Geometry and Concentration in Polynomial Sign Recovery

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Abstract

We study the degree needed for polynomial squares to recover negative directions of a signed measure. For a fixed number of negative atoms collapsing at arbitrary relative rates, we determine the Gaussian minimax degree of every fixed negative index from the corresponding Vandermonde eigenvalue, with leading coefficient $1/(8cd^2)$. A discrete monic minimax formulation identifies the separate geometric scales, and an exponential-tail counterpart determines their logarithmic recovery exponent. A Gaussian extension permits a shrinking negative probability measure with continuous or mixed support. Without a support gap, a local mass bound of order Rr^q yields sharp degree orders $R^{p/[q(p-1)]}$ for tails $e^{c|x|^p}$ with $p \geq 2$, and a sharp logarithmic degree order $R^{1/q}$ at $p = 1$, for every $q > 0$. We give constructive upper bounds for $1 < p < 2$. Classical orthogonal-polynomial and Christoffel estimates are separated from the signed-measure minimax deductions; exact rational finite controls accompany the proofs.

1 Introduction

A negative atom can be hidden from low-degree polynomial tests by a positive or signed background. Tail decay controls how far a polynomial can localize without excessive cost. The geometry of several negative atoms controls how many independent directions can be recovered at that cost. If background mass is allowed arbitrarily close to a negative atom, a local nonconcentration condition becomes a separate resource.

This paper develops these two mechanisms. Section 2 resolves every fixed negative index in a finite collapsing packet, including unequal collision scales, and extends the Gaussian result to a negative probability measure. Section 4 replaces the gap by a local mass bound and proves the stated sharp recovery orders. Statements are minimax assertions over explicitly budgeted measure classes. They are not uniform asymptotics for every individual background.

The predecessors [4, 3] establish sharp single-defect and collision barriers and the subexponential masking construction. We retain their normalization of ordinary polynomial degree: a degree-N certificate is the square of a polynomial of degree at most N. The present geometric contribution is the index-by-index dependence on the actual packet, rather than a single separation statistic or only the last negative direction. The local-mass contribution concerns a different admissible class and includes a critical exponential lower bound for all real $q > 0$. Orthogonal-polynomial identities, reproducing-kernel variational principles, and established Christoffel estimates are classical inputs.

2 Exact geometry and recovery of each negative index

For a real signed measure with all absolute moments, let

$$\mathcal{Q}_\mu(P) = \int P^2 \, \mathrm{d}\mu, \qquad n_s(\mu) = \inf\{N : \mathrm{ind}_-(\mathcal{Q}_\mu|_{\mathcal{P}_N}) \geq s\}.$$

Here $\mathcal{P}_N$ denotes real polynomials of ordinary degree at most N, $\inf \varnothing = \infty$, and negative index counts strictly negative directions. In particular, a null square is not a negative direction. All recovery statements below concern the original measure and its complete square form, including mixed polynomial terms.

Fix $k \geq 2$, $s \in \{2, \dots, k\}$, $m = s - 1$, and $B, c, d > 0$. For a vector of distinct nodes $b = (b_1, \dots, b_k) \in [-B, B]^k$, write

$$V(b) = (b_i^j)_{\substack{1 \leq i \leq k, \\ 0 \leq j < k}}, \qquad \lambda_1(b) \geq \cdots \geq \lambda_k(b) > 0$$

for the decreasing eigenvalues of $V(b)V(b)^{\mathsf{T}}$. The monomial basis in this definition is fixed. An equivalent geometry statistic is

$$t_j(b) = \min_{\substack{A \text{ monic} \\ \deg A = j}} \max_{1 \leq i \leq k} |A(b_i)|, \qquad 0 \leq j < k. \tag{1}$$

In particular $t_0 = 1$ and $t_j > 0$. We will use both descriptions, proving their uniform equivalence rather than assuming that a small minimum separation describes every recovery stage.

The Gaussian exact-node class $\mathcal{G}(R; b)$ consists of

$$\mu = -\frac{1}{k} \sum_{i=1}^{k} \delta_{b_i} + \nu, \qquad R \geq 1, \qquad \operatorname{supp} \nu \subset \bigcap_i \{|x - b_i| \geq d\}, \tag{2}$$

$$\|\nu\|_{\mathrm{TV}} \leq R, \qquad \int e^{cx^2} \, \mathrm{d}|\nu| \leq CR, \qquad C > e^{cd^2}. \tag{3}$$

Let $D_s^G(R; b) = \sup_{\mu \in \mathcal{G}(R;b)} n_s(\mu)$. For the exponential class $\mathcal{E}(R; b)$ we instead use

$$\mu = -\frac{1}{k} \sum_{i} \delta_{b_i} + \nu, \quad \operatorname{supp} \nu \subset \bigcap_i \{|x - b_i| \geq d\}, \quad \int e^{c|x|} \, \mathrm{d}|\mu| \leq C(1 + R), \qquad C > e^{cd}, \tag{4}$$

and put $D_s^E(R; b) = \sup_{\mu \in \mathcal{E}(R;b)} n_s(\mu)$. There is no additional remainder-mass condition in (4). Both upper classes allow signed ν. These two budget conventions are different and are retained throughout the proofs.

Theorem 2.1 (Exact-node recovery hierarchy)**.** *Fix the declared parameters and $s \in \{2, \dots, k\}$. Along every sequence of distinct node vectors with $\max_i |b_i| \to 0$, and arbitrary jointly varying $R \geq 1$,*

$$\frac{D_s^G(R; b)}{[\log(R/\lambda_s(b))]^2} \longrightarrow \frac{1}{8cd^2}, \tag{5}$$

$$\frac{\log D_s^E(R; b)}{\log(R/t_{s-1}(b)^2)} \longrightarrow \frac{\pi}{2cd}. \tag{6}$$

In (5) one may replace λ_s by t_{s-1}^2; in (6) one may replace t_{s-1}^2 by λ_s. For each lower bound, a positive reference law can be fixed before the contrast and node geometry vary within its stated small interval.

Thus no fixed scaled cluster shape or bound on relative gap ratios is required. The result is a minimax at the exact nodes, not an asymptotic for every individual background. The convergence holds for fixed k and s in the stated collapsing-center regime. Since $V(b)$ tends to a rank-one matrix,

$\lambda_s(b) \to 0$ for $s \geq 2$; also $t_{s-1} \leq (\max_i |b_i|)^{s-1}$. All displayed logarithmic scales therefore diverge on the allowed sequences.

The preceding minimum-separation results in [4, 3] concern the last negative direction over a coarse geometry class. Theorem 2.1 distinguishes every fixed stage at each exact geometry. Finite-rank kernel formulas, orthogonal-polynomial extremality and clustered Vandermonde spectra are classical subjects; see [5, 10, 8, 9]. The statements here additionally require degree-uniform estimates for the original signed-measure recovery problem. A fixed-degree confluent kernel expansion alone does not provide those estimates.

2.1 The spectral and finite minimax descriptions

Proposition 2.2 (Uniform geometry bridge)**.** *Let $r = \max_i |b_i| \to 0$, with k fixed and the nodes distinct. In the unnormalized inner product $\sum_i P(b_i)Q(b_i)$, let π_j be the monic orthogonal polynomial of degree $j < k$ and $u_j^2 = \sum_i \pi_j(b_i)^2$. Then, uniformly over all relative spacings,*

$$\frac{\lambda_s(b)}{u_{s-1}^2} \longrightarrow 1, \qquad t_j(b) \leq u_j \leq \sqrt{k}\, t_j(b). \tag{7}$$

Consequently $\log(R/\lambda_s) = \log(R/t_{s-1}^2) + O_k(1)$ uniformly for $R \geq 1$ and all sufficiently small r.

Proof. The monic orthogonal polynomial minimizes the discrete square norm. The trial polynomial $x\pi_{j-1}$ gives $u_j \leq r u_{j-1}$. For $r < 1$, the column norms $u_0, \ldots, u_{k-1}$ are therefore decreasing; in this unnormalized convention $u_0 = \sqrt{k}$. Every zero of π_j lies in the convex hull of the nodes, since $j < k$ and the discrete positive inner product is nondegenerate through that degree. This is the usual orthogonality sign-change argument [5, Chapter III].

Let A have as columns the monomial coefficients of $\pi_0, \ldots, \pi_{k-1}$. It is triangular with diagonal one. The coefficient of x^ℓ in π_j, $\ell < j$, has magnitude at most $\binom{j}{\ell} r^{j-\ell}$, by its root factorization. Thus $\|A - I\| = O_k(r)$, uniformly in the spacings, and both $\|A\|$ and $\|A^{-1}\|$ are $1 + O_k(r)$. The columns of VA are orthogonal with lengths u_j. Singular-value product inequalities give

$$\frac{u_{s-1}^2}{\|A\|^2} \leq \lambda_s(VV^{\mathsf{T}}) \leq \|A^{-1}\|^2 u_{s-1}^2.$$

This is a multiplicative comparison even for the smallest eigenvalues. Finally the Euclidean norm of a k-vector lies between its maximum norm and $\sqrt{k}$ times that norm. Apply this to the two monic minimization problems to get the second assertion in (7). □

Only a fixed nonsingular change of the monomial basis is harmless in the limiting logarithms. A singular node-dependent rescaling would change the normalization and is not part of the theorem. For the last index, an especially simple scalar statistic survives. If

$$c_i = \prod_{j \neq i} (b_i - b_j)^{-1}, \qquad S(b) = \sum_i c_i^2,$$

then

$$\lambda_k(b) S(b) \longrightarrow 1 \tag{8}$$

uniformly as $r \to 0$. Indeed, the last row of V^{-1} is $(c_i)_i$, so $\|V^{-1}\|^2 \geq S$. Every lower row is bounded componentwise by $\binom{k-1}{j} r^j |c_i|$ for the corresponding missing degree j. Hence

$$S \leq \lambda_k^{-1} \leq \|V^{-1}\|_F^2 \leq S \left[1 + \sum_{j=1}^{k-1} \binom{k-1}{j}^2 r^{2j} \right].$$

The statistic S describes the last stage, whereas the other λ_s or t_{s-1} are needed for intermediate stages.

