Geometric Blind Modes in the Weyl Asymptotics of Twisted Tubes

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Luminary Oddities

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Abstract

We determine the leading Dirichlet eigenvalue count for rapidly twisted Euclidean tubes whose inverse twisting speed is a power times a smooth positive logarithmically periodic function. The actual operator count is an explicit periodic profile, without a regular-variation hypothesis. The Fourier multiplier factors into a nonvanishing universal factor and a complex angular-width moment. This gives a necessary and sufficient criterion for recovery of the inverse-speed modulation from the complete leading profile. A fixed connected two-width cross-section hides every odd logarithmic harmonic. In particular, smooth strictly increasing non-regularly-varying speeds on this same geometry have the pure-power law $N(\Lambda) = \Lambda^2/(6\pi) + o(\Lambda^2)$ for two identical ends. The ambiguity is infinite dimensional, whereas visible modes admit conditional Sobolev recovery. For constant-step sections and inverse speeds with derivative bounded above and away from zero, we also prove an $O(\Lambda^{7/4})$ counting remainder. A mixed-homogeneity inverse speed then has a genuine second periodic spectral profile: every mode of its lower-order modulation is visible on the example geometry.

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1 Introduction and statement

Rapid rotation can confine an infinite-volume Dirichlet tube [3]. Eigenvalue-moment bounds were obtained in [2]. The author's earlier papers [7, 8] give exact counting laws for polar-strip tubes, including finite-band geometry, regular and rapid variation, and stability under convergent angle perturbations. Here we extend the actual counting argument to persistent logarithmic modulation outside regular variation, and classify what its leading profile detects when the cross-section is fixed and known.

The mechanism of transform-zero cancellation is classical. In particular, Lapidus–Maier's inverse problems for fractal strings [5], and the explicit oscillatory construction recalled in [6, Section 6], already show how a geometric oscillation can disappear after a spectral transform. Log-periodic asymptotic theory is also established [4]. Our contribution is the actual Euclidean twisted-tube realization and its width-moment criterion within a smooth strictly increasing speed class. For constant-step geometry we further obtain an explicit remainder exponent. It yields an actual second periodic counting term for an inverse speed with two homogeneities, and distinguishes blindness at the critical homogeneity from recovery of a lower-order modulation. The quantitative theorem has narrower geometric hypotheses than the general leading-profile theorem. The zeta factor here is evaluated strictly to the right of 1, so all zeros responsible for blindness come from geometry. There is no connection here to

a Riemann-hypothesis criterion. Nonclassical counting with transverse-mode coefficients also has classical horn antecedents [9].

Definition 1.1 (Admissible cross-section). Let $0 < a < b < \infty$. A nonempty bounded Lipschitz open $\omega \subset \mathbb{R}^2$ is admissible if its closure lies in $a \le r \le b$ and a finite partition $a = r_0 < \cdots < r_M = b$ has the following property. On $I_m = (r_{m-1}, r_m)$ the angular fiber is a finite disjoint union

$$\omega_r = \bigcup_{j=1}^{k_m} (l_{mj}(r), u_{mj}(r)) \pmod{2\pi}.$$

The number k_m may be zero. The endpoint lifts extend continuously to $\overline{I}_m$ and, when present, have positive widths and positive intercomponent gaps on I_m. All their closed-band values lie in one fixed real interval J_m of length $A_m < 2\pi$. Widths and gaps may vanish at the band endpoints. Fibers at partition radii are subject only to the openness and Lipschitz hypothesis on ω itself.

Write $\alpha_{mj} = u_{mj} - l_{mj}$. Define the finite, unweighted width measure by

$$\int f(\alpha)\,\mathrm{d}\mu(\alpha) = \sum_{m,j} \int_{I_m} r f(\alpha_{mj}(r))\,\mathrm{d}r, \qquad M_z = \int \alpha^z\,\mathrm{d}\mu(\alpha) \quad (\Re z > 0).$$

Absent components contribute nothing. The measure is supported in $[0, A]$ for some $A < 2\pi$. This convention differs from an area-weighted width measure; in particular $M_1 = |\omega|$.

Fix $\gamma, L > 0$ and a real smooth L-periodic function h satisfying

$$h > 0, \qquad q := \gamma h + h' > 0. \tag{1.1}$$

Set $R(y) = y^\gamma h(\log y)$ for $y > 0$, and $b = R^{-1}$. On a remote end let the magnitude of the angular speed be $b(x)$, where x is outward axial distance. The sign is fixed on that end; bounded smooth modifications in a fixed core are permitted. The tube is $\Omega_\theta = \{(R_{\theta(s)}y, s) : y \in \omega, s \in \mathbb{R}\}$. Here R_θ denotes planar rotation, distinct from the scalar inverse speed R. Let H be its Friedrichs Dirichlet Laplacian, and let $N(\Lambda)$ count eigenvalues strictly below Λ with multiplicity. For a single end, retain original lateral Dirichlet conditions and allow either a Dirichlet or a free artificial axial cap.

Put $\tau = 2\pi/L$, $z_k = \gamma + ik\tau$ and $h_k = L^{-1}\int_0^L h(t)e^{-ik\tau t}\,\mathrm{d}t$.

