Frequency Localization for Weil Operator Certificates

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Abstract

We give a frequency-block enclosure for a constrained theta-weighted Weil operator. The retained paired-prime operator is translation invariant before the theta multipliers are applied. Splitting at that intermediate stage eliminates the cross-frequency blocks exactly. A finite-rank approximation of the remaining low-frequency map then has a complete operator error whose high-frequency term is proportional to the reciprocal digamma symbol, rather than its inverse square root. We retain the signed finite interaction matrix and enclose its resolvent Gram by full residuals. We prove matching leading asymptotics for both the raw resolvent tail and the positive spectral edge of the actual signed high-frequency block. The retained arithmetic coefficient is therefore asymptotically exact at every fixed prime cutoff and positive resolvent parameter. A compact-torus recurrence argument supplies the arithmetic peaks; an abstract conditional extension covers valid null-space deflations. An elementary theta estimate makes the multiplier constant self-contained.

1 Introduction

A finite matrix can certify a lower bound for an infinite-dimensional operator only when its omitted and mixed terms are paid. In a weighted Weil realization, the positive reference controls a logarithmic Fourier moment, while the arithmetic perturbation is a sum of paired translations between theta multipliers. Approximating the entire resolvent output first loses the particular algebra of those translations. We instead split the retained arithmetic form in frequency before spatial localization.

The main result is an explicit finite-rank enclosure in operator norm, with a signed one-sided refinement. The finite matrix retains its negative part. Its resolvent Gram can be bounded using the entire residual covariance, without assuming that a Galerkin residual equals a full residual. The high-frequency error has a matching positive spectral asymptotic, proved by arithmetic phase recurrence and constrained Fourier packets. The raw-output asymptotic is also sharp. Together these identify both the improvement supplied by the frequency split and the leading high-band cost that remains at a fixed arithmetic cutoff.

The constrained realization, its chosen core closure and paired arithmetic tail were developed in the author's working notes [3]. The signed finite interaction matrix and full residual-Gram enclosure already occur in [4]; they are reproduced here to make the certificate self-contained. The present contribution is to split the arithmetic sandwich before localization, orient the finite approximation as a frequency projection after a spatial cutoff, and obtain a reciprocal-symbol error with matching signed and raw high-frequency asymptotics. The elementary unit multiplier bound below supplies the same useful constant directly from the full theta series; its role is to make that input self-contained.

The ingredients are classical: Fourier multipliers and orthogonal projections, real-phase Taylor remainders, Hilbert–Schmidt bounds, resolvent identities and positive-form representation; see [2, 1].

For other operator realizations of Weil's form, see Suzuki [5]; full-arithmetic constructions also appear in [6]. Those constructions are context, rather than proof dependencies for the estimates below. We define the weighted realization and prove the bounds needed here directly. The resulting contribution is a quantitative localization theorem for this specified operator.

2 The weighted realization and its explicit bounds

All Hilbert spaces are complex, with inner products conjugate-linear in the first argument. Fourier transformation is

$$\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-iu\xi} f(u)\,\mathrm{d}u, \qquad \|f\|^2 = \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \frac{\mathrm{d}\xi}{2\pi}.$$

Let $\vartheta(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}$. Define the Riemann theta kernel, in the present spatial normalization, by

$$\psi(u) = 2\Phi_{\mathrm{RT}}(u/2), \qquad \Phi_{\mathrm{RT}}(t) = \sum_{n=1}^{\infty} (2\pi^2 n^4 e^{9t} - 3\pi n^2 e^{5t}) e^{-\pi n^2 e^{4t}}. \tag{1}$$

The series and all derivatives converge locally uniformly for real arguments. It is even after the indicated scaling. To check this normalization, put $b(u) = e^{u/2}(\vartheta(e^{2u}) - 1)$. Poisson summation gives

$$b(u) - b(-u) = e^{-u/2} - e^{u/2}.$$

Direct differentiation gives $\psi = \frac{1}{2}(\partial_u^2 - 1/4)b$, proving evenness. For $u \geq 0$ the individual terms of ψ are positive, since $\pi n^2 e^{2u} > 3/2$. Hence ψ is smooth and strictly positive everywhere.