2.2 Reference kernels and original negative index

Lemma 2.3 (Finite-section kernel criterion)**.** *Let $G > 0$ be the Gram matrix of a positive reference measure τ on $\mathcal{P}_N$, in any fixed basis. Let E be evaluation at the nodes and let $W = \operatorname{diag}(w_i) > 0$. The original form of $R\tau - \sum_i w_i \delta_{b_i}$ has negative index equal to the number of eigenvalues of*

$$W^{1/2} E G^{-1} E^{\mathsf{T}} W^{1/2}$$

strictly greater than R. Its nullity is the number equal to R.

Proof. The original matrix is $RG - E^{\mathsf{T}} W E$. Congruence by $G^{-1/2}$ gives $RI - B^{\mathsf{T}} B$, with $B = W^{1/2} E G^{-1/2}$. The nonzero eigenvalues of $B^{\mathsf{T}} B$ and $B B^{\mathsf{T}}$ agree. Extra zero eigenvalues do not affect comparisons with $R > 0$. Sylvester's law of inertia proves both statements. This is the usual finite-rank kernel interface; compare [10]. □

Let ρ be a positive probability with all moments and infinite support in $[D^2, \infty)$, and write $K_{\rho,l}(z,z) = \sum_{j=0}^{l} |p_j(z)|^2$ for its orthonormal evaluation kernel. At real points this is the squared evaluation norm:

$$K_{\rho,l}(t,t) = \sup_{0 \neq H \in \mathcal{P}_l} \frac{|H(t)|^2}{\int H^2 \, \mathrm{d}\rho}. \tag{9}$$

Write $K_l = K_{\rho,l}(0,0)$ and let τ_* be the even lift $x = \pm\sqrt{v}$, $v \sim \rho$, with fair signs.

Lemma 2.4 (Uniform evaluation in the reference gap)**.** *For $0 \le t < D^2$,*

$$K_{v\rho,l}(t,t) \le D^{-2} K_{\rho,l}(t,t) \le D^{-2} K_l.$$

For every real $P \in \mathcal{P}_N$ and $|x| \le h < D$,

$$|P(x)|^2 \le (1 + h^2/D^2) K_{\lfloor N/2 \rfloor} \int P^2 \, \mathrm{d}\tau_*. \tag{10}$$

Proof. The first inequality follows from $v \ge D^2$ and (9). All roots r_i of p_j lie in $[D^2, \infty)$. Thus for $0 \le t < D^2$, $|p_j(t)/p_j(0)| = \prod_i (1 - t/r_i) \le 1$, proving the second inequality. Write $P(x) = F(x^2) + xH(x^2)$. Its positive reference norm is $\int F^2 \, \mathrm{d}\rho + \int v H^2 \, \mathrm{d}\rho$. Apply the two evaluation bounds, then Cauchy–Schwarz on this direct sum, to obtain (10). Enlarging the degree of H to $\lfloor N/2 \rfloor$ only enlarges its evaluation bound. □

We will also use two standard Laguerre facts, with their normalization specified:

$$L_n(u) = \sum_{j=0}^{n} \binom{n}{j} \frac{(-u)^j}{j!}, \quad \int_0^\infty L_n L_j e^{-u} \, \mathrm{d}u = \delta_{nj}, \quad |L_n(u)| \le e^{u/2} \quad (u \ge 0). \tag{11}$$

The envelope is the classical Laguerre inequality [2, Eq. 18.14.8]; see also [5]. For each fixed $a > 0$,

$$\log L_n(-a) = 2\sqrt{an} + O_a(\log(n+2)), \tag{12}$$

$$\log \sum_{j=0}^{n} L_j(-a)^2 = 4\sqrt{an} + O_a(\log(n+2)). \tag{13}$$

Here is a short proof of the amount of asymptotics needed. Coefficient positivity and $\binom{n}{j} \le n^j/j!$ bound $L_n(-a)$ by $\sum_{j\ge 0}(na)^j/(j!)^2 \le e^{2\sqrt{na}}$. The last inequality follows by comparing the even coefficients of the exponential and using $\binom{2j}{j} \le 4^j$. For a lower bound retain the term $j = \lfloor\sqrt{na}\rfloor$. For large n, $j \le n/2$ and $\binom{n}{j} \ge n^j e^{-j^2/n}/j!$, since $\log(1-u) \ge -2u$ on $[0, 1/2]$. Stirling's logarithmic formula for this term gives $2\sqrt{na} + O_a(\log(n+2))$. The kernel sum is between its last term squared and $(n+1)e^{4\sqrt{an}}$, proving (13).

2.3 Gaussian upper bound retaining the spectrum

Fix $0 < \gamma < c$ and $0 < r_0 < d/2$. For nodes with $r = \max_i |b_i| \le r_0$, every background point satisfies $|x| \ge d - r$. Use the single common window

$$H_n(x) = L_n(\gamma[x^2 - (d-r)^2]), \qquad a_0 = \gamma(d^2 - 2dr_0) > 0.$$

For $p_u(x) = \sum_{j=0}^{k-1} u_j x^j$, the background envelope gives

$$\int (p_u H_n)^2 \,\mathrm{d}|\nu| \le AR\|u\|_2^2, \quad A = C \sup_x \left(\sum_{j=0}^{k-1} x^{2j}\right) e^{-(c-\gamma)x^2} < \infty. \tag{14}$$

The discarded factor $e^{-\gamma(d-r)^2}$ is at most one. This pays every polynomial multiplier and mixed coefficient term. At every atom, $\gamma[b_i^2 - (d-r)^2] \le -a_0$, so $H_n(b_i) \ge L_n(-a_0) > 0$. With $D_n = \mathrm{diag}(H_n(b_i))$,

$$V^{\mathsf{T}} D_n^2 V \;\ge\; L_n(-a_0)^2 V^{\mathsf{T}} V.$$

Select the top s coefficient eigenspace of the matrix on the left. On that whole space, the original signed form obeys

$$\mathcal{Q}_\mu(p_u H_n) \le \left[AR - \frac{1}{k} L_n(-a_0)^2 \lambda_s(b)\right] \|u\|_2^2.$$

Multiplication by the nonzero polynomial H_n is injective. It therefore suffices that

$$L_n(-a_0)^2 > 2kA\, R/\lambda_s(b). \tag{15}$$

Writing $X_s = R/\lambda_s(b)$, equation (12) shows that, for fixed $\theta > 0$, the integer

$$n = \left\lceil \frac{(1+\theta)(\log X_s)^2}{16 a_0} \right\rceil$$

satisfies (15) for all sufficiently large X_s. Its ordinary degree is at most $2n + k - 1$. The threshold is uniform over the factorization of X_s into contrast and geometry. First take the sequence limit, then $\theta \downarrow 0$, $r_0 \downarrow 0$ and $\gamma \uparrow c$. This proves the upper coefficient in (5).

The construction has a strict common margin. When the input data are certified computable, it can be replaced by a rational basis without evaluating original moments. Indeed, with $J = e^{cB^2} + CR$ a coefficient error $\sum_j \epsilon_j x^j$ has norm at most

$$\sqrt{J} \sum_j |\epsilon_j| M_j, \qquad M_0 = 1, \quad M_j = (j/(ce))^{j/2} \quad (j \ge 1) \tag{16}$$

in $L^2(|\mu|)$, by maximizing $|x|^{2j} e^{-cx^2}$. For a finite basis $f_1, \dots, f_s$ with common negative margin δ, let F bound their absolute-measure norms. If each is perturbed by norm at most η, every unit coefficient combination changes its square integral by at most $2sF\eta + s\eta^2$. Taking this below $\delta/2$ preserves the full negative subspace at unchanged degree.

2.4 A uniform growing-degree lower comparison

The next estimate pays the degree dependence explicitly. Fix $0 < h < D$, and let the reference probability τ_* of Lemma 2.4 have all moments. For nodes in $[-h, h]$, let $I_b P$ be the degree-$k-1$ interpolant of P, and let $T_b P$ be its monomial coefficient vector. Evaluation factors exactly as $E_b P = V(b) T_b P$. Newton interpolation [6] gives

$$I_b P(x) = \sum_{j=0}^{k-1} [b_1, \dots, b_{j+1}] P \prod_{i=1}^{j} (x - b_i).$$

The real divided-difference mean-value formula bounds the coefficient by $\|P^{(j)}\|_{[-h,h]}/j!$. Repeated Markov inequality on this interval [7] gives $\|P^{(j)}\| \le (N^2/h)^j \|P\|$ for $\deg P \le N$; for $j > N$ the derivative vanishes. Each Newton product has coefficient norm at most $(1+h)^j$. It follows that

$$\|T_b P\|_2 \le A_h (N+1)^{2(k-1)} \|P\|_{[-h,h]}, \quad A_h = \sum_{j=0}^{k-1} \frac{((1+h)/h)^j}{j!}.$$

Combining with (10) proves

$$\|T_b\|^2_{L^2(\tau_*)\to\ell^2} \le A_h^2 (1 + h^2/D^2)(N+1)^{4(k-1)} K_{\lfloor N/2 \rfloor}. \tag{17}$$

This bounds interpolated data that actually arise from a polynomial with a controlled background norm. It is not a uniform Euclidean bound on V^{-1} for arbitrary data.