Theorem 1.2. *The operator has compact resolvent. For one prescribed end,*

$$N_{\mathrm{end}}(T^2) = T^{3+\gamma}F_h(\log T) + o(T^{3+\gamma}), \qquad T \longrightarrow \infty, \tag{1.2}$$

where F_h is smooth, positive and bounded away from zero, and

$$F_h(t) = \sum_{k\in\mathbb{Z}} h_k K_\mu(z_k) e^{ik\tau t}, \qquad K_\mu(z) = \frac{z\zeta(1+z)M_{1+z}}{2\pi^{2+z}(1+z)(3+z)}. \tag{1.3}$$

For two identical ends the leading term doubles; independently specified ends add. Recovery statements below concern a prescribed end or identical ends.

The local comparison machinery below is inherited from [8] and is reproduced to make the proof self-contained. The new use of it replaces regular-variation energy squeezing by uniform phase continuity.

2 The form, confinement and bounded cores

For a sector $a < r < b$, $0 < t < \alpha < 2\pi$, pullback uses $L^2(r\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s)$ and gives

$$q_\beta[v] = \int \left(|v_r|^2 + r^{-2}|v_t|^2 + |X_\beta v|^2 \right) r\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s, \qquad X_\beta = \partial_s - \beta(s)\partial_t. \tag{2.1}$$

The physical axial coordinate and sector opening make the map injective. Original Dirichlet domains are form closures of compact tests. Auxiliary radial-Neumann domains permit free radial traces and retain angular Dirichlet traces. Finite axial caps are either Dirichlet or free. Locally bounded β controls ordinary H^1 on bounded axial pieces.

Lemma 2.1 (Half-tail estimate)**.** *Suppose $\beta \geq B > 0$ on $s > S$. For either radial boundary convention, restrictions of form vectors satisfy*

$$\int_{s>S} |Xv|^2 r\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s \geq \frac{\pi^2 B^2}{4\alpha^2} \int_{s>S} |v|^2 r\,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}s. \tag{2.2}$$

There is an analogous estimate on the negative half-tail when its twisting speed has the same positive lower bound.

Proof. At fixed physical polar angle $\varphi = t + \theta(s)$, X becomes differentiation in s. The change from (t, s) to (φ, s) has Jacobian one, with φ interpreted modulo 2π. Since $\theta' = \beta > 0$, each connected permitted vertical-line component in the half-tail corresponds to a θ-interval of width at most α. Its length in s is at most α/B. The genuine lateral endpoints are Dirichlet. At most one component can meet the artificial cap $s = S$; that component has at least one Dirichlet endpoint and may have a free trace at the cap. The one-dimensional Dirichlet–Neumann Poincaré inequality on an interval of length L has constant $\pi^2/(4L^2)$. Apply it on each component and integrate in r, φ. The argument first applies to locally smooth vectors with angular zero trace and then passes to the form domain by approximation. The radial boundary convention does not enter the estimate. The negative tail is handled in the same way. □

The same bound applies to a fixed negative sign after reversing the angular orientation. In particular diverging reference speed supplies uniformly vanishing tail mass for form-bounded families.

Bounded axial cores have compact resolvent and count $O(\Lambda^{3/2})$ under either axial cap convention. Indeed, on a fixed piece the form and norm are comparable to the ordinary gradient and volume norms. Transverse Dirichlet zero extension into a fixed annulus, followed by relaxing its artificial boundaries to Neumann conditions, bounds the count by a fixed rectangular Neumann lattice count. The same comparison proves compactness. A fixed squared IMS partition into a cap collar and a part vanishing near the cap gives

$$N_D(\Lambda) \leq N_N(\Lambda) \leq N_D(\Lambda + C) + O(\Lambda^{3/2}). \tag{2.3}$$

The second part extends by zero into the Dirichlet end. The localization error is $\sum |\chi'|^2$, independent of the twist. Min–max proves the inequality. The same argument compares fixed cap positions and combines the two end counts with a bounded core. We will justify removal of its fixed energy shift without invoking regular variation.

3 An exact frozen-box comparison

The following comparison, used in [7], is proved here in full.

Fix ρ in a compact positive interval, a constant $B > 0$, radial length $h > 0$ and longitudinal length $L > 0$. On the rectangular box use Lebesgue measure and the constant-coefficient form

$$q_{\rho,B}[v] = \int \left(|v_r|^2 + \rho^{-2}|v_t|^2 + |v_s - Bv_t|^2 \right) \mathrm{d}r\, \mathrm{d}t\, \mathrm{d}s. \tag{3.1}$$

Angular faces are always Dirichlet. The radial condition is either Dirichlet on both faces, denoted D, or Neumann on both, denoted N. Longitudinal faces are Dirichlet. Denote the corresponding count by $D_{\mathsf{b}}(h, L, \rho, B; E)$. Put

$$\mathcal{J}_{\mathsf{D}} = \{1, 2, \ldots\}, \qquad \mathcal{J}_{\mathsf{N}} = \{0, 1, \ldots\}, \qquad \nu_n = \frac{\pi n}{\alpha}, \tag{3.2}$$

and define

$$J_{\mathsf{b}}(h, \rho, B; E) = \frac{\sqrt{1 + B^2\rho^2}}{\pi} \sum_{\substack{j \in \mathcal{J}_{\mathsf{b}} \\ n \geq 1}} \left[E - (\pi j/h)^2 - (B^2 + \rho^{-2})\nu_n^2 \right]_+^{1/2}. \tag{3.3}$$

All sums at a fixed energy are finite. For negative energy, all counts and positive-part sums in this section are defined to be zero.