Put

$$c(u) = \cosh(u/2), \quad w = \psi c, \quad s = \sqrt{\psi/c}, \quad \mathcal{K} = \{h \in L^2(\mathbb{R}) : \langle \sqrt{w}, h\rangle = \langle \sqrt{w}\tanh(u/2), h\rangle = 0\}.$$

The two row vectors are in L^2 and linearly independent. Under the weighted unitary transformation $h = \sqrt{w}\,g$, the auxiliary physical test is $\psi g = sh$. We retain exactly these two rows.

Write $\psi_0(z) = \Gamma'(z)/\Gamma(z)$ for the digamma function, to distinguish it from the theta kernel. Define

$$m_0 = \psi_0(1/4) - \log\pi, \qquad a(\xi) = \Re\psi_0(1/4 + i\xi/2) - \psi_0(1/4). \tag{2}$$

The standard digamma series and asymptotic [1, §§5.7,5.11] give, with $t = |\xi|/2$,

$$a(\xi) = \sum_{j=0}^{\infty} \frac{t^2}{(j+1/4)((j+1/4)^2 + t^2)}, \qquad a(\xi) = \log|\xi| + O(1) \quad (|\xi| \to \infty). \tag{3}$$

In particular a is nonnegative, even, and strictly increasing in $|\xi| > 0$.

Let $\mathcal{C} = C_c^\infty(\mathbb{R}) \cap \mathcal{K}$. It is dense in $\mathcal{K}$: approximate in L^2 by compact smooth functions and repair their two row errors using two fixed compact smooth bumps whose row matrix is invertible. Such bumps exist by linear independence of the smooth row functions. Define on $\mathcal{C}$

$$\mathfrak{a}[h] = \int_{\mathbb{R}} a(\xi)|\widehat{sh}(\xi)|^2 \frac{\mathrm{d}\xi}{2\pi}.$$

This form is closable. Indeed the maximal graph map $h \mapsto a(D)^{1/2}(sh)$ is closed, since multiplication by s is bounded and the Fourier multiplier is closed. Restriction of its nonnegative closed form is closable. Let $\mathcal{V}$ be the closure of $\mathcal{C}$ in its form norm, and let $A \geq 0$ be its associated selfadjoint

operator. We use this chosen core closure throughout, without asserting equality with a maximal form domain.

Let $S_b f(u) = f(u+b)$ and $c_n = \Lambda(n)/\sqrt{n}$, where Λ is the von Mangoldt function. On $\mathcal{K}$ define

$$B = P_{\mathcal{K}} \left[m_0 M_s^2 - \sum_{n=2}^{\infty} c_n M_s (S_{\log n} + S_{-\log n}) M_s \right] \big|_{\mathcal{K}}, \qquad H = A + B. \tag{4}$$

The bounds below prove convergence in operator norm. This defines the theta-weighted constrained realization directly. All results concern this specified H and its bounded perturbations.

Lemma 2.1 (Explicit theta and arithmetic budgets)**.** *For $u \geq 0$,*

$$\psi(u) \leq 8\pi^2 e^{9u/2 - \pi e^{2u}}, \qquad s(u)^2 \leq 16\pi^2 e^{4u - \pi e^{2u}}.$$

Moreover $0 < \psi(u) < 11161963/11500000 < 1$ for every real u. Consequently $\|s\|_\infty < 1$ and, for $R > 0$,

$$\sup_{|u|>R} s(u) \leq \sigma_R := \min\{1, 13 e^{2R - (3/2)e^{2R}}\}.$$

If B_M retains the same m_0 term and $2 \leq n \leq M$, then, for integer $M \geq 1$,

$$\|B - B_M\| \leq \delta_M := \frac{640(M+1)^2}{20^{M+1}}. \tag{5}$$

Moreover $|m_0| < 6$ and $a(L) \geq \frac{1}{2}\log(2L)$ for $L \geq 1/2$.

Proof. Put $r = \pi e^{2u} \geq \pi$. Discarding the negative coefficient in (1) gives

$$\psi(u) \leq 4\pi^2 e^{9u/2 - r} \sum_{n \geq 1} n^4 e^{-r(n^2-1)}.$$

For $n \geq 2$, $n^2 - 1 \geq 3(n-1)$. With $q = e^{-3\pi} < 1/100$, the sum is at most

$$\sum_{n \geq 1} n^4 q^{n-1} = \frac{1 + 11q + 11q^2 + q^3}{(1-q)^5} < 2.$$

This proves the first bound. Divide by $c(u) \geq e^{u/2}/2$ for the second. Taking its square root and using $4\pi < 13$, $\pi > 3$ gives the displayed decaying spatial bound.