In an orthonormal basis of $\mathcal{P}_N$ for τ_*, write $M_N = E_b E_b^{\mathsf{T}}$ for the unweighted evaluation kernel matrix. The factorization and $T_b T_b^{\mathsf{T}} \le \|T_b\|^2 I$ yield the complete Loewner comparison

$$M_N \le A_h^2 (1 + h^2/D^2)(N+1)^{4(k-1)} K_{\lfloor N/2 \rfloor} V V^{\mathsf{T}}. \tag{18}$$

It controls every ordered eigenvalue, with no reciprocal minimum separation in its constants.

For the Gaussian lower, choose $h > 0$ small with $D = d + h$ and $e^{cD^2} < C$. Fix $\sigma^2 < 1/c$ and take the even probability

$$X = \pm\sqrt{D^2 + \sigma^2 U}, \qquad U \text{ with density } e^{-u}, \quad u \ge 0,$$

as τ_*. Its Gaussian moment is $E_* = e^{cD^2}/(1 - c\sigma^2)$. Mix it once with the endpoint probability $\tau_D = (\delta_D + \delta_{-D})/2$:

$$\tau = \eta \tau_* + (1 - \eta)\tau_D, \qquad \eta = \min\left(1, \frac{C - e^{cD^2}}{2(E_* - e^{cD^2})}\right) > 0. \tag{19}$$

Thus $\int e^{cx^2} \, d\tau \le C$ and $\tau \ge \eta \tau_*$. This one law fits every $R \ge 1$ and every node vector in $[-h, h]^k$: $R\tau$ has variation R, satisfies the same CR budget, and stays at least d from all the atoms.

By reference-norm domination, the kernel for τ is at most η^{-1} times the kernel for τ_* in Loewner order. Lemma 2.3 and (18) therefore make the degree-N index at most $s - 1$ whenever

$$A_h^2 (1 + h^2/D^2)(N+1)^{4(k-1)} K_{\lfloor N/2 \rfloor} \lambda_s(b) \le k \eta R. \tag{20}$$

For this reference,

$$K_l = \sum_{j=0}^{l} L_j(-a)^2, \qquad a = D^2/\sigma^2.$$

Take $0 < \theta < 1$ and $N = \lfloor (1-\theta)(\log X_s)^2/(8a) \rfloor$. The logarithm of the kernel factor in (20), including the displayed degree power, is $\sqrt{1-\theta}\log X_s + O(\log\log X_s)$. It is below $\log X_s$ plus the required fixed constant for all sufficiently large X_s, uniformly in its factorization. Thus $n_s > N$. Floors and any bounded initial range do not affect the limiting ratio. Taking the limit first, then $\theta \downarrow 0$, $\sigma^2 \uparrow 1/c$ and $h \downarrow 0$, proves the lower coefficient $1/(8cd^2)$. Each reference law was fixed before contrast or nodes varied. This proves (5); Proposition 2.2 gives its t_{s-1} formulation.

2.5 Finite alternation and a factor that preserves the gap

We record an elementary finite minimax fact used in the exponential proof. A monic minimizer in (1) exists: bounded values at any $m+1$ distinct nodes bound its remaining coefficients by interpolation, so a minimizing sequence has a convergent subsequence. On the ordered finite node set, its extremal values alternate at $m+1$ selected nodes $a_1 < \cdots < a_{m+1}$. To see the necessity, group consecutive extremal nodes having the same sign. If there were at most m alternating groups, a polynomial of degree at most $m-1$, with one root between successive groups, could have the same sign as the minimizer at every extremal node. Subtracting a sufficiently small multiple would reduce all extremal absolute values, while the finitely many other nodes retain a strict margin below the old maximum. This contradicts minimality. Conversely, alternating extrema at $m+1$ nodes prevent a smaller monic norm: the difference, of degree at most $m-1$, would have at least m sign-changing zeros between them.

Put $t = t_m(b)$, $\omega(x) = \prod_{j=1}^{m+1}(x-a_j)$, and $\ell_j = 1/\omega'(a_j)$. Leading coefficients in Lagrange interpolation give

$$1 = \sum_j A(a_j)\ell_j = t\sum_j |\ell_j|, \qquad \sum_j |\ell_j| = 1/t. \tag{21}$$

The signs agree because both the alternating extrema and the derivatives $\omega'(a_j)$ alternate; their common product sign is fixed by the positive leading coefficient. Selected cardinals $L_j(x) = \ell_j \prod_{i\neq j}(x-a_i)$ consequently have coefficient sums at most

$$A_L = (1+B)^m/t. \tag{22}$$

The minimizing polynomial A has one real root between each successive alternating pair. Snap each of its m roots ξ_j to a nearest actual node q_j in the full node set, and put $B_m(x) = \prod_j (x-q_j)$. Repeated snapped roots are allowed. For every actual atom b_i, $|b_i - q_j| \le 2|b_i - \xi_j|$, so

$$|B_m(b_i)| \le 2^m t. \tag{23}$$

At every admissible background point, on the other hand,

$$|B_m(x)| \ge d^m, \tag{24}$$

since each root is an actual atom. This is why a factor with unsnapped roots in the convex hull cannot simply be used for an atomic-support gap: a root could lie in a permitted hole between atoms.

2.6 The exponential upper with all absolute norms paid

We give the quantitative construction, since selecting only s atoms changes an absolute-norm estimate relative to the all-atom proof. The classical continuous Chebyshev window used in [4] is

$$w_T(x) = \frac{\cos(T\sqrt{x^2-d^2})}{\cosh(Td)}.$$

It is real on $\mathbb{R}$, with $w_T(0) = 1$, $|w_T| \le 1$, and $|w_T| \le \operatorname{sech}(dT)$ on $|x| \ge d$. It has a positive Fourier probability representation on $[-T, T]$; this classical window and its representation are used in precisely the normalization of [4]. Define $f_j(x) = L_j(x)w_T(x - a_j)$ for the $s = m + 1$ selected cardinals above. Let

$$V_{2m}(c) = 2^{2m}\left[1 + \left(\frac{2m}{ce}\right)^{2m}\right], \qquad H_0 = sA_L^2 V_{2m}(c), \qquad J_0 = C(1 + R).$$

Then $\sum_j \sup_x e^{-c|x|}L_j(x)^2 \le H_0$, since $\sup_{u\ge 0}(1 + u)^{2m}e^{-cu} \le V_{2m}(c)$. Choose rational T at least

$$\max\left(1, \frac{1}{d}\operatorname{arcosh}\max(1, 2\sqrt{kJ_0H_0})\right) \tag{25}$$

and at most one above that value. At the selected atoms the evaluation matrix is the identity. Contributions from unselected negative atoms can only improve the signed upper bound. For every $u \in \mathbb{R}^s$, the full remainder Gram is bounded in absolute measure:

$$\mathcal{Q}_\mu\left(\sum_j u_j f_j\right) \le [-1/k + \operatorname{sech}^2(dT)J_0H_0]\|u\|_2^2 \le -\frac{3}{4k}\|u\|_2^2. \tag{26}$$

However, unselected atoms need not have small values of these targets. Their absolute norms must also be paid. The global bound $|w_T| \le 1$ gives

$$\left\|\sum_j u_j f_j\right\|_{L^2(|\mu|)} \le C_0\|u\|_2, \qquad C_0 = \sqrt{\max(1, J_0H_0)}. \tag{27}$$

For $Y = 2+R/t^2$, we have $C_0 \le K\sqrt{Y}$, with a fixed K. No bounded interpolation values at omitted atoms are assumed.

Here is the critical-scale polynomial approximation needed for these targets. Set $h = \pi/c$ and let $q_n(x) = P_n^{(1/2)}(x/h; \pi/2)$. Their classical generating function [2, Eq. 18.23.7] is

$$\sum_{n\ge 0} q_n(x)z^n = (1 - iz)^{-1/2+ix/h}(1 + iz)^{-1/2-ix/h}, \qquad |z| < 1.$$

For $|z| = \rho < 1$, Cauchy's estimate gives

$$|q_n(x)| \le \rho^{-n}(1 - \rho^2)^{-1/2}e^{2\arctan(\rho)|x|/h}. \tag{28}$$

For $n \ge 1$ take $\rho = n/(n + 1)$. The residual exponential margin is

$$\frac{\pi/2 - 2\arctan\rho}{h} = \frac{1}{h}\int_\rho^1 \frac{2}{1 + u^2}\,\mathrm{d}u \ge \frac{1}{h(n + 1)}.$$

Thus a degree-m multiplier L of coefficient sum at most A_L obeys

$$|L(x)q_n(x)| \le eA_L U_m(h)(n + 1)^{m+1/2}e^{c|x|/2}, \quad U_m(h) = \max\left(1, \max_{1\le j\le m}(jh/e)^j\right). \tag{29}$$

Indeed $\sup_{v\ge 0} v^j e^{-\delta v} = (j/(e\delta))^j$. For $n = 0$ the same estimate follows from $q_0 = 1$ and $2/c < h$. This retains the original rate c.