Lemma 3.1 (Uniform longitudinal boundary comparison)**.** *For $\eta \in (0, 1)$ there is $C_\eta > 0$, independent of B, ρ, h, L, E and the radial condition, such that*

$$\frac{L}{1+\eta} J_{\mathsf{b}}\left(h, \rho, B; E - \frac{C_\eta}{L^2}\right) \leq D_{\mathsf{b}}(h, L, \rho, B; E) \leq L J_{\mathsf{b}}(h, \rho, B; E). \tag{3.4}$$

Proof. First replace the longitudinal boundary condition by $v(L) = e^{i\vartheta} v(0)$. The longitudinal frequencies are $\xi = (2\pi k + \vartheta)/L$, $k \in \mathbb{Z}$. At a fixed frequency, completion of the angular square gives

$$\rho^{-2}|w_t|^2 + |i\xi w - Bw_t|^2 = (B^2 + \rho^{-2}) \left| w_t - \frac{iB\xi}{B^2 + \rho^{-2}} w \right|^2 + \frac{\xi^2}{1 + B^2\rho^2} |w|^2. \tag{3.5}$$

The constant-modulus angular gauge preserves the two angular Dirichlet conditions. Its phase is independent of r, since ρ is frozen. The exact quasiperiodic eigenvalues are therefore

$$(\pi j/h)^2 + (B^2 + \rho^{-2})\nu_n^2 + \frac{\xi^2}{1 + B^2\rho^2}. \tag{3.6}$$

The longitudinal Dirichlet form domain is contained in every quasiperiodic form domain. Its count is at most the count for each ϑ. Averaging the latter over $[0, 2\pi)$ converts the frequency lattice exactly into $L\, \mathrm{d}\xi/(2\pi)$. Integrating the intervals allowed by (3.6) proves the upper bound in (3.4).

For the lower bound set $d = L/(1 + \eta)$. On a circle of length Kd, for K sufficiently large, choose K translates of a smooth square partition of unity, with spacing d, supports of length L, and

$$\sum_{k=1}^{K} \chi_k^2 = 1, \qquad \sum_{k=1}^{K} |\chi_k'|^2 \leq C_\eta L^{-2}. \tag{3.7}$$

Such a partition follows by normalizing translated nonnegative bumps which are positive on the spacing intervals and supported in their slightly enlarged intervals. The IMS identity is

$$\sum_k q_{\rho,B}[\chi_k v] = q_{\rho,B}[v] + \int \sum_k |\chi_k'|^2 |v|^2. \tag{3.8}$$

Indeed $(\partial_s - B\partial_t)(\chi_k v) = \chi_k X_B v + \chi_k' v$; the localization error has no factor depending on B.

Let $N_{\mathrm{per}}(Kd, E_0)$ be the periodic count on that circle. The isometric localization $v \mapsto (\chi_1 v, \dots, \chi_K v)$ and min–max give

$$N_{\mathrm{per}}(Kd, E_0) \le K D_{\mathsf{b}}(h, L, \rho, B; E_0 + C_\eta L^{-2}). \tag{3.9}$$

For completeness, if the left spectral subspace had greater dimension than the direct sum of the right spectral subspaces, some nonzero vector would localize orthogonally to every right low-energy subspace. Its summed localized energy would be at least the right threshold times its squared norm, contradicting (3.8) and the strict energy bound on the left finite-dimensional spectral subspace. At fixed E_0, h, ρ, B, d, only finitely many bands in (3.6) are active. The count for each band is a lattice count in one bounded frequency interval, with an error at most two. Consequently

$$\lim_{K\to\infty} \frac{1}{K} N_{\mathrm{per}}(Kd, E_0) = d J_{\mathsf{b}}(h, \rho, B; E_0). \tag{3.10}$$

Divide (3.9) by K, pass to this limit, and take $E_0 = E - C_\eta L^{-2}$. This proves the lower bound. □

Lemma 3.2 (Uniform radial sum)**.** *Let ρ range in a fixed compact positive interval. Uniformly in $B > 0$, $h > 0$, $E \ge 0$ and $\mathsf{b} \in \{\mathsf{D}, \mathsf{N}\}$,*

$$J_{\mathsf{b}}(h, \rho, B; E) = \frac{h}{4\pi} \sqrt{1 + B^2\rho^2} \sum_{n\ge 1} \Big[E - (B^2 + \rho^{-2})\nu_n^2\Big]_+ + O(E). \tag{3.11}$$

The implied constant is independent of the growing twist B.