To obtain the unit cap, write the individual theta terms as

$$\psi_n(u) = e^{u/2}(4r_n^2 - 6r_n)e^{-r_n}, \qquad r_n = \pi n^2 e^{2u}.$$

Direct differentiation gives

$$\psi_n'(u) = -e^{u/2 - r_n} r_n q(r_n), \qquad q(r) = 8r^2 - 30r + 15.$$

For $r \geq \pi > 157/50$, $q'(r) > 0$, and $q(16/5) = 23/25 > 0$. The first term therefore attains its maximum at a point with $r_1 \leq 16/5$, either at the boundary or at the unique possible preceding zero of q. Its location satisfies

$$u_* \leq \tfrac{1}{2}\log(160/157) < 3/314, \qquad e^{u_*/2} < e^{3/628} \leq 628/625.$$

Here $e^z \le (1-z)^{-1}$ for $0 \le z < 1$. The numerator is at most $4(16/5)^2 - 6(157/50) = 553/25$, while $e^{r_1} \ge e^{\pi} > 23$. For the last elementary inequality, the Taylor sum through degree eight proves $e^3 > 20$, and, with $z = 7/50$, one has $20(1 + z + z^2/2 + z^3/6) > 23$. Thus

$$\sup_{u \ge 0} \psi_1(u) < \frac{628 \cdot 553}{625 \cdot 25 \cdot 23} = \frac{347284}{359375}.$$

For $n \ge 2$, $r_n \ge 4\pi > 16/5$, so every remaining term decreases on $[0, \infty)$. Since $\pi^2 < 10$,

$$\sum_{n \ge 2} \psi_n(u) < 40 \sum_{n \ge 2} n^4 e^{-3n^2} \le 40 e^{-12} \sum_{k \ge 0} (k+2)^4 e^{-15k}.$$

We used $(k+2)^2 \ge 4 + 5k$. Also $e^{12} > 20^4$ and $e^{15} > 1000$. Differentiating the geometric series gives

$$\sum_{k \ge 0} (k+2)^4 z^k = \frac{16 + z + 11z^2 - 5z^3 + z^4}{(1-z)^5},$$

whose value at $z = 1/1000$ is less than 17 by exact rational arithmetic. Its positive series coefficients imply monotonicity for $0 \le z < 1$. Hence the remaining theta tail is less than $17/4000$, and

$$\psi(u) < \frac{347284}{359375} + \frac{17}{4000} = \frac{11161963}{11500000} < 1.$$

Reflection covers negative u, and division by $c(u) \ge 1$ proves the unit multiplier bound and the cap in σ_R.

For $b \ge 0$, a weighted translation has norm $\sup_u s(u)s(u+b)$. Write $v = |u| + |u+b| \ge b$. The two theta bounds and arithmetic–geometric mean give

$$s(u)s(u+b) \le 16\pi^2 e^{2v - \pi e^v} \le 16\pi^2 e^{2b - \pi e^b},$$

since the latter exponent decreases for $v \ge 0$. Thus the channel with $b = \log n$ has norm at most $16\pi^2 n^2 e^{-\pi n}$. Since $c_n \le 1$, the paired tail is at most $32\pi^2 \sum_{n > M} n^2 e^{-\pi n}$. Put $k = M + 1 \ge 2$ and use $e^{-\pi} < 1/20$:

$$\sum_{n \ge k} n^2 e^{-\pi n} \le k^2 20^{-k} \sum_{j \ge 0} (j+1)^2 20^{-j} < 2k^2 20^{-k}.$$

Together with $\pi^2 < 10$, this proves (5), including norm convergence and bounded selfadjointness of B. The bound $c_n \le 1$ follows from $\Lambda(n) \le \log n \le \sqrt{n}$. The classical value $\psi_0(1/4) = -\gamma - \pi/2 - 3\log 2$ gives $|m_0| < 6$. Finally, for $L \ge 1/2$ put $t = L/2$, $J = \lfloor t - 1/4 \rfloor \ge 0$. Retaining $0 \le j \le J$ in (3) gives

$$a(L) \ge \frac{1}{2} \sum_{j=0}^{J} \frac{1}{j + 1/4} \ge \frac{1}{2} \log(4J + 5) \ge \frac{1}{2} \log(2L).$$