For a Fourier measure of variation at most one on $[-T,T]$, substitution $z=i\tanh(ht/2)$ in the generating function gives

$$e^{itx}=\operatorname{sech}(ht/2)\sum_{n\ge 0}q_n(x)[i\tanh(ht/2)]^n.$$

Integration gives real coefficients of magnitude at most ρ_T^n, where $\rho_T=\tanh(hT/2)$. Translation of the window preserves its Fourier variation. Absolute weighted convergence from (29) justifies integrating the series. For $p=m+1$, the resulting error for each selected target, after truncation at n_0 and before rational rounding, is at most

$$eA_LU_m(h)\sqrt{J_0}\,p!(n_0+2)^p\frac{\rho_T^{n_0+1}}{(1-\rho_T)^{p+1}}. \tag{30}$$

To verify the elementary tail sum, write $n=n_0+1+j$ and use $(j+1)^p\le p!\binom{j+p}{p}$ to bound $\sum_{n>n_0}(n+1)^p\rho_T^n$.

Take

$$\eta=\min\left(\frac{1}{4\sqrt{ks}},\frac{1}{16kC_0\sqrt{s}}\right).$$

There is a fixed $K_\eta>0$ with $\eta\ge K_\eta Y^{-1/2}$. If approximation and rounding give each target norm error below η, then the error of every combined square is at most $\sqrt{s}\eta(2C_0+\sqrt{s}\eta)\|u\|_2^2\le 3\|u\|_2^2/(16k)$. Hence (26) remains strictly negative on the entire span.

We now pay the degree in this approximation. With $\alpha=\pi/(2cd)$ and $\delta=1-\rho_T$, (25) gives

$$e^{hT}\le K_1(1+J_0H_0)^\alpha\le K_2Y^\alpha,\quad \delta^{-1}=\frac{e^{hT}+1}{2}\le K_3Y^\alpha,\quad A_L\sqrt{J_0}\le K_4\sqrt{Y}.$$

Choose fixed $L>2+\alpha(2p+1)+p$ and then fixed $L_0\ge 1$, and put $n_0=\lceil\delta^{-1}(L\log Y+L_0)\rceil$. Since $\log\rho_T\le-\delta$, $\rho_T^{n_0+1}\le e^{-L_0}Y^{-L}$. Also $(n_0+2)^p\le K_5(L,L_0)\delta^{-p}(\log Y)^p$, where K_5 grows polynomially in L_0 at fixed L,p. The error (30) is therefore at most

$$K_6(L,L_0)e^{-L_0}Y^{1/2+\alpha(2p+1)-L}(\log Y)^p\le K_7(L,L_0)e^{-L_0}Y^{-3/2}.$$

A sufficiently large fixed L_0 makes it less than $\eta/2$ for all $Y\ge 2$. Rational rounding costs another $\eta/2$ without changing the degree: the exponential moment gives $\int|x|^{2j}\,\mathrm{d}|\mu|\le J_0(2j)!/c^{2j}$, so a coefficient-error sum has norm at most $\sqrt{J_0}\sum_j|\epsilon_j|\sqrt{(2j)!}/c^j$. The resulting ordinary polynomials have degree at most n_0+m and form a strictly negative s-space. In particular,

$$D_s^E(R;b)\le A(2+R/t_m^2)^\alpha\log(2+R/t_m^2), \tag{31}$$

with a constant depending only on the fixed declared data. The smaller approximation tolerance caused by omitted atoms costs a logarithm, not an extra power of the geometry statistic.

2.7 Exponential lower on the whole divisibility subspace

For clarity we specify the classical continuous dual Hahn input. The probability W_a with parameters $(a,1/2,1/2)$, $a>0$, on $u>0$ has density

$$\frac{2u\tanh(\pi u)|\Gamma(a+iu)|^2}{\Gamma(a+1/2)^2}.$$

Its standard polynomials $S_j(u^2)$ satisfy $S_j(-a^2) = (a+1/2)_j^2$ and $\|S_j\|^2 = (j!)^2(a+1/2)_j^2$; these are classical continuous dual Hahn orthogonality data [2, Section 18.25]. For the even lift $x = \pm\sigma\sqrt{u^2+a^2}$ its squared-variable kernel at zero is exactly

$$K_l = \sum_{j=0}^{l} \left[\frac{(a+1/2)_j}{j!}\right]^2 \sim \frac{l^{2a}}{2a\Gamma(a+1/2)^2} \le K_0(l+1)^{2a} \tag{32}$$

for an appropriate fixed K_0 in the final inequality. The kernel sum asymptotic follows from the gamma-ratio asymptotic and summation of j^{2a-1}. Stirling's formula on a vertical line gives density of order $u^{2a}e^{-\pi u}$, so its exponential moment at rate c is finite if $c\sigma < \pi$.

Fix $0 < \beta < \alpha$. Choose a small fixed $h_0 > 0$ with $h_0 \le B$, $D = d + h_0$, $e^{cD} < C$, and $cD/\pi < 1/(2\beta)$. Choose $cD/\pi < a < 1/(2\beta)$, and $\sigma = D/a$. The preceding lift τ_* is supported on $|x| \ge D$ and has finite exponential moment E_*. Mix it with τ_D by one fixed

$$\eta_0 = \min\left(1, \frac{C - e^{cD}}{2(E_* - e^{cD})}\right) > 0$$

to obtain $\tau \ge \eta_0\tau_*$ with moment at most C. For every node vector in $[-h_0, h_0]^k$, $\mu_{R,b} = R\tau - k^{-1}\sum_i \delta_{b_i}$ belongs to (4): the background contributes at most CR and the normalized atoms at most $e^{ch_0} < C$. Every support distance is at least $D - h_0 = d$.

Use the snapped factor B_m from (23). For $N \ge m$, the subspace $\{B_m Q : \deg Q \le N - m\}$ has codimension exactly m in $\mathcal{P}_N$. Its positive contribution is at least $R\eta_0 d^{2m}\|Q\|^2_{L^2(\tau_*)}$, by (24). Lemma 2.4 bounds its complete negative contribution by $4^m t_m^2(1 + h_0^2/D^2)K_{\lfloor (N-m)/2\rfloor}\|Q\|^2_{L^2(\tau_*)}$. Thus the whole subspace is nonnegative if

$$4^m t_m^2(1 + h_0^2/D^2)K_{\lfloor (N-m)/2\rfloor} \le R\eta_0 d^{2m}. \tag{33}$$

A negative subspace of dimension $s = m+1$ must intersect a codimension-m subspace nontrivially, so cannot exist under this condition. For $N < m$ the ambient dimension already excludes it.

For completeness, the floor adjustment can be made uniformly. Put $X = R/t_m^2$, $\gamma_0 = 1/(2a) > \beta$, and

$$q_0 = \left[\frac{\eta_0 d^{2m}}{4^m(1 + h_0^2/D^2)K_0}\right]^{\gamma_0}.$$

The sufficient condition $N + 1 \le q_0 X^{\gamma_0}$ pays (33). If $q_0 X^{\gamma_0} \ge 4$, take $N = \lfloor q_0 X^{\gamma_0}/2\rfloor$; it gives $n_s > N$ and hence $n_s > q_0 X^{\gamma_0}/2$. If $q_0 X^{\gamma_0} < 4$, use $n_s \ge 1$. Consequently, whenever $X \ge 1$,

$$n_s(\mu_{R,b}) \ge (q_0/4)X^{\gamma_0} \ge (q_0/4)X^{\beta}.$$

Along total coalescence $X \to \infty$, and every fixed-β lower law eventually applies. Combine with (31), then let $\beta \uparrow \alpha$ after taking sequence limits. This proves (6), and the bridge gives its spectral version. The proof gives no constants uniform at $\beta = \alpha$ and no optimal logarithmic factor.

Remark 2.5 (An alternate Gaussian lower). The same snapped factor and codimension argument give a shorter lower proof directly in t_m. Replace the CDH reference by the shifted exponential reference in (19); then $\log K_l = 4\sqrt{(D^2/\sigma^2)l} + O(\log(l+2))$. Condition (33), with the corresponding fixed positive constants, excludes the sth negative direction up to $(1-\theta)\sigma^2[\log(R/t_m^2)]^2/(8D^2)$. Taking the fixed-law limits gives the lower coefficient $1/(8cd^2)$. The interpolation estimate (18) proves more: it retains the entire finite evaluation spectrum in one comparison and explicitly pays its degree dependence. These are two proofs within the same recovery theorem, not independent results.

2.8 A four-node recovery staircase

Consider the ordered nodes $-(a+h), -(a-h), a-h, a+h$, with $0 < h < a$. Finite alternation gives the exact values

$$t_1 = a + h, \qquad t_2 = 2ah, \qquad t_3 = 2h(a^2 - h^2). \tag{34}$$

The respective monic minimizers are x, $x^2 - a^2 - h^2$, and $x^3 - (a^2 + 3h^2)x$. Their extrema alternate on two, three, and four selected nodes, respectively, proving optimality by the preceding alternation argument. These formulas hold for every $0 < h < a$.

