Finite theta data and global Laguerre inequalities

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Luminary Oddities

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Abstract

We construct positive finite reweightings of the actual integer-square theta atoms which reproduce any prescribed finite set of modular endpoint jets, converge to the complete theta density in every fixed source derivative norm, and nevertheless eventually violate the first Laguerre inequality. The first missed odd jet has an explicit strictly positive Gaussian-tail formula. More generally, every nonzero finite signed compact-scale mixture of these atoms has a nonzero odd endpoint jet, finitely many real transform zeros and infinitely many nonreal zeros. For the exact split into a finite surrogate and its complete arithmetic remainder, we determine the leading cancellation of both diagonal Laguerre densities and their cross term, uniformly on bounded horizontal strips including the real axis. Each diagonal has only finitely many absolute moments, although their complete combination decreases rapidly. Explicit source-seminorm bounds give a sufficient negativity onset. The positive interpolation mechanism is classical; the conclusions concern its explicit realization and global-sign obstruction in the actual theta atom family.

1 Theta atoms and the approximation problem

For $q > 0$ put

$$h_q(u) = q(2qe^{4u} - 3)e^{5u - qe^{4u}}, \qquad \Phi(u) = \sum_{n \geq 1} h_{\pi n^2}(u), \quad u \geq 0, \tag{1}$$

and write $\mathcal{C}f(z) = \int_0^\infty f(u)\cos(zu)\,\mathrm{d}u$. For a real entire function H define

$$\mathcal{L}_1[H](x) = H'(x)^2 - H(x)H''(x).$$

The first Laguerre inequality is $\mathcal{L}_1[H] \geq 0$ on $\mathbb{R}$. Our construction matches finite data of Φ exactly while its cosine transform fails this global inequality.

Here is the normalization and symmetry used throughout. Set $\omega(v) = \sum_{n \geq 1} e^{-\pi n^2 v}$ and $b(u) = e^u \omega(e^{4u})$. Jacobi inversion gives

$$\omega(v^{-1}) = v^{1/2}\omega(v) + (v^{1/2} - 1)/2, \qquad b(-u) = b(u) + \sinh u.$$

Termwise differentiation gives $(D^2 - 1)b = 8\Phi$, so Φ extends evenly across zero. The series is locally normally convergent near every real u (in particular in $|\Im u| < \pi/8$), justifying these differentiations and analytic evenness. On either real half-line, every derivative has double-exponential decay. With $\Xi(t) = \xi(1/2 + it)$, the classical theta Mellin formula reads

$$F(z) := \mathcal{C}\Phi(z) = \Xi(z/2)/8. \tag{2}$$

For example this normalization follows by substituting $t = 2u$ in Haglund's theta representation [4, Section1].

Arithmetic truncations and their incomplete-gamma transforms have long been studied; see Haglund [4]. Related nonreal-zero phenomena for truncated theta transforms occur in Csordas [2]. The direct raw equal-weight obstruction, with its first endpoint leakage and full Laguerre remainder, is already proved in Teal-sea's open research note [8, Section3]. The result below concerns positive repairs of arbitrarily many exact jets, not the raw truncation special case. Finite positive interpolation is classical: Bayer–Teichmann [1] treat arbitrary integrable finite observations; Glaubitz [3, Theorem3.5] and Neumayer–Schäfer–Ullrich [6] control positive exact rules near reference weights. Our formulas keep the actual integer-square scales and identify the first missed modular jet together with its global transform consequence.

Theorem 1.1 (Positive exact repairs with global failure)**.** *For each integer $K \geq 0$ and every sufficiently large integer N, there are positive real weights $c_n^{(N)}$, $1 \leq n \leq N$, such that*

$$f_N(u) = \sum_{n \leq N} c_n^{(N)} h_{\pi n^2}(u), \qquad f_N^{(j)}(0) = \Phi^{(j)}(0) \quad (0 \leq j \leq K).$$

Only the fixed indices $1, \dots, K+3$ are modified, and each modified weight tends to one. The convergence $f_N \to \Phi$ holds in every fixed $C^j([0,\infty))$ norm, and the transforms converge locally uniformly on $\mathbb{C}$ with every fixed derivative. Nevertheless each $\mathcal{C}f_N$ has eventually strictly negative $\mathcal{L}_1$, finitely many real zeros and infinitely many nonreal zeros.

The first nonzero odd jet of f_N is exactly the least odd integer $j_* > K$, and it is strictly positive. Its formula and a sufficient positive-weight bound are given in Section 3.

The reflected half-line function $f_N(|u|)$ has only finite matched smoothness at zero. The finite surrogate has the regularity stated in the theorem. Local entire-transform convergence also does not imply a uniform relative estimate on an unbounded horizontal strip.