At endpoints the same weak inequalities suffice. This completes all budgets. □

3 The frequency-block enclosure

Initially let $\mathcal{S} = \mathcal{K}$, $A_{\mathcal{S}} = A$, and $B_{\mathcal{S}} = B$. More generally the following proof works on any closed receiving subspace with a nonnegative reference satisfying the same form identity and the compressed factorization (4); valid null-space compression is addressed in Section 5. Fix $\varepsilon > 0$ and put

$$R_\varepsilon = (A_{\mathcal{S}} + \varepsilon)^{-1/2}, \quad J = M_s R_\varepsilon, \quad j_\varepsilon = 1/\sqrt{\varepsilon}, \quad K_\varepsilon = R_\varepsilon(-B_{\mathcal{S}})R_\varepsilon.$$

For unit $h \in \mathcal{S}$,

$$\|Jh\| \le j_\varepsilon, \qquad \int a(\xi)|\widehat{Jh}(\xi)|^2 \frac{\mathrm{d}\xi}{2\pi} = \mathfrak{a}[R_\varepsilon h] \le 1. \tag{6}$$

The form inequality $H|_{\mathcal{S}} \ge -\varepsilon I$ is equivalent to $K_\varepsilon \le I$, by the invertible form-domain parametrization through R_ε.

Let Q_L be Fourier projection to $[-L, L]$. For fixed M set

$$V_M = -m_0 I + \sum_{2\le n\le M} c_n (S_{\log n} + S_{-\log n}), \quad v_M(\xi) = -m_0 + 2\sum_{2\le n\le M} c_n \cos(\xi \log n),$$

$$d_M = |m_0| + 2\sum_{2\le n\le M} c_n \le 2M + 4, \qquad b_M(L) = \max\left\{0, \sup_{|\xi|>L} \frac{v_M(\xi)}{a(\xi)}\right\}.$$

Thus $\|V_M\| \le d_M$ and $b_M(L) \le d_M/a(L)$.

For $R, L > 0$ and integer $d \ge 0$, define the finite-rank map E_d by

$$\widehat{E_d f}(\xi) = 1_{[-L,L]}(\xi) \sum_{k=0}^{d} \frac{(-i\xi)^k}{k!} \int_{-R}^{R} u^k f(u)\,\mathrm{d}u, \qquad F_d = E_d J. \tag{7}$$

Write

$$b_d = \sqrt{\frac{2}{\pi}} \frac{(RL)^{d+3/2}}{(2d+3)(d+1)!}, \quad \tau_0 = \frac{\sigma_R}{\sqrt{\varepsilon}} + b_d j_\varepsilon, \quad e_0 = d_M \tau_0 (2j_\varepsilon + \tau_0). \tag{8}$$

Theorem 3.1 (Complete frequency-block error)**.** *With the preceding definitions,*

$$\|K_\varepsilon - F_d^* V_M F_d\| \le \frac{\delta_M}{\varepsilon} + \frac{d_M}{a(L)} + e_0, \tag{9}$$

$$K_\varepsilon \le F_d^* V_M F_d + \left(\frac{\delta_M}{\varepsilon} + b_M(L) + e_0\right) I. \tag{10}$$

Every bound acts on the entire receiving Hilbert space, including its infinite complement.

Proof. The retained arithmetic sandwich is $K_{\varepsilon,M} = J^* V_M J$, and (5) gives $\|K_\varepsilon - K_{\varepsilon,M}\| \le \delta_M/\varepsilon$. The multiplier v_M commutes with $Q = Q_L$. Therefore

$$J^* V_M J = (QJ)^* V_M (QJ) + ((I-Q)J)^* V_M ((I-Q)J). \tag{11}$$

The cross-frequency terms are exactly zero. From (6), the high summand has norm at most $d_M/a(L)$, while its quadratic form is at most $b_M(L)\|h\|^2$, directly by integrating $v_M|\widehat{Jh}|^2$ on $|\xi| > L$.