Proof. For $A \ge 0$, decreasing-sum comparison gives

$$\sum_{j\ge 1} \Big[A - (\pi j/h)^2\Big]_+^{1/2} = \frac{h}{4} A + O(\sqrt{A}). \tag{3.12}$$

The integral is $hA/4$ and the sum-integral error is bounded by $\sqrt{A}$. The inclusion of $j = 0$ changes only that error bound. Put $a_B = B^2 + \rho^{-2}$. A further decreasing-sum comparison yields

$$\sum_{n\ge 1} \Big[E - a_B \nu_n^2\Big]_+^{1/2} \le \int_0^\infty \Big[E - a_B(\pi x/\alpha)^2\Big]_+^{1/2} \,\mathrm{d}x = \frac{\alpha E}{4\sqrt{a_B}}. \tag{3.13}$$

The identity $\sqrt{1 + B^2\rho^2} = \rho\sqrt{a_B}$ cancels the possible large coefficient in the summed error of (3.12). Substitution into (3.3) proves the result. □

4 Local clocks and physical phase

Lemma 4.1. *On an interval I of length ℓ let $B = \ell^{-1}\int_I b$. If $\sup_I |b/B - 1| \le \epsilon$, the form and its constant-speed counterpart differ in Rayleigh quotients by relative $O(\epsilon)$, uniformly in B. Freezing a radial cell of length at most d adds relative $O(d)$ on a fixed positive annulus.*

Proof. Set $x(s) = B^{-1}\int_{\inf I}^s b$. This maps I onto $(0, \ell)$. If $u(r, t, x) = v(r, t, s(x))$ and $g = ds/dx = B/b$, exact substitution yields

$$\|v\|^2 = \int g|u|^2 r \,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}x, \qquad q_b[v] = \int \{g(|u_r|^2 + r^{-2}|u_t|^2) + g^{-1}|u_x - Bu_t|^2\} r \,\mathrm{d}r\,\mathrm{d}t\,\mathrm{d}x.$$

Thus $g = 1 + O(\epsilon)$ proves the comparison on positive terms and the norm. The map preserves Dirichlet traces and form closures. No derivative of a square-root density is taken. Both r and r^{-2} vary relatively by $O(d)$, proving the last assertion. □

Since h, q have positive minima,

$$R(y) \asymp y^{\gamma}, \quad R'(y) \asymp y^{\gamma-1}, \quad b'/b^2 = O(b^{-1-\gamma}), \quad (\log b)' = O(b^{-\gamma}). \tag{4.1}$$

In particular $R(cT) = O_c(R(T))$ for every fixed $c > 0$. For a sector define $a_s = b(s)^2 + r^{-2}$, $w = r\sqrt{a_s}$ and

$$I(E) = \frac{1}{4\pi} \int_{\mathrm{end}} \int_a^{b_{\mathrm{rad}}} w \sum_{n \geq 1} [E - a_s(\pi n/\alpha)^2]_+ \, \mathrm{d}r \, \mathrm{d}s. \tag{4.2}$$

All active points for $T^2/2 \leq E \leq 2T^2$ lie below $s = R(CT)$ for a fixed C depending on the opening. The number of modes active at the larger of E_1, E_2 is at most $\alpha\sqrt{\max E_i}/(\pi\sqrt{a_s})$. The elementary Lipschitz inequality for the positive part therefore gives

$$|I(E_2) - I(E_1)| \leq C|E_2 - E_1| T R(T), \qquad I(E) = O(T^3 R(T)). \tag{4.3}$$

These bounds also hold for finite sums of sectors and for the reduced phase introduced below. In particular an $o(T^2)$ energy change costs $o(T^3 R(T))$. A fixed bounded axial region costs $O(T^3)$.

Proposition 4.2. *For a constant-opening sector and either fixed cap, $N_{\mathrm{end}}(T^2) = I(T^2) + o(T^3 R(T))$.*

Proof. Fix $\eta > 0$. The half-tail lemma excludes $s > R(CT)$ at energy T^2 for sufficiently large fixed C, using free radial and axial traces. It also proves compact embedding by uniformly vanishing tail mass, together with compactness on bounded cores. On the active range put

$$D_T = \sup_{s_0 \leq s \leq R(2CT)} |(\log b)'(s)|, \quad \rho_T = \min\{T^{1/4}, [T/(1+D_T)]^{1/2}\}, \quad \ell = \rho_T/T, \quad d = T^{-1/2}.$$

By (4.1), D_T is bounded. Thus $\ell \to 0$, $\ell^{-2} = o(T^2)$, $D_T \ell = o(1)$ and $dT \to \infty$. Use disjoint radial cells of size d with free radial traces for the upper bound and a squared axial partition with spacing ℓ, supports of length $(1+\eta)\ell$, and IMS error $C_\eta \ell^{-2}$. All contributing supports lie below $R(2CT)$: reaching that level from the active range in length $O_\eta(\ell)$ would require a logarithmic speed change at least $\log 2$, contradicting $D_T \ell = o(1)$. Lemma 4.1 and radial freezing distort each quotient by $o(1)$, uniformly in the growing speed. The upper bound of Lemma 3.1 and Lemma 3.2 now apply.

There are $O_\eta(R(CT)/(d\ell))$ boxes. The radial-sum error per box is $O(T^2)$ times its axial length, so its total is $O_\eta(R(CT)T^2/d) = o(T^3 R(T))$. Radial and axial sums of the principal terms are squeezed into (4.2). Explicitly, if $(1-\delta)a_0 \leq a \leq (1+\delta)a_0$, then $[E - a\nu^2]_+$ lies between $(1 \pm \delta)[E/(1 \pm \delta) - a_0\nu^2]_+$ with the matching inequality directions. This controls all thresholds simultaneously; the weight w has the same uniform relative continuity. The partition overlap costs factor $1 + \eta$. Equation (4.3) removes the $o(T^2)$ energy changes, and gives $N \leq (1+\eta) I(T^2) + o(T^3 R(T))$.