For example let $h = a^\rho$ with fixed $\rho > 1$, $a \downarrow 0$, and $R = 1$. The Gaussian degrees for recovering indices two, three and four have respective leading factors

$$4, \qquad 4(\rho+1)^2, \qquad 4(\rho+2)^2$$

multiplying $\log^2(1/a)/(8cd^2)$. The coarse four-atom minimum-separation expression of [3] uses $\Delta = 2h$ for sufficiently small a and pays factor $36\rho^2$ for its last-index logarithmic scale. This is an improvement at a fixed multiscale geometry, not a contradiction of that earlier worst-geometry minimax theorem. Full-spectrum cluster analysis in [8, 9] supplies relevant spectral ancestry; it is the signed-tail degree minimax problem and its uniform joint limit that are asserted here.

3 Continuous negative packets with a convex-hull gap

The negative support need not consist of a fixed number of equal atoms. The natural receiving statistic is now a monic square norm for the actual negative measure.

Theorem 3.1 (Probability-packet hierarchy)**.** *Fix $s \geq 2$, $c, d > 0$ and $C > e^{cd^2}$, and put $m = s-1$. For a probability κ supported in $[-\varepsilon, \varepsilon]$ with at least s distinct support points, define*

$$h_m(\kappa) = \min_{\substack{A \text{ monic} \\ \deg A = m}} \int A^2 \, \mathrm{d}\kappa > 0.$$

Let $D_s^G(R; \kappa, \varepsilon)$ be the supremum of $n_s(-\kappa + \nu)$ over real signed remainders satisfying

$$\operatorname{supp} \nu \subset \{|x| \geq d + \varepsilon\}, \qquad \|\nu\|_{\mathrm{TV}} \leq R, \qquad \int e^{cx^2} \, \mathrm{d}|\nu| \leq CR, \qquad R \geq 1.$$

Along every family $\varepsilon \downarrow 0$ of such probabilities, with arbitrary jointly varying $R \geq 1$,

$$\frac{D_s^G(R; \kappa, \varepsilon)}{[\log(R/h_m(\kappa))]^2} \longrightarrow \frac{1}{8cd^2}. \tag{35}$$

No density, finite atom count, lower bound on atomic weights, or comparable internal scales are assumed.

Proof. Let $M = (\int x^{i+j} \, \mathrm{d}\kappa)_{0 \leq i,j \leq m}$. It is positive definite by the support assumption. Its monic orthogonal polynomials have roots in $[-\varepsilon, \varepsilon]$, and squared norms h_j with $h_0 = 1$ and $h_j \leq \varepsilon^2 h_{j-1}$. Exactly the coefficient-matrix argument of Proposition 2.2 therefore gives, for $\lambda = \lambda_{\min}(M)$,

$$\lambda / h_m \longrightarrow 1 \tag{36}$$

uniformly over all these moving probability measures. This remains a multiplicative comparison when h_m is extremely small.

For the upper fix $\gamma < c$ and use $H_n(x) = L_n(\gamma[x^2 - (d+\varepsilon)^2])$. On the remainder support its argument is nonnegative; on the negative support it is at most $-\gamma(d^2 + 2d\varepsilon)$. For $p_u \in \mathcal{P}_m$, the Gaussian margin gives $\int H_n^2 p_u^2 \, \mathrm{d}|\nu| \le AR\|u\|_2^2$, with fixed A independent of κ, n, R and ε in a fixed bounded interval. Coefficient positivity and (36) give

$$\int H_n^2 p_u^2 \, \mathrm{d}\kappa \ge L_n(-\gamma d^2)^2 \lambda \|u\|_2^2.$$

Thus $H_n\mathcal{P}_m$ is a strictly negative s-dimensional space when $L_n(-\gamma d^2)^2 > 2AR/\lambda$. Equations (12) and (36) give the upper coefficient $1/(8\gamma d^2)$, and then $\gamma \uparrow c$ after the limit.

For the lower fix small $\varepsilon_0 > 0$ with $D = d + \varepsilon_0$ and $e^{cD^2} < C$. Choose a fixed shifted-exponential reference and endpoint mixture $\tau \ge \eta\tau_*$ as in (19). It fits every $\varepsilon \le \varepsilon_0$ and every such κ. Let A be its monic κ-orthogonal polynomial of degree m. Every root is in the negative packet's convex hull, so on τ support $|A| \ge d^m$. On the codimension-m space $P = AQ$, $\deg Q \le N - m$, the positive contribution is at least $R\eta d^{2m}\|Q\|^2_{L^2(\tau_*)}$. Lemma 2.4 bounds the complete negative contribution by

$$\int A^2 Q^2 \, \mathrm{d}\kappa \le h_m(1 + \varepsilon_0^2/D^2) K_{\lfloor (N-m)/2 \rfloor} \|Q\|^2_{L^2(\tau_*)}.$$

Hence that entire subspace is nonnegative when its kernel is at most a fixed constant times R/h_m. The codimension argument excludes index s; for $N < m$ ambient dimension suffices. For the shifted-exponential law, $\log K_l = 4\sqrt{(D^2/\sigma^2)l} + O(\log(l+2))$. With any fixed $0 < \theta < 1$, take $N = m + 2\lfloor (1-\theta)[\log(R/h_m)]^2/(16D^2/\sigma^2) \rfloor$. The strict leading deficit pays all fixed constants for large R/h_m, giving lower coefficient $\sigma^2/(8D^2)$. Finally take $\theta \downarrow 0$, $\sigma^2 \uparrow 1/c$ and $\varepsilon_0 \downarrow 0$ after the fixed-law limits. Every allowed family eventually enters each such fixed interval. The scale diverges since $h_m \le \int x^{2m} \, \mathrm{d}\kappa \le \varepsilon^{2m}$. This proves (35). □

In particular, the theorem includes arbitrary unequal moving atomic weights and arbitrary atom counts, as long as each negative probability has at least the fixed number s of support points. It also includes absolutely continuous, singular continuous and mixed negative packets. For continuous negative parts the total negative index can be infinite; the assertion concerns each fixed s.

The support exclusion is from the entire containing interval and thus from the convex hull of the negative support. Replacing it by a gap only from individual points of a disconnected negative support would not justify the lower proof: an orthogonal root could lie in a hole. For atomic gaps, the snapped-root construction (23)–(24) is what pays that distinction. No stronger disconnected-support statement is inferred here.

4 Recovery from local nonconcentration

A support gap is a strong form of control near a negative atom. We now replace it by an upper bound on local background mass. The resource changes from logarithmic to polynomial or stretched-exponential degree. The classical localization and Christoffel estimates used below are credited explicitly; the conclusions concern uniform recovery over the stated signed-measure classes, rather than asymptotics for every individual background.

Fix $p, q, c, C, L, r_0 > 0$. For $R \ge 1$, let $\mathcal{A}_{p,q}(R)$ consist of real signed measures

$$\mu = -\delta_0 + \nu, \qquad \mathrm{TV}(\nu) \le R, \quad \int_{\mathbb{R}} e^{c|x|^p} \, \mathrm{d}|\nu|(x) \le CR, \quad |\nu|([-r,r]) \le LRr^q \quad (0 < r \le r_0). \tag{37}$$

All parameters except R are fixed. No density, positivity, doubling condition, lower mass bound, or support gap is assumed. In particular, $\nu(\{0\}) = 0$. Write

$$n(\mu) = \min\{\deg P : P \in \mathbb{R}[x],\ \int P^2 \, \mathrm{d}\mu < 0\}, \qquad D_{p,q}(R) = \sup_{\mu \in \mathcal{A}_{p,q}(R)} n(\mu),$$

with $n(\mu) = \infty$ if the set is empty. Degrees are ordinary polynomial degrees; the square has twice that degree.

Theorem 4.1 (Tail and local-concentration orders)**.** *For every fixed set of positive class parameters, as $R \to \infty$,*

$$D_{p,q}(R) = \Theta(R^{p/(q(p-1))}), \qquad p \geq 2, \tag{38}$$

$$\log D_{1,q}(R) = \Theta(R^{1/q}), \qquad q > 0, \tag{39}$$

$$D_{p,q}(R) = \infty \quad \textit{for all sufficiently large } R, \qquad 0 < p < 1. \tag{40}$$

For $1 < p < 2$ we prove only $D_{p,q}(R) = O(R^{p/(q(p-1))})$. The constants may depend on all fixed parameters. These statements assert minimax orders, not sharp leading constants or rates for every member of the class. The upper bounds hold for signed and singular remainders.

The range distinction is part of the theorem. The subcritical conclusion is a rescaling corollary of the masking theorem in [3].