Let P_R be spatial projection to $[-R, R]$. The real-phase integral Taylor remainder gives

$$\left| e^{-iu\xi} - \sum_{k=0}^{d} \frac{(-iu\xi)^k}{k!} \right| \le \frac{|u\xi|^{d+1}}{(d+1)!}.$$

Integrating its square on $[-R, R] \times [-L, L]$ with measure $\mathrm{d}u\,\mathrm{d}\xi/(2\pi)$ proves $\|Q_L P_R - E_d\| \le b_d$. In particular no exponential phase factor is needed. The ordering of the two cutoffs matters:

$$Q_L J - F_d = Q_L(I - P_R)J + (Q_L P_R - E_d)J.$$

The theta tail bounds the first term by $\sigma_R/\sqrt{\varepsilon}$, hence $\|Q_L J - F_d\| \le \tau_0$ and $\|F_d\| \le j_\varepsilon + \tau_0$. Expanding the difference of the two low-band products bounds it by $d_M \tau_0 (2j_\varepsilon + \tau_0)$. Combining with the arithmetic tail and (11) proves both inequalities. Spatial localization is applied to J, not to an assumed spatially localized $Q_L J$. □

4 The signed finite certificate and full residuals

Define $Y : \mathbb{C}^{d+1} \to L^2$ and $Q_0 : \mathbb{C}^{d+1} \to \mathcal{S}$ by

$$\widehat{Ye_k}(\xi) = 1_{[-L,L]}(\xi)(-i\xi)^k/k!, \qquad Q_0 e_k = P_{\mathcal{S}}[s(u)1_{[-R,R]}(u)u^k].$$

All rows and columns are in their declared Hilbert spaces, and $F_d = YQ_0^* R_\varepsilon$. Put

$$C = Y^* V_M Y, \qquad G = Q_0^*(A_{\mathcal{S}} + \varepsilon)^{-1} Q_0. \tag{12}$$

Then

$$C_{jk} = \int_{-L}^{L} v_M(\xi) \frac{(i\xi)^j(-i\xi)^k}{j!k!} \frac{\mathrm{d}\xi}{2\pi}, \qquad F_d^* V_M F_d = R_\varepsilon Q_0 C Q_0^* R_\varepsilon.$$

These finite polynomial–cosine integrals retain all off-diagonal entries and the sign of C.

Lemma 4.1 (Residual Gram enclosure)**.** *For any $X : \mathbb{C}^{d+1} \to \mathrm{Dom}(A_{\mathcal{S}})$, let*

$$E = Q_0 - (A_{\mathcal{S}} + \varepsilon)X, \quad D = Q_0^* X + X^* Q_0 - X^*(A_{\mathcal{S}} + \varepsilon)X.$$

Then

$$G - D = E^*(A_{\mathcal{S}} + \varepsilon)^{-1} E, \qquad 0 \le G - D \le E^* E/\varepsilon.$$

Proof. Expand the right-hand product. Its four terms are respectively G, $-Q_0^* X$, $-X^* Q_0$ and $X^*(A_{\mathcal{S}} + \varepsilon)X$. The spectral bound $0 \le (A_{\mathcal{S}} + \varepsilon)^{-1} \le \varepsilon^{-1} I$ proves the enclosure. No Galerkin orthogonality is assumed. □

Corollary 4.2 (A complete sufficient lower certificate)**.** *Suppose Hermitian matrices $\overline{C} \ge C$ and $\overline{G} \ge D + E^* E/\varepsilon$ include all enclosure errors, with $\overline{G} > 0$. If $\beta \ge 0$ and*

$$\overline{C} \le \beta \overline{G}^{-1}, \qquad \beta + \frac{\delta_M}{\varepsilon} + b_M(L) + e_0 \le 1,$$

then $H|_{\mathcal{S}} \ge -\varepsilon I$. One may replace $b_M(L)$ by the explicit $d_M/a(L)$.

Proof. The residual lemma gives $G \le \overline{G}$. Hence $U = R_\varepsilon Q_0 \overline{G}^{-1/2}$ is a contraction. The finite operator is at most

$$U[\overline{G}^{1/2} \overline{C}\, \overline{G}^{1/2}] U^* \le \beta U U^* \le \beta I.$$

The last step uses $\beta \ge 0$. Apply (10) and the receiving equivalence. □

A full residual in this statement is a vector in the complete receiving space. Its projection onto a finite trial space is insufficient. Likewise the covariance $E^* E$ must be enclosed as a positive matrix; entrywise optimistic endpoints are not Loewner bounds. The same condition applies to numerical errors in C and G. Evaluating $b_M(L)$ also requires coverage of the entire high half-line, rather than sampling its multiplier.