For the lower bound insert disjoint radial and axial Dirichlet boxes of lengths d, ℓ. Lemma 3.1 costs $C_\eta \ell^{-2} = o(T^2)$ and factor $(1+\eta)^{-1}$. The total radial error is unchanged. Discarded radial layers cost $O(dR(CT)T^3)$ and axial end layers $O(\ell T^3)$; the uniform phase bound makes both negligible. The same coefficient squeeze gives $N \geq (1+\eta)^{-1} I(T^2) - o(T^3 R(T))$. The uniform phase bound allows $\eta \downarrow 0$ after $T \to \infty$. Fixed core removal and free caps follow from (2.3) and (4.3). No regular-variation equivalent was used. □

5 Reduced phase and exact periodic evaluation

Define

$$P_R(T) = \frac{1}{4\pi} \int \mathrm{d}\mu(\alpha) \sum_{n \geq 1} \int_0^{\infty} y[T^2 - (\pi n y/\alpha)^2]_+ \, \mathrm{d}R(y). \tag{5.1}$$

For a sector its measure is $(\int r\,\mathrm{d}r)\delta_\alpha$. The continuation to all $y > 0$ below a fixed remote cap costs $O(T^3)$.

Lemma 5.1. *For a sector, $I(T^2) - P_R(T) = o(T^3 R(T))$.*

Proof. Fix $B > 0$. The region $b(s) < B$ in either expression has size $O(T^3 R(B))$, negligible after division by $T^3 R(T)$. On its complement, with $\delta = CB^{-2}$,

$$1 \le a_s/b(s)^2 \le 1+\delta, \qquad 1 \le w/(rb(s)) \le (1+\delta)^{1/2}.$$

The physical tail lies above $(1+\delta)$ times the reduced tail at energy $T^2/(1+\delta)$ and below $(1+\delta)^{1/2}$ times the reduced tail at T^2. The reduced version of (4.3) bounds their differences by $O(\delta T^3 R(T)) + O_B(T^3)$. First send T to infinity and then B to infinity. This proves the claim. □

For $\Re z = \gamma > 0$ direct integration gives

$$\int_0^{T/\nu} z y^z (T^2 - \nu^2 y^2)\,\mathrm{d}y = \frac{2zT^{3+z}\nu^{-1-z}}{(1+z)(3+z)}.$$

Summing $\nu = \pi n/\alpha$ and integrating the widths proves (1.3) for a complex-power inverse speed. For general smooth h, its Fourier series and the series of its derivative are absolutely convergent. Since $\mathrm{d}R = y^{\gamma-1} q(\log y)\,\mathrm{d}y$ and q is bounded, the real majorant has convergent sum $\sum n^{-1-\gamma}$ and finite moment $M_{1+\gamma}$. Tonelli and dominated convergence justify every exchange. Thus

$$P_R(T) = T^{3+\gamma} F_h(\log T) \tag{5.2}$$

exactly. Positivity and uniform upper/lower bounds on q compare this phase with that of y^γ, proving $0 < c \le F_h \le C$. The moment is uniformly bounded on the Fourier lattice and the rational factor in (1.3) is $O((1+|k|)^{-1})$; smoothness of h therefore also proves smooth convergence of the profile series.

6 Finite-band passage and completion of the count

Lemma 6.1 (Lateral zero extension)**.** *Restrict an original form vector to a radial slab and an axial half-line. If its physical support there lies in a comparison sector tube, its lateral zero extension preserves norm and energy and permits free radial and axial cap traces.*

Proof. On bounded axial pieces the original transverse Dirichlet zero extension is H^1, first for compact smooth tests and then by completion. Restrict this extension to the slab and half-line; no extension is made through artificial caps. Its gradient is the original gradient on the support and zero elsewhere, with no distributional Dirichlet-wall term. This argument in physical coordinates does not differentiate continuous polar endpoints. Exhaustion passes to finite-energy vectors on a half-line. If a union of disjoint outer sectors contains the support with open gaps, restriction to those sectors gives the corresponding direct-sum injection. □

Proof of Theorem 1.2. The sector result follows from Proposition 4.2, Lemma 5.1 and (5.2). For general admissible geometry first cut at radial partition radii. The envelope sectors with free radial faces and axial caps contain the support and give compactness using the half-tail estimate and lateral zero extension. Compactness on bounded cores completes this argument.

Fix $\epsilon > 0$, trim neighborhoods of radial partition endpoints of total length at most ϵ, and keep trimming fixed. On retained compact bands, every present width and gap has a positive minimum. Uniform continuity permits a fixed fine radial mesh C with inner angular intervals $(\max_C l, \min_C u)$ and outer intervals $(\min_C l, \max_C u)$, each of positive length and separated

by gaps. Contract or enlarge them by an additional fixed angular amount $\delta > 0$ if needed. Original physical-domain inclusions give a lower bound by inserting radial Dirichlet walls and an upper bound by free radial restriction and the zero-extension lemma. Trimmed layers in the upper bound use envelope sectors and cost at most $C\epsilon T^{3+\gamma}$ in their leading phase. The core costs $O(T^3)$.