4.1 A local-mass detector and a global Taylor transfer

Lemma 4.2 (Local-mass detector)**.** *Choose an integer $\ell \geq 1$ with $\beta = 2\ell > q$, and set*

$$B = \frac{2\ell L}{2\ell - q} + r_0^{-2\ell}, \qquad T = \max\{1, r_0^{-1}, (16BR)^{1/q}\}, \qquad f_T(x) = \left(\frac{\sin Tx}{Tx}\right)^{\ell}.$$

The value at zero is one. Then f_T is real, is the Fourier transform of a symmetric probability measure π_T on $[-W, W]$, $W = \ell T$, and

$$\int |f_T|^2 \, \mathrm{d}|\nu| \leq \frac{1}{16}, \qquad \int f_T^2 \, \mathrm{d}\mu \leq -\frac{15}{16}. \tag{41}$$

Proof. The probability is the convolution of ℓ uniform laws on $[-T, T]$. Also

$$|f_T(x)|^2 \leq \min\{1, (T|x|)^{-2\ell}\}.$$

If $F(r) = |\nu|([-r, r])$, layer cake gives

$$\int |f_T|^2 \, \mathrm{d}|\nu| \leq 2\ell T^{-2\ell} \int_{1/T}^{\infty} F(r) r^{-2\ell-1} \, \mathrm{d}r.$$

Use $F(r) \leq LRr^q$ up to r_0, and $F(r) \leq R$ afterwards. Since $T \geq 1$, the result is at most BRT^{-q}. □

Proposition 4.3 (Constructive superlinear-tail upper)**.** *For $p > 1$ the upper bound in Theorem 4.1 is realized by an original-measure polynomial with $P(0) = 1$.*

Proof. Put $a_p = 1 - 1/p$ and $H = e^{1-1/p}(2/(cp))^{1/p}$. Maximization of $r^n e^{-cr^p/2}$, followed by $n! \geq (n/e)^n$, gives

$$\sup_x e^{-c|x|^p/2} \frac{W^n |x|^n}{n!} \leq \left(\frac{HW}{n^{a_p}} \right)^n. \tag{42}$$

Integrate the Taylor polynomial of e^{itx} against π_T:

$$P_N(x) = \int \sum_{j=0}^{N} \frac{(itx)^j}{j!} \, \mathrm{d}\pi_T(t).$$

Symmetry makes P_N real, kills its odd coefficients and gives $P_N(0) = 1$. For $N \geq (2HW)^{1/a_p}$ every term of the weighted remainder has bound 2^{-n}, so the full-line weighted error is at most 2^{-N}. Consequently

$$\|P_N - f_T\|^2_{L^2(|\nu|)} \leq CR4^{-N}.$$

Choose the integer N also at least 1 and $\log_4 \max\{1, 16CR\}$. Minkowski and Lemma 4.2 imply $\|P_N\|_{L^2(|\nu|)} \leq 1/2$. Thus $\int P_N^2 \, \mathrm{d}\mu \leq -3/4$. Since $W = O(R^{1/q})$, $N = O(R^{p/(q(p-1))})$. The estimate is on the original measure, not on a compact restriction or a convolution. □

Strict signs are stable under rational coefficient approximation. Put

$$J = \int e^{c|x|^p} \, \mathrm{d}|\mu| \leq 1 + CR.$$

For $E(x) = \sum_{j=0}^{N} \epsilon_j x^j$, we have

$$\begin{aligned} \|E\|_{L^2(|\mu|)} &\leq \sqrt{J} \sum_{j=0}^{N} |\epsilon_j| M_j, \\ M_0 = 1, \qquad M_j &= \left(\frac{2j}{cpe} \right)^{j/p} \quad (j \geq 1). \end{aligned} \tag{43}$$

For certified computable class data, the finite Fourier integrals and polynomial coefficients can therefore be rounded with a certified margin at unchanged degree.

4.2 Gaussian localization and exact lower models

There is a shorter classical proof of the Gaussian upper. In the one-dimensional normalization of [12, Lemma 2.1], a nonnegative smooth cutoff Hermite-function kernel Φ_n satisfies, for every fixed integer $S > 1$,

$$|\Phi_n(x, y)| \leq C_S n \min\{1, (n|x - y|)^{-S}\}.$$

It is a polynomial in y of degree less than n^2 times $e^{-y^2/2}$. Moreover $\Phi_n(0, 0) \geq c_0 n$: the cutoff equals one for $j \leq n^2/4$, and $\psi_{2j}(0)^2 = \pi^{-1/2}(2j)!/(4^j(j!)^2)$, whose partial sums have order n. With $\gamma = c/2$, the function

$$P_n(x) = e^{\gamma x^2/2} \frac{\Phi_n(0, \sqrt{\gamma} x)}{\Phi_n(0, 0)}$$

is an actual real polynomial, $P_n(0) = 1$, of degree less than n^2. The tilted positive measure $\mathrm{d}\sigma = e^{\gamma x^2} \, \mathrm{d}|\nu|$ has mass at most CR and local mass at most $e^{\gamma r_0^2} LRr^q$. Choosing $2S > q$ and integrating the kernel bound over dyadic annuli yields

$$\int P_n^2 \, \mathrm{d}|\nu| \leq KRn^{-q}.$$

The central ball contributes $O(Rn^{-q})$, the successive annuli form the convergent series $\sum_j 2^{j(q-2S)}$, and the region beyond r_0 contributes $O(Rn^{-2S})$. Taking $n \asymp R^{1/q}$ proves the upper $O(R^{2/q})$. This applies classical localization; compare also [13, 14].

For a matching model put $\alpha = q/2$, choose $a > c$, and let

$$\mathrm{d}\tau(x) = \frac{a^\alpha}{\Gamma(\alpha)}|x|^{q-1}e^{-ax^2}\,\mathrm{d}x. \tag{44}$$

This is a probability with

$$\int e^{cx^2}\,\mathrm{d}\tau = \left(\frac{a}{a-c}\right)^\alpha, \qquad \tau([-r,r]) \le \frac{a^\alpha r^q}{\Gamma(\alpha+1)}.$$

Fix

$$0 < \eta \le \min\left\{1, \frac{L\Gamma(\alpha+1)}{a^\alpha}, C\left(\frac{a-c}{a}\right)^\alpha\right\}.$$

Then $\nu_R = \eta R\tau$ belongs to the class. Under $v = ax^2$, its even norm is the gamma probability norm with shape α. The classical Laguerre value and norm identities [1, 5] give the exact degree-N origin kernel

$$K_N^\tau(0,0) = \sum_{j=0}^{\lfloor N/2\rfloor} \frac{(\alpha)_j}{j!} = \frac{(\alpha+1)_{\lfloor N/2\rfloor}}{\lfloor N/2\rfloor!}. \tag{45}$$

Odd polynomials vanish at zero and have positive orthogonal norm. Thus Christoffel extremality gives the exact strict threshold

$$n(-\delta_0 + \eta R\tau) = 2\min\left\{m \ge 0 : \frac{(\alpha+1)_m}{m!} > \eta R\right\}. \tag{46}$$

Equality produces a null square, not a negative square. The gamma-ratio asymptotic therefore gives

$$n(-\delta_0 + \eta R\tau) \sim 2\big(\Gamma(q/2+1)\eta R\big)^{2/q}.$$

This exact coefficient belongs to this model, not to the class minimax.

4.3 Classical Freud lower bounds for $p \ge 2$

Lemma 4.4 (Restricted singular-Freud receiver)**.** *For fixed $p \ge 2$, $q > 0$ and $a > 0$, let*

$$\mathrm{d}\tau(x) = A|x|^{q-1}e^{-a|x|^p}\,\mathrm{d}x, \qquad A = \frac{pa^{q/p}}{2\Gamma(q/p)}.$$

There is $b > 0$ such that for all $N \ge 0$ and all $\deg P \le N$,

$$\int P^2\,\mathrm{d}\tau \ge b(N+1)^{-q(1-1/p)}|P(0)|^2. \tag{47}$$

Proof. Apply the lower Christoffel bound of [11, Chapter I, Theorem 1.3.2] with integrability exponent 2, singular root exponent $(q-1)/2$, and $Q(x) = (a/2)|x|^p$. Its squared weight is exactly $|x|^{q-1}e^{-a|x|^p}$, and the singular integrability condition is $q - 1 > -1$. The scale determined by $b_nQ'(b_n) = n$ is $b_n = (2n/(ap))^{1/p}$. The source lower estimate, after multiplying its normalized weight factor, is

$$\lambda_{n,2}(w^2;x) \ge b_0 e^{-2Q(x)}(b_n/n)(|x| + b_n/n)^{q-1}$$

near zero, with the degree convention $\deg P < n$. This specialization has continuous nonnegative Q'', $tQ''/Q' = p-1$ and $Q'(2t)/Q'(t) = 2^{p-1} > 1$. In particular, the curvature estimate used in the quadratic-multiplier proof is valid on $|\xi| \le Bb_n$:

$$Q''(\xi) \le (p-1)B^{p-2}n/b_n^2.$$

The finite-dimensional Christoffel function is the reciprocal of a positive polynomial kernel, hence is continuous in its evaluation point. Letting $x \to 0$ avoids any ambiguity from a zero or infinite weight value and gives $\lambda_{n,2}(w^2;0) \ge b_1(b_n/n)^q$. Probability normalization multiplies this by A, and $n = N+1$ gives (47). We use this restricted classical input only for $p \ge 2$. □

For $a > c$ the same law satisfies

$$E = \int e^{c|x|^p}\, \mathrm{d}\tau = (a/(a-c))^{q/p}, \qquad \tau([-r,r]) \le \frac{a^{q/p} r^q}{\Gamma(q/p+1)}.$$

Choose $\eta = \min\{1, C/E, L\Gamma(q/p+1)/a^{q/p}\} > 0$. Then $\eta R\tau$ satisfies every budget. If $\eta Rb(N+1)^{-q(1-1/p)} \ge 1$, Lemma 4.4 makes the entire degree-N square form of $-\delta_0 + \eta R\tau$ nonnegative. Taking N of order $R^{p/(q(p-1))}$ proves the lower in (38). The law and η are fixed before R varies.