5 Sharp high-frequency asymptotics

We first establish the raw-output asymptotic, then recover the full retained arithmetic coefficient at the positive spectral edge of the signed high-frequency block.

Theorem 5.1 (Sharp raw tail)**.** *For the realization of Section 2 and every fixed $\varepsilon > 0$,*

$$\lim_{L\to\infty} \sqrt{a(L)}\, \|(I - Q_L)M_s(A+\varepsilon)^{-1/2}\|_{\mathcal{K}\to L^2} = 1. \tag{13}$$

The same limit holds with A replaced by its reference on $\mathcal{S} = Z^\perp \cap \mathcal{K}$ under the following additional hypotheses: Z is a closed subspace of $\ker H$, its orthogonal projection Π preserves the form domain, $Z \subset \mathrm{Dom}(A)$, and $A\Pi$ is a bounded compact operator on $\mathcal{K}$. The reference on $\mathcal{S}$ is the operator of the restricted form. This second assertion is conditional on these precise null-space facts, rather than a claim that a selected family exhausts the kernel.

Proof. The energy estimate (6) gives the upper bound $a(L)^{-1/2}$ in both cases. Choose nonzero $\phi \in C_c^\infty(\mathbb{R})$. Fix two compact smooth functions whose matrix of the two constraint rows is invertible, as in the density argument. Correct $e^{iNu}\phi(u)$ by their linear combination to obtain $v_N \in \mathcal{C}$. Each correction coefficient decreases faster than every inverse power of N: integrate the compact smooth row integrands repeatedly by parts. In particular v_N is bounded in L^2 and converges weakly to zero.

Set $b = \|s\phi\| > 0$. The Fourier transform of $s\phi$ is Schwartz. By (3), translation of that transform gives

$$\mathfrak{a}[v_N] = b^2 \log N + O(1), \qquad \|(I - Q_{N/2})sv_N\| = b + o(1). \tag{14}$$

Here is the needed control of the energy asymptotic. On the translated region $|\xi - N| \le N/2$, $a(\xi) = \log N + O(1)$. On its complement, use $a(\xi) \le C + \log(1 + |\xi|)$ and the Schwartz tail. The lost mass times $\log N$ and the remaining logarithmic integral tend to zero faster than a fixed inverse power of N. This gives the upper and lower leading coefficient b^2. Fixed smooth correction terms have rapidly decreasing form norm, and their mixed terms are bounded by form Cauchy–Schwarz. The norm assertion follows from the translated Schwartz mass outside $[-N/2, N/2]$.

For the full constrained space take $w_N = v_N$. Under the additional null-space hypotheses take $w_N = (I - \Pi)v_N$. Compactness gives $\|A\Pi v_N\| \to 0$. Consequently $\mathfrak{a}[\Pi v_N] \to 0$ and

$$\mathfrak{a}[w_N] = \mathfrak{a}[v_N] + o(1),$$

since the mixed term equals $\langle v_N, A\Pi v_N\rangle$. Furthermore

$$\|(I - Q_{N/2})s\Pi v_N\| \le \sqrt{\mathfrak{a}[\Pi v_N]/a(N/2)} = o((\log N)^{-1/2}).$$

Thus (14) holds for w_N as well, and its norm remains bounded. Form-domain preservation makes this a valid restricted-form vector. In either case the unit vector

$$u_N = \frac{(A_{\mathcal{S}} + \varepsilon)^{1/2} w_N}{\sqrt{\mathfrak{a}[w_N] + \varepsilon\|w_N\|^2}}$$

has image under J equal to sw_N divided by the same denominator. Take $N = 2L$. Equations (14) give a tail norm at least $(1 + o(1))/\sqrt{\log L}$. Since $a(L)/\log L \to 1$, this matches the upper bound and proves the theorem. □