The sector profile depends continuously on its width $\alpha \in [0, A]$ in the supremum norm of the phase variable. Indeed substitute $y = \alpha T v/(\pi n)$ in (5.1). Up to a constant the profile is a sum of $\alpha^{1+\gamma} n^{-1-\gamma}$ times integrals of bounded periodic q at shifted logarithmic arguments. Uniform continuity of q and dominated convergence prove continuity for positive width; the bound $C\alpha^{1+\gamma}$ proves it at zero. These observations also cover pinching widths at band endpoints. Taking the energy limit first, then $\delta \downarrow 0$, refining the fixed radial mesh, and finally removing trimming identifies the inner and outer profile sums with the stated width measure, uniformly in $\log T$. This proves (1.2).

Reversing the axial coordinate and reflecting angular orientation handles either remote speed sign. A fixed IMS partition combines independent ends and a bounded core. Its fixed energy shifts are negligible by (4.3); no regular variation is assumed. □

7 Detectability and conditioning

Theorem 7.1. *Two admissible modulations have the same complete leading profile if and only if*

$$(h_{1,k} - h_{2,k}) M_{1+\gamma+ik\tau} = 0 \quad (k \in \mathbb{Z}). \tag{7.1}$$

Consequently recovery throughout this smooth positive monotone class is injective exactly when no nonzero lattice moment vanishes. If, on the visible modes, $|M_{1+\gamma+ik\tau}| \geq c(1+|k|)^{-a}$ for some $a \geq 0$, then, after projection away from the blind modes,

$$\|h_1 - h_2\|_{H^s} \leq C \|F_{h_1} - F_{h_2}\|_{H^{s+a+1}} \quad (s \in \mathbb{R}). \tag{7.2}$$

Here the Sobolev norms are periodic Fourier norms. For a single constant nonzero width the inverse loses exactly one derivative.

Proof. A continuous periodic leading profile is unique: if two such profiles differ by $o(1)$ as $t \to \infty$, evaluation at $t_0 + nL$ proves equality at every t_0. The factors of K_μ other than its moment are nonzero on $\Re z = \gamma > 0$. In particular the Euler product shows $\zeta(1+z) \neq 0$ there. Fourier uniqueness gives (7.1). The zero mode is visible since $M_{1+\gamma} > 0$. If a nonzero moment vanishes, small real cosine or sine perturbations of a positive constant h preserve both strict inequalities (1.1), but are invisible. This proves the criterion.

Absolute convergence of the zeta and reciprocal-zeta Dirichlet series gives

$$\zeta(1+\gamma)^{-1} \leq |\zeta(1+\gamma+it)| \leq \zeta(1+\gamma).$$

The rational factor $z/((1+z)(3+z))$ is comparable in modulus to $(1+|k|)^{-1}$ on the lattice. The moment is uniformly bounded above. Thus $|K_\mu(z_k)| \geq c'(1+|k|)^{-a-1}$ on visible modes, proving (7.2) by summing squared Fourier coefficients. For a constant width the moment modulus is a positive constant; the multiplier is bounded above and below by constant multiples of $(1+|k|)^{-1}$. Testing individual high modes proves that the loss of one derivative cannot be improved. Their amplitudes may be chosen small enough to retain (1.1). □

Absence of zeros alone gives no quantitative bound for general geometries: small divisors must be controlled. The theorem concerns an ideal complete leading profile, rather than recovery from finitely many eigenvalues or from a remainder with an unspecified rate.

8 A connected fixed-geometry counterexample

Let ω be the interior of the closure of the union of polar sectors $(1,3)\times(0,1)$ and $(3,\sqrt{11})\times(0,2)$. They share an open angular interface, so the domain is connected. It is a finite step polygon under the smooth bi-Lipschitz polar map on this positive annulus and opening $2<2\pi$, and is therefore Lipschitz and admissible. Its width measure is $4\delta_1+\delta_2$. Choose $\gamma=1$, $L=2\log 2$, $\tau=\pi/\log 2$. Then

$$M_{2+ik\tau}=4+2^{2+ik\tau}=4[1+(-1)^k].$$

Exactly the odd harmonics are blind. The smooth kernel consists of functions g with $g(t+\log 2)=-g(t)$, an infinite-dimensional space. Every even mode has moment 8, so its recovery loses exactly one derivative as in Theorem 7.1.

For an explicit example set

$$h_\epsilon(t)=1+\epsilon\cos(\tau t),\qquad 0<|\epsilon|<(1+\tau^2)^{-1/2}.$$

Then $h_\epsilon>0$ and $h_\epsilon+h_\epsilon'=1+\epsilon(\cos\tau t-\tau\sin\tau t)>0$. Thus $R_\epsilon(y)=yh_\epsilon(\log y)$ and its inverse are smooth strictly increasing. On two identical remote ends Theorem 1.2 gives

$$N_\epsilon(\Lambda)=\frac{\Lambda^2}{6\pi}+o(\Lambda^2), \tag{8.1}$$

the same law as $\epsilon=0$ on this same cross-section. Indeed $M_2=8$ and $K_\mu(1)=1/(12\pi)$ per end. But

$$\frac{R_\epsilon(2y)}{R_\epsilon(y)}=2\frac{1-\epsilon\cos(\tau\log y)}{1+\epsilon\cos(\tau\log y)}$$

has distinct subsequential limits. Hence R_ϵ is not regularly varying. Nor is b_ϵ: its comparability to x forces index 1 if it is regularly varying, and monotone inversion then makes R_ϵ regularly varying of index 1 [1].