4.4 Critical exponential tails: universal upper

The following normalization records the classical MP identities used in the critical argument [1, 4]. Define $Q_n(x) = P_n^{(1/2)}(x/h;\pi/2)$ by

$$\sum_{n\ge 0} Q_n(x) z^n = (1-iz)^{-1/2+ix/h}(1+iz)^{-1/2-ix/h}, \qquad |z| < 1. \tag{48}$$

These are orthonormal for $\mathrm{d}x/[h\cosh(\pi x/h)]$. At $h = \pi/c$, Cauchy's coefficient estimate on $|z| = n/(n+1)$ gives

$$|Q_n(x)| \le e\sqrt{n+1}\, e^{c|x|/2}. \tag{49}$$

Indeed the algebraic modulus on this circle is at most $(1-r^2)^{-1/2}$, the difference of the two arguments has modulus at most $\pi/2$, and $r^{-n} \le e$. The case $n = 0$ is immediate. Substitution $z = i\tanh(ht/2)$ gives

$$e^{itx} = \operatorname{sech}(ht/2) \sum_{n\ge 0} Q_n(x)[i\tanh(ht/2)]^n.$$

For the detector of Lemma 4.2, put $r = \tanh(hW/2)$, $\delta = 1-r$, $J = 1+CR$, and $U = 1+\log J + \log\delta^{-1}$. Integrating the plane-wave series against its Fourier probability gives real coefficients a_n with $|a_n| \le r^n$. Absolute weighted convergence and (49) yield

$$\left\| f_T - \sum_{n=0}^{N} a_n Q_n \right\|_{L^2(|\mu|)} \le e\sqrt{J}\, r^{N+1}\left(\frac{N+2}{\delta} + \frac{r}{\delta^2}\right). \tag{50}$$

Take $N = \lceil 16\delta^{-1}U \rceil$. Since $\log r \le -\delta$ and $N+2 \le \delta^{-1}(16U+3)$, the last quantity is at most

$$e(16U+4)e^{-14U} \le 20e^{-13} < 1/32.$$

Here $\sqrt{J}\delta^{-2} \le e^{2U}$; thus every norm loss is included. Also $\delta^{-1} = (e^{hW}+1)/2 \le e^{hW}$, and hence

$$N \le 16e^{hW}(1+\log(1+CR)+hW)+1 = \exp(O(R^{1/q})). \tag{51}$$

Rational rounding using (43), if desired, costs less than $1/32$ in $L^2(|\mu|)$ at unchanged degree. Since $\|f_T\|^2_{L^2(|\mu|)} \le 17/16 < 5/4$, the resulting polynomial satisfies

$$\left|\int (P^2 - f_T^2)\, \mathrm{d}\mu\right| \le \frac{1}{16}\left(2\sqrt{5/4} + \frac{1}{16}\right) < \frac{1}{4}.$$

Thus it is strictly negative on the original form. This proves the upper half of (39) for every $q > 0$.

4.5 Critical necessity for all real $q \ge 1$

Choose $0 < h < \pi/c$ and let $\mathrm{d}\tau = \mathrm{d}x/[h\cosh(\pi x/h)]$. Its exponential moment is at most $E_0 = 4/(\pi - ch)$. Write $a_n = (1/2)_n/n!$ and $H_n = \sum_{j=0}^{n-1}(j+1/2)^{-1}$, with $H_0 = 0$. The generating function (48) gives the coefficient bound

$$|[x^m]Q_n(x)| \le \frac{2^{m+1}}{h^m m!} a_n H_n^m. \tag{52}$$

To check it coefficientwise, replace $(1+z^2)^{-1/2}$ by $(1-z^2)^{-1/2} \le 2(1-z)^{-1/2}$ and the absolute logarithmic series by $2\log(1/(1-z))$. The first inequality follows from $a_j \le 2a_{2j}$. The remaining coefficient is the mth derivative of $(a)_n/n!$ at $a = 1/2$, at most $a_n H_n^m$: its derivative is the sum of products of distinct positive reciprocals, bounded by the unrestricted power of their sum. We use $H_0^0 = 1$. The elementary bounds $a_n \le (n+1)^{-1/2}$ and $H_n \le 3 + \log(n+1)$ now imply, with $H = 3 + \log(N+1)$,

$$K_N^\tau(x,x) \le 4e^{4|x|H/h} H. \tag{53}$$

Indeed sum (52) first over m and then square and sum over n.

Put $\kappa = \min\{1, h/64\}$ and $\delta = \kappa/H$. The reproducing-kernel inequality and the density bound $\tau' \le 1/h$ give, for every $\deg P \le N$,

$$\int_{|x|<\delta} P^2\, \mathrm{d}\tau \le (8\kappa/h)e^{4\kappa/h}\|P\|_\tau^2 < \tfrac{1}{2}\|P\|_\tau^2.$$

Consequently

$$\int_{|x|\ge\delta} P^2\, \mathrm{d}\tau \ge \tfrac{1}{2}\|P\|_\tau^2 \ge \frac{|P(0)|^2}{8H}. \tag{54}$$

Fix $0 < a \le \min\{1, C/E_0, Lh/2\}$ and, for $q \ge 1$, set

$$\nu_{R,N} = aR\delta^{q-1}\mathbf{1}_{\{|x|\ge\delta\}}\tau.$$

Its variation and exponential moment satisfy (37). For $r < \delta$ its local mass is zero; for $r \ge \delta$ it is at most $(2a/h)R\delta^{q-1}r \le LRr^q$. Thus every radius is paid. By (54),

$$\int P^2\, \mathrm{d}\nu_{R,N} \ge \frac{aR\kappa^{q-1}}{8H^q}|P(0)|^2 \qquad (\deg P \le N).$$

Let $U_R = (a\kappa^{q-1}R/16)^{1/q}$ and take $N = \lfloor e^{U_R - 3}\rfloor - 1$ for large R. Then $H \le U_R$, and the whole original degree-N form is nonnegative. Therefore $\log D_{1,q}(R) \ge bR^{1/q}$ with $b > 0$. The lower law is allowed to move with R in a minimax class; its central hole shrinks to zero. This proof treats the varying-law regime specified above.

4.6 A fixed cusp law for $0 < q < 2$

For completeness we give a complementary lower proof that covers the remaining range $0 < q < 1$ with a fixed positive law. Let $p_n(t) = P_n^{(1/2)}(t; \pi/2)$. The classical terminating hypergeometric formula and Euler integral [1] are

$$p_n(t) = i^n \frac{\cosh(\pi t)}{\pi} \int_0^1 u^{-1/2+it}(1-u)^{-1/2-it}(1-2u)^n \, \mathrm{d}u. \tag{55}$$

Fix $0 < \theta < \pi/2$. For $t \geq 0$ deform the interval to the arc $u = e^{i\theta}s/(1+e^{i\theta}s)$, $0 < s < \infty$; for $t < 0$ use its reflection. The powers are continued from $(0,1)$, the arc avoids their cuts, and the endpoint arcs vanish because their real exponents are $-1/2$. Since $(1-2u)^n$ is a polynomial, it adds no cut. The substitution transforms the remaining differential into

$$e^{i\theta/2-\theta t} s^{-1/2+it}(1+e^{i\theta}s)^{-1} \, \mathrm{d}s.$$

Consequently

$$|p_n(t)| \leq \frac{\cosh(\pi t)}{\pi} e^{-\theta|t|} I_n(\theta), \quad I_n(\theta) = \int_0^\infty \frac{s^{-1/2}}{|1+e^{i\theta}s|} \left| \frac{1-e^{i\theta}s}{1+e^{i\theta}s} \right|^n \mathrm{d}s.$$

Inversion $s \mapsto 1/s$ equates the two halves of this integral. On $0 < s \leq 1$, its denominator is at least one and the squared ratio is $1 - 4s\cos\theta/(1+2s\cos\theta+s^2) \leq 1 - s\cos\theta$. Thus for $n \geq 1$,

$$I_n(\theta) \leq 2\int_0^1 s^{-1/2} e^{-ns\cos\theta/2} \, \mathrm{d}s \leq 2\sqrt{2\pi/(n\cos\theta)},$$

and we have the globally uniform, deliberately nonoptimal envelope

$$|p_n(t)| \leq C_\theta n^{-1/2} e^{(\pi-\theta)|t|}. \tag{56}$$

Choose $b > c$, $a > 2b$, and $\theta = \pi/4$. For $\tau_a'(x) = a/(\pi\cosh ax)$ its orthonormal polynomials are $p_n(ax/\pi)$. Put $\Lambda = \log(N+2)$ and let $K_N(x,0)$ be its reproducing kernel. The origin values are $p_{2j}(0) = (-1)^j(1/2)_j/j!$ and $p_{2j+1}(0) = 0$. Central-binomial bounds and (56) imply $K_N(0,0) \leq C\Lambda$ and $K_N(x,x) \leq Ce^{3a|x|/2}\Lambda$. Kernel Cauchy–Schwarz gives a bound $Ce^{3a|x|/4}\Lambda$. The MP recurrence has off-diagonal coefficient $\pi(n+1)/(2a)$, so Christoffel–Darboux also gives, for $x \neq 0$ and $N \geq 1$,

$$K_N(x,0) = \frac{\pi(N+1)}{2ax}\big[p_{N+1}(ax/\pi)p_N(0) - p_N(ax/\pi)p_{N+1}(0)\big].$$

The two factors of order $N^{-1/2}$ cancel $N+1$ up to a fixed factor. Combining the two bounds yields

$$|K_N(x,0)| \leq Ce^{3a|x|/4} \min\{\Lambda, |x|^{-1}\}. \tag{57}$$

For $0 < q < 2$ let

$$\sigma_q'(x) = Z^{-1}|x|^{q-1}/\cosh(bx), \qquad Z = \int_{\mathbb{R}} |x|^{q-1}/\cosh(bx) \, \mathrm{d}x.$$

