v/1.

[2] teal-sea/zeta-lab, *Theta sums and Laguerre inequalities: first checkpoint*, hunts/rh_theta_laguerre/RESULTS.md, Section 4.1, repository commit ef4e0564349c19c9e6ca91e78ebe3e4f5e224997, October 3, 2026, https://github.com/teal-sea/zeta-lab/blob/ef4e0564349c19c9e6ca91e78ebe3e4f5e224997/hunts/rh_theta_laguerre/RESULTS.md.

[3] G. Csordas, *Fourier transforms of positive definite kernels and the Riemann ξ-function*, Comput. Methods Funct. Theory **15** (2015), 373–391, doi:10.1007/s40315-014-0105-8; https://arxiv.org/abs/1309.0055.

[4] G. Gasper, *Using integrals of squares of certain real-valued special functions to prove that the Pólya $\Xi^*(z)$ function, the functions $K_{iz}(a)$, $a > 0$, and some other entire functions have only real zeros*, arXiv:0801.2996, version 1 (2008), https://arxiv.org/abs/0801.2996v1.

[5] Y. Liu and J. Mao, *Infinite log-concavity of the Taylor coefficients of the Riemann xi-function*, arXiv:2610.04972, version 1, October 4, 2026, https://arxiv.org/abs/2610.04972v1.

[6] M. H. Y. Xie, C. X. T. Zhang and P. B. Zhang, *A unified approach to infinite log-concavity of combinatorial sequences*, arXiv:2610.12237, version 1, October 8, 2026, https://arxiv.org/abs/2610.12237v1.

[7] T. Craven and G. Csordas, *Iterated Laguerre and Turán inequalities*, J. Inequal. Pure Appl. Math. **3** (2002), no. 3, Article 39; author manuscript at https://math.hawaii.edu/~tom/mathfiles/iteratedLT.pdf.