The strict monotonicity is essential to the intellectual distinction: the convergent-angle perturbations in [8] already permit highly irregular or reversing speeds with the same leading law. The present example retains monotonicity and persistent order-one inverse modulation. It is a failure of recovery from the leading count, not an equality of entire spectra.

9 A quantitative remainder for constant-step sections

We now restrict the geometry: on each of finitely many radial bands the angular fibers are fixed finite unions of intervals, of openings below a fixed $A<2\pi$. The sectors may join across radial interfaces, provided that their union is Lipschitz. This includes the connected example above.

Theorem 9.1. *For such a constant-step section suppose the remote inverse speed is C^1, strictly increasing, and satisfies*

$$0<c\le R'(y)\le C<\infty\qquad(y\ge y_0>0).$$

Then, for one end and either fixed cap,

$$N_{\mathrm{end}}(T^2)=P_R(T)+O(T^{7/2}). \tag{9.1}$$

In defining P_R, any increasing C^1 extension to zero with bounded derivative may be used; different choices below a fixed threshold change the phase by $O(T^3)$. For two ends, add their phases and errors. Constants may depend on the fixed geometry, derivative bounds and core; they are not uniform as the lower derivative bound tends to zero.

Proof. Here $b' = (R' \circ b)^{-1}$ is bounded above and below, $b(s) \asymp s$ on the remote end, and $(\log b)'$ is bounded after a fixed positive-speed cap. Both physical and reduced phases are $O(T^4)$. The proof of (4.3) now gives

$$|I(E_2) - I(E_1)| \le C|E_2 - E_1|T^2 \quad (T^2/2 \le E_i \le 2T^2), \tag{9.2}$$

also on arbitrary restricted axial subsets. The integrand per unit axial length is at most CT^3, by the decreasing integral bound on the angular positive-part sum. The same bounds hold for P_R.

We first make the overlap constants explicit. For axial spacing ℓ and support length $(1+\eta)\ell$, $0 < \eta \le 1/2$, choose sine/cosine transitions on overlaps of length $\eta\ell$. At most two cutoffs overlap, their squares sum to one, and

$$\sum_j |\chi_j'|^2 \le C\eta^{-2}\ell^{-2}.$$

Smooth approximations with the same bound suffice if desired. This construction on the circle in Lemma 3.1 proves that its constant can be chosen $C_\eta \le C\eta^{-2}$. It also controls the global axial IMS error, with no twist-dependent factor.

Put $B = T^{1/2}$. The region $b \le 2B$ has axial length $O(B)$. To bound its count uniformly, use a squared partition into axial Dirichlet supports of length at most 2 and spacing comparable to 1, with uniformly bounded derivative-square sum, and a fixed finite radial mesh with cell sizes bounded above and below. On these supports b varies by a bounded multiplicative factor, since $(\log b)'$ is bounded. The exact local clock gives a fixed quotient distortion independent of the speed size. The frozen upper bound and radial-sum lemma give $O(T^3)$ eigenvalues per support at energy CT^2; the radial errors are $O(T^2)$ per fixed cell. There are $O(B)$ supports, so this entire growing region costs $O(BT^3) = O(T^{7/2})$. Use squared low/high cutoffs across $B \le b \le 2B$: because b' is bounded above and below, their transition has length comparable to B and derivatives $O(B^{-1})$. Composing with the local partition keeps energy errors bounded. A genuine fixed free cap uses a fixed collar, with fixed coefficients and count $O(T^3)$. Thus no growing-speed free-box estimate is being assumed.

On $b \ge B$, choose axial length $\ell = 1$, overlap $\eta = T^{-1/2}$ and radial mesh $d = T^{-1/2}$. The relative speed variation on an axial support is $O(B^{-1})$, and radial freezing costs $O(d)$. The global and frozen-box IMS energy errors are $O(\eta^{-2}) = O(T)$. The half-tail bound confines all active supports to an axial range of length $O(T)$, allowing a bounded enlargement of the cutoff range. The quotient errors change energy by $O((B^{-1} + d)T^2) = O(T^{3/2})$; by (9.2), their total phase cost is $O(T^{7/2})$. Overlap factors cost $O(\eta T^4) = O(T^{7/2})$, and IMS shifts cost $O(T^3)$. There are $O(T/d)$ radial-axial boxes, so the summed radial error is $O(T^3/d) = O(T^{7/2})$.

For the upper bound use the small-overlap squared partition and free radial interfaces. For the lower bound insert disjoint axial Dirichlet boxes of length 1 and radial Dirichlet cells. The lower frozen comparison costs the same $O(T)$ energy shift and factor $(1+\eta)^{-1}$. Missing axial end intervals have length $O(1)$ and phase $O(T^3)$; the low-speed omission was already controlled. The positive-part squeeze from Proposition 4.2 includes every angular threshold crossing. These comparisons give

$$N_{\mathrm{end}}(T^2) = I(T^2) + O(T^{7/2})$$

for each constant sector with either radial convention. At a step interface insert Dirichlet walls for the lower comparison and relax traces on entire radial faces for the upper comparison. This is an exact domain enlargement even where sectors join only along part of a face. Each comparison splits into the same finite sector list, with the same physical phase and error. No radial trimming is needed.