This is an atomless probability; its integrable density is unbounded at zero when $q < 1$. Reproduction in τ_a, followed by Cauchy–Schwarz in σ_q, gives for $\deg P \leq N$

$$|P(0)|^2 \leq \|P\|_{\sigma_q}^2 \int_{\mathbb{R}} K_N(x,0)^2 \frac{\tau_a'(x)^2}{\sigma_q'(x)} \, \mathrm{d}x.$$

The density ratio is at most $C|x|^{1-q}e^{-(2a-b)|x|}$. By (57), the last integral is at most

$$C\int_0^\infty \min\{\Lambda^2, x^{-2}\}x^{1-q}e^{-(a/2-b)x}\,\mathrm{d}x \le C\left(\frac{1}{2-q}+\frac{1}{q}\right)\Lambda^q.$$

The last step splits at $1/\Lambda$ and discards the exponential only in the outer integral. Thus a fixed $B_q > 0$ satisfies

$$|P(0)|^2 \le B_q[\log(N+2)]^q\|P\|_{\sigma_q}^2. \tag{58}$$

All constants are independent of P, N, R. Moreover

$$\sigma_q([-r,r]) \le \frac{2r^q}{qZ}, \qquad \int e^{c|x|}\,\mathrm{d}\sigma_q \le \frac{4\Gamma(q)}{Z(b-c)^q}.$$

Choose $0 < \eta \le \min\{1, LqZ/2, CZ(b-c)^q/(4\Gamma(q))\}$. Then $\nu_R = \eta R\sigma_q$ satisfies all budgets and its entire degree-N original form is nonnegative whenever $B_q[\log(N+2)]^q \le \eta R$. Taking $N = \lfloor \exp((\eta R/(2B_q))^{1/q})\rfloor - 2$ proves the required lower for $0 < q < 2$. Together with the preceding $q \ge 1$ proof this establishes (39) for every real $q > 0$.

4.7 Explicit critical models and fixed odd exponents

The basic sech law already gives an exact model at $q = 1$. For $\tau_h'(x) = 1/(h\cosh(\pi x/h))$, $h < \pi/c$, choose $0 < \eta \le \min\{1, Lh/2, C(\pi - ch)/4\}$. Then

$$K_N^{\tau_h}(0,0) = \sum_{j=0}^{\lfloor N/2\rfloor}\left(\frac{(1/2)_j}{j!}\right)^2 = \frac{1}{\pi}\log N + O(1), \quad n(-\delta_0+\eta R\tau_h) = \min\{N : K_N^{\tau_h}(0,0) > \eta R\}. \tag{59}$$

Hence $\log n = \pi\eta R + O(1)$. Equality is again only a null threshold.

A useful fixed-law alternative for each odd $q = 2m+1$ is $\mathrm{d}\tau_m = x^{2m}\,\mathrm{d}\tau_h/Z_m$, $Z_m = \int x^{2m}\,\mathrm{d}\tau_h$. Its local mass is at most $2r^{2m+1}/((2m+1)hZ_m)$ and it has a finite exponential moment. The substitution $Q = x^mP$ gives $P(0) = [x^m]Q$ and $\|P\|_{\tau_m}^2 = \|Q\|_{\tau_h}^2/Z_m$. Dropping the lower-jet constraints on Q can only increase the norm of this coefficient functional. By (52),

$$K_N^{\tau_m}(0,0) \le Z_m\sum_{n=0}^{N+m}|[x^m]Q_n|^2 \le C_m[\log(N+m+2)]^{2m+1}.$$

After one fixed mass rescaling to meet the budgets, this yields the same critical lower order with a fixed law.

At $q = 3$ there is an elementary exact refinement. Put $\mathrm{d}\tau_3 = 4x^2\,\mathrm{d}\tau_h/h^2$, and let $c_j = (1/2)_j/j!$, $S_j = \sum_{l=0}^j c_l^2$, $w_j = ((2j+1)c_j)^{-1}$. The recurrence $(n+1)Q_{n+1} = (2x/h)Q_n - nQ_{n-1}$ gives

$$Q_{2j+1}'(0) = (-1)^j(2/h)w_jS_j.$$

Indeed, dividing its differentiated recurrence by the homogeneous solution w_j adds $(2/h)c_j^2$ at the jth step. Multiplication $Q = xP$ maps the even evaluation problem bijectively to the odd derivative problem, while odd P do not affect evaluation at zero. Thus

$$K_N^{\tau_3}(0,0) = \sum_{j=0}^{\lfloor N/2\rfloor} w_j^2S_j^2 = \frac{(\log N)^3}{12\pi} + O((\log(N+2))^2). \tag{60}$$

The coefficient follows from $c_j^2 = 1/(\pi j) + O(j^{-2})$ and $w_j^2 = \pi/(4j)(1+O(j^{-1}))$. The local and tail bounds are $\tau_3([-r,r]) \le 8r^3/(3h^3)$ and $\int e^{c|x|}\,\mathrm{d}\tau_3 \le 32/(\pi-ch)^3$. For a fixed admissible $\eta > 0$, $\log n(-\delta_0 + \eta R\tau_3) \sim (12\pi\eta R)^{1/3}$. These exact coefficients describe the models, not the minimax constant.

4.8 Why local control is needed

Proposition 4.5. *Fix $c > 0$, $R > 1$ and $C > 1$. For every finite N there is a strictly positive smooth Gaussian-mixture density ν of mass R with $\int e^{cx^2}\,\mathrm{d}\nu \le CR$ such that the square form of $-\delta_0 + \nu$ is positive definite on degree at most N. Nevertheless each such measure has a negative square at a later finite degree.*

Proof. Choose $0 < \epsilon < 1 - 1/R$ and a centered Gaussian probability τ with sufficiently small positive variance that its Gaussian moment is less than C. Let τ_σ be a centered Gaussian of variance σ^2, and put $\nu_\sigma = R((1-\epsilon)\tau_\sigma + \epsilon\tau)$. Its budget holds for all small σ, since $\int e^{cx^2}\,\mathrm{d}\tau_\sigma = (1 - 2c\sigma^2)^{-1/2} \to 1$. In a fixed degree-$N$ monomial basis the original Gram tends to

$$[R(1-\epsilon) - 1]e_0 e_0^T + R\epsilon G_N^\tau,$$

which is positive definite. Finite-dimensional continuity proves the first claim, including all mixed terms.

If one fixed σ had no negative polynomial square, Hamburger's theorem would provide a positive measure ρ with the same moments. Its positive even moments equal those of ν_σ and its mass is $R-1$. Monotone convergence applied to the nonnegative series for $e^{c'x^2}$, $0 < c' < c$, gives $\int e^{c'x^2}\,\mathrm{d}\rho < \infty$. The Fourier transforms of ρ and $-\delta_0 + \nu_\sigma$ are analytic near the real axis, have identical Taylor coefficients, and hence coincide by analytic continuation and Fourier uniqueness. This is impossible because the latter measure has mass -1 at zero. Thus eventual detection holds. □

The example violates any one fixed quantitative local-mass bound along the sequence of increasingly narrow peaks. It does not produce one Gaussian-tail measure invisible at every degree.

4.9 Subcritical tails: inherited all-degree masking

We finish the proof of Theorem 4.1 by recording the exact receiver of the earlier masking theorem [3, Section 6]. For $0 < p < 1$ choose any $d > 0$ and an auxiliary $C_0 > e^{cd^p}$. That theorem gives a fixed positive probability τ supported on $|x| \ge d$, with $\int e^{c|x|^p}\,\mathrm{d}\tau \le C_0$ and a constant $\lambda > 0$ such that

$$\int P^2\,\mathrm{d}\tau \ge \lambda |P(0)|^2 \qquad (P \in \mathbb{R}[x]).$$

Fix $\eta = \min\{1, C/C_0, Ld^q\} > 0$ and let $\nu_R = \eta R\tau$. Below radius d its local mass is zero, and for $r \ge d$ it is at most $\eta R \le LRr^q$. Its other budgets also hold. If $R \ge 1/(\eta\lambda)$, every polynomial square has nonnegative original integral. Thus $D_{p,q}(R) = \infty$. The auxiliary requirement on C_0 imposes no restriction on the receiving parameter $C > 0$.

The conclusions separate three effects: exact geometry under a fixed gap, local background concentration when no gap is required, and the tail regime that permits or prevents uniform polynomial recovery. The classical inputs are essential in each regime.

5 Reproduction and limitations

The accompanying standard-library Python program `verify_finite_controls.py` reconstructs rational moments and compares original signed Hankel inertia with finite reproducing-kernel predictions. It checks strict negativity and exact-null endpoints, including Gaussian, hyperbolic-secant and weighted critical examples. It can be run with ordinary Python and with optimization enabled; its explicit checks remain active in both modes. These computations verify bounded algebraic controls. They do not replace the proofs of asymptotic estimates, uniformity, or all-degree masking.

The geometric results fix the number of atoms and the recovered index. The continuous-packet extension fixes the index and assumes a gap from the convex hull. The exponential class has a full absolute exponential-moment budget and is stated separately from the Gaussian background budget. The gapless results determine orders, not exact leading minimax constants. In particular, the matching lower order for $1 < p < 2$ and growing-index extensions remain separate questions.

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