Lemma 5.2 (Relatively dense phase returns). *For finitely many real numbers $\alpha_1, \dots, \alpha_m$ and every $\rho > 0$, there is $T_\rho < \infty$ such that every interval $[b, b + T_\rho]$ contains N satisfying*

$$\max_j |e^{iN\alpha_j} - 1| < \rho.$$

Proof. Put $z(t) = (e^{it\alpha_j})_j$ and let G be the compact closure of this orbit in the finite torus. Choose a finite cover of G by open ρ-balls centered at orbit points $z(t_1), \dots, z(t_k)$, in the maximum coordinate metric. Let t_- and t_+ be the smallest and largest chosen times. At $t = b + t_+$ choose a covering center $z(t_l)$. Then $N = t - t_l \in [b, b + t_+ - t_-]$ and division by the unit coordinates of $z(t_l)$ gives the claimed return. Enlarge T_ρ if necessary. No independence hypothesis on the frequencies is used. □

Theorem 5.3 (Sharp signed high-frequency block). *Fix $\varepsilon > 0$ and an integer $M \geq 1$, and let*

$$K_{\mathrm{hi}}(L) = J^*(I - Q_L)V_M(I - Q_L)J.$$

For $\mathcal{S} = \mathcal{K}$,

$$\lim_{L\to\infty} a(L) \sup \sigma(K_{\mathrm{hi}}(L)) = \lim_{L\to\infty} a(L)\|K_{\mathrm{hi}}(L)\| = d_M. \tag{15}$$

The same limits hold on a null complement under the additional hypotheses of Theorem 5.1. Moreover

$$\lim_{L\to\infty} a(L)b_M(L) = d_M. \tag{16}$$

All limits are for fixed M and ε.

Proof. Here $m_0 < 0$, so $d_M = -m_0 + 2\sum c_n$. Since $|v_M| \leq d_M$, the energy estimate gives

$$\sup \sigma(K_{\mathrm{hi}}(L)) \leq \|K_{\mathrm{hi}}(L)\| \leq d_M/a(L).$$

We construct the matching signed lower bound. Fix a small $\eta > 0$. Apply Lemma 5.2 to the retained frequencies $\log n$ with $c_n > 0$. Choose its radius ρ so small that $2\rho\sum c_n \leq \eta$. At every return N one then has $v_M(N) \geq d_M - \eta$. Since $|v_M'| \leq D_M := 2\sum c_n \log n$, choose $r > 0$ with $D_M r \leq \eta$. It follows that

$$v_M(N + t) \geq d_M - 2\eta \quad (|t| \leq r). \tag{17}$$

When all retained coefficients vanish, $V_M = d_M I$; choose any $r > 0$ and the same bound is immediate without phase returns.

Choose $f \in C_c^\infty(\mathbb{R})$ with $\|f\| = 1$ and

$$\int_{-r}^{r} |\widehat{f}(t)|^2 \frac{\mathrm{d}t}{2\pi} \geq 1 - e_\eta, \qquad e_\eta \leq \frac{\eta}{1 + 2d_M}.$$

Such an f is obtained by a sufficiently large physical dilation of one normalized compact smooth function, by Plancherel. Put $\phi = f/s \in C_c^\infty$. The envelope and its potentially large norm are fixed after η is fixed. For each large L, select a return

$$N_L \in [L + r + 1, L + r + 1 + T_\rho].$$

In the constant-multiplier case take $N_L = L + r + 1$. Thus $N_L = L + O_\eta(1)$, and the window $[N_L - r, N_L + r]$ is entirely above L.

Repair $e^{iN_L u}\phi(u)$ with the same two fixed constraint vectors used in Theorem 5.1, obtaining $v_L \in \mathcal{C}$. The correction is smaller than every inverse power of N_L in both norm and form norm. The translated-Schwartz calculation of that theorem gives

$$sv_L = e^{iN_L u} f + o(1) \text{ in } L^2, \qquad \mathfrak{a}[v_L] = \log N_L + O(1), \qquad \|v_L\| = O_\eta(1), \quad v_L \rightharpoonup 0.$$

Write $T_L = (I - Q_L)V_M(I - Q_L)$. On the central translated window use (17); on the rest of the high band use $v_M \geq -d_M$. Therefore, for sufficiently small fixed η,

$$\begin{aligned}\langle sv_L, T_L sv_L\rangle &\geq (d_M - 2\eta)(1 - e_\eta) - d_M e_\eta + o(1)\\ &\geq d_M - 4\eta + o(1).\end{aligned}$$

The repair error is $o(1)$ because $\|T_L\| \leq d_M$. All frequency mass outside the central window has been included in the signed estimate.