Finally, when $y = b(s) \geq \max(1, y_0)$ the relative difference of $y^2 + r^{-2}$ from y^2, and of its square-root weight from y, is $O(y^{-2})$. The per-unit-length phase is $O(T^3)$ and its energy derivative is $O(T)$. A shift $O(T^2 y^{-2})$ in the positive-part squeeze therefore costs $O(T^3 y^{-2})$ pointwise. Since $\mathrm{d}s = R'(y)\,\mathrm{d}y$ with bounded derivative,

$$|I(T^2) - P_R(T)| \leq CT^3 \left(1 + \int_1^\infty y^{-2}\,\mathrm{d}y\right) = O(T^3).$$

The bounded low-speed interval contributes the first $O(T^3)$ term. This proves (9.1). Fixed cap/core IMS comparisons cost $O(T^3)$ by (9.2) and complete the proof. □

In particular, for the original pure blind example (8.1) improves to

$$N_\epsilon(\Lambda) = \frac{\Lambda^2}{6\pi} + O(\Lambda^{7/4}). \tag{9.3}$$

This exponent is sufficient for the next theorem.

10 An actual lower-order profile and its recovery

Write $F_\sigma[f]$ for the multiplier profile in (1.3), with γ replaced by $\sigma > 0$ and periodic input f. The definition is linear and makes sense for signed smooth inputs.

Theorem 10.1. *On a constant-step section suppose, eventually,*

$$R(y) = yh(\log y) + y^\delta g(\log y), \qquad \tfrac{1}{2} < \delta < 1,$$

where h, g are real smooth L-periodic functions with $h > 0$ and $h + h' > 0$. Then

$$N_{\mathrm{end}}(T^2) = T^4 F_1[h](\log T) + T^{3+\delta} F_\delta[g](\log T) + O(T^{7/2}). \tag{10.1}$$

Both periodic profiles are uniquely determined by this asymptotic. For the connected two-width section and $L = 2\log 2$, every Fourier mode of g is recoverable from the second profile, with exactly one Sobolev derivative lost, although odd modes of h are blind in the first profile.

Proof. Differentiation gives $R' = h + h' + y^{\delta-1}(\delta g + g')$. Hence R' is eventually bounded above and away from zero. Choose any positive increasing bounded-core extension and apply Theorem 9.1. Linearity of the reduced phase and the complex-power integration give the two displayed profiles. Smooth Fourier convergence and the convergent majorants $\sum n^{-1-\sigma}$ for $\sigma = 1, \delta$ justify both evaluations, including signed g. Cutting off either model expression on a fixed initial interval costs $O(T^3)$, absorbed in the remainder.

Set $t = \log T$ and sample $t = t_0 + nL$. Division by T^4 first determines the leading periodic profile. Subtract it exactly; division by $T^{3+\delta}$ then determines the second profile because $7/2 < 3 + \delta$. For the stated section,

$$M_{1+\delta+ik\tau} = 4 + 2^{1+\delta}(-1)^k, \qquad 4 - 2^{1+\delta} > 0.$$

These moments are uniformly bounded above and away from zero. The proof of Theorem 7.1 therefore gives $c_\delta(1 + |k|)^{-1} \leq |K_\mu(\delta + ik\tau)| \leq C_\delta(1 + |k|)^{-1}$. It proves the recovery estimate and sharp derivative loss. Constants deteriorate as $\delta \uparrow 1$. □

For example, let $h = h_0 + u$ with smooth anti-periodic $u(t + \log 2) = -u(t)$ satisfying the strict positivity conditions, and take $g(t) = a\cos(\tau t + \phi)$. Per end,

$$N_{\mathrm{end}}(T^2) = \frac{h_0 T^4}{12\pi} + aT^{3+\delta}\Re\{e^{i\phi} K_\mu(\delta + i\tau) e^{i\tau \log T}\} + O(T^{7/2}).$$

Here the multiplier is nonzero since its moment is $4 - 2^{1+\delta} > 0$. This is a genuine lower-order spectral signature at the same logarithmic frequency as a leading blind mode. It describes a separate lower-homogeneity perturbation.

If a prescribed model couples the two orders, for example

$$R_\epsilon(y) = y[1 + \epsilon \cos(\tau \log y)] + \epsilon y^\delta \cos(\tau \log y)$$

on a remote tail, with $|\epsilon| < (1 + \tau^2)^{-1/2}$, then the second profile determines ϵ, while the first profile does not. This recovery uses the explicitly imposed coupling.

11 Limits and further questions

The error in the general finite-band theorem (1.2) is qualitative. The constant-step estimate (9.1) gives an explicit exponent and proves the mixed-order profile (10.1), but is still too large to determine whether lower-order terms detect the original pure critical blind modulation. In particular, expanding the radial correction in a formal phase is insufficient: the operator-to-phase error would first have to be controlled below the proposed lower order, with radial interfaces and bounded cores treated at the same accuracy. No second term for the pure critical blind example, heat expansion, meromorphic spectral-zeta continuation or exact isospectrality follows. Improving the remainder below the radial boundary scale and finite-spectrum recovery remain separate problems.

AI assistance. AI tools assisted with mathematical exploration, drafting and independent internal checks. The mathematical statements and full arguments are given here for independent scrutiny.

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