For the full space put $w_L = v_L$. For an allowed null complement put $w_L = (I - \Pi)v_L$. As in Theorem 5.1, compactness gives

$$\mathfrak{a}[\Pi v_L] \to 0, \qquad \mathfrak{a}[w_L] = \mathfrak{a}[v_L] + o(1), \qquad \|(I - Q_L)s\Pi v_L\| = o((\log L)^{-1/2}).$$

The bounded multiplier then gives $\langle sw_L, T_L sw_L\rangle = \langle sv_L, T_L sv_L\rangle + o(1)$. Normalize with the valid unit vector

$$u_L = \frac{(A_{\mathcal{S}} + \varepsilon)^{1/2} w_L}{\sqrt{\mathfrak{a}[w_L] + \varepsilon\|w_L\|^2}}.$$

Since $a(L)/\log N_L \to 1$, its Rayleigh quotient satisfies

$$\liminf_{L\to\infty} a(L)\langle u_L, K_{\mathrm{hi}}(L)u_L\rangle \geq d_M - 4\eta.$$

The fixed norm cost $\varepsilon\|\phi\|^2$ disappears against $\log N_L$. Send $\eta \downarrow 0$ only after taking this limit inferior. The preceding upper bound proves both limits in (15).

Finally choose the same relatively dense returns just above L. Then $a(N_L)/a(L) \to 1$ and $v_M(N_L) \geq d_M - \eta$, while $b_M(L) \leq d_M/a(L)$. Taking the limit inferior and then $\eta \downarrow 0$ proves (16). □

Remark 5.4 (A sufficient compact-null mechanism). The compactness hypothesis in the conditional assertion has a source-specific sufficient condition in the present model. The map $M_s : \mathcal{V} \to L^2$ is compact: bounded form sets have complete spatial tails bounded by σ_R and frequency tails bounded by $a(L)^{-1/2}$, and a finite space–frequency rectangle is Hilbert–Schmidt. Finite weighted-shift truncations then show $B(A + I)^{-1/2}$ compact, using (5). If a closed null subspace has $Z \subset \mathrm{Dom}(A)$ and $A\Pi = -B\Pi$, its null-energy identity bounds $(A + I)^{1/2}\Pi$. Therefore

$$A\Pi = -B(A + I)^{-1/2}(A + I)^{1/2}\Pi$$

is compact. The null identities and form-domain preservation are hypotheses to verify before compression.

6 Localization budgets and further applications

For fixed M and ε, the coarse frequency term in (9) is at most a tolerance η once $a(L) \geq d_M/\eta$. Since $a(L) = \log L + O(1)$, this supplies $\log L = O(d_M/\eta)$ as $\eta \downarrow 0$. In contrast, approximating J

first in norm places $a(L)^{-1/2}$ inside the first-order product error and gives a sufficient cutoff budget of inverse-square tolerance order. Theorem 5.1 shows that this raw-output loss cannot be removed by a better uniform raw-frequency estimate. The improvement in Theorem 3.1 instead uses exact cross-frequency cancellation.

This compares sufficient cutoff bounds. A fixed M leaves a nonzero arithmetic error; achieving an arbitrarily small complete budget requires increasing M. Parameters can be chosen in the order M, L, R, d, paying respectively the arithmetic tail, high band, spatial tail and finite-phase remainder. The logarithmic frequency reference can still make the resulting rank enormous.

Theorem 5.3 identifies the leading fixed-cutoff cost of the high band: its positive spectral edge reaches $d_M/a(L)$ asymptotically. The finer scalar cap $b_M(L)$ can improve finite-cutoff enclosures, but has the same leading coefficient. Further certificate development can therefore concentrate on the signed low-frequency matrix and full residual covariance in Corollary 4.2, or on arithmetic truncations that vary with the target tolerance. The present asymptotics keep M and ε fixed; joint cutoff limits require their own estimates.

The accompanying standard-library program `verify_constants.py` checks the finite rational endpoints used in the elementary unit multiplier proof. Its acceptance checks remain active under optimized Python; the analytic series and asymptotic arguments are proved above.

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