2 Signed compact-scale mixtures

Theorem 2.1. *Let $\mu \neq 0$ be a finite real signed Borel measure supported in $[a,b] \subset (0,\infty)$, and set*

$$f(u) = \int h_q(u)\, \mathrm{d}\mu(q), \qquad C(z) = \mathcal{C}f(z).$$

There is a least nonzero odd endpoint derivative $f^{(2r+1)}(0)$. Put $m = 2r+2$ and $c = (-1)^{r+1} f^{(2r+1)}(0) \neq 0$. Then, as real $x \to \infty$,

$$C^{(j)}(x) = (-1)^j (m)_j c x^{-m-j} + O(x^{-m-j-2}), \quad j = 0, 1, 2, \tag{3}$$

$$\mathcal{L}_1[C](x) = -mc^2 x^{-2m-2} + O(x^{-2m-4}) < 0 \quad \textit{eventually.} \tag{4}$$

The transform is a nonzero real even entire function of order at most one. It has finitely many real zeros and infinitely many nonreal zeros.

Here $(m)_j$ is the rising factorial, with $(m)_0 = 1$. A point mass is allowed by placing it in any such compact interval.

Proof. Compact scale support makes f entire in u. For $v = e^{4u}$ write

$$f(u) = e^{5u} A(v), \quad A(v) = \int q(2qv - 3) e^{-qv}\, \mathrm{d}\mu(q).$$

This A is entire and, with $M_j = \int q^j \, \mathrm{d}\mu(q)$,

$$[v^k]A(v) = \frac{(-1)^{k+1}(2k+3)}{k!} M_{k+1}.$$

If $A = 0$, the finite signed measure $q \, \mathrm{d}\mu$ annihilates all polynomials. Polynomial density in $C([a,b])$ forces $q \, \mathrm{d}\mu = 0$, and $a > 0$ forces $\mu = 0$, a contradiction. Let k be the first nonzero Taylor coefficient a_k of A. Then

$$f(u) = a_k e^{(5+4k)u}(1 + O(e^{4u})) \quad (u \to -\infty), \qquad |f(u)| \le C_0 e^{9u - ae^{4u}} \quad (u \to \infty).$$

These behaviors are incompatible with evenness. If all odd endpoint derivatives vanished, entire analyticity would make f even. The least nonzero odd jet therefore exists.

Every derivative of $u^j f(u)$ is integrable on $[0,\infty)$ for fixed j, and tends to zero at infinity. Repeated integrations by parts give

$$C(x) = \sum_{\ell=0}^{R} (-1)^{\ell+1} f^{(2\ell+1)}(0) x^{-2\ell-2} + O(x^{-2R-4}), \tag{5}$$

where taking two more integrations supplies the claimed remainder from a finite derivative L^1 norm. Applying the same integrations to $C' = -\int uf \sin(xu) \, \mathrm{d}u$ and $C'' = -\int u^2 f \cos(xu) \, \mathrm{d}u$ proves (3). Thus no uncontrolled real O term is differentiated. Substitution gives the coefficient $[m^2 - m(m+1)]c^2 = -mc^2$ in (4).

The double-exponential tail pays every complex exponential moment, so C is real even entire. If $|z| \le R$, split the integral at $u = \log(R+e)$. On the first part, $e^{Ru} \le e^{R\log(R+e)}$; on the second, $Ru + 9u - ae^{4u}$ decreases sufficiently fast for large R. This gives $\log \max_{|z|\le R} |C(z)| = O(R\log(R+e))$, hence order at most one. The nonzero algebraic leading term proves that $C \not\equiv 0$ and that it has no real zeros for sufficiently large $|x|$. It is not a polynomial, since $C(x) \to 0$ on the real axis.

If there were only finitely many zeros, finite-order Hadamard factorization would give $C(z) = e^{Az+B}p(z)$ for a polynomial p. Evenness implies $e^{2Az} = p(-z)/p(z)$; a rational function equal to an exponential must be constant, so $A = 0$. This would make C a nonzero polynomial, a contradiction. Thus infinitely many zeros are nonreal. The standard factorization used here is described, for example, in Levin [5, ChapterI]. □

2.1 A complete sufficient negativity onset

The endpoint proof has a useful quantitative version. All norms in this subsection are on $[0,\infty)$.

Proposition 2.2. *In Theorem 2.1 set*

$$\begin{aligned}
A_0 &= |f^{(m+1)}(0)| + \left\| f^{(m+2)} \right\|_1, \\
A_1 &= |(uf)^{(m+2)}(0)| + \left\| (uf)^{(m+3)} \right\|_1, \\
A_2 &= |(u^2 f)^{(m+3)}(0)| + \left\| (u^2 f)^{(m+4)} \right\|_1, \\
B_1 &= |c|\{2mA_1 + A_2 + m(m+1)A_0\}, \qquad B_2 = A_1^2 + A_0 A_2.
\end{aligned}$$

For $x > 0$,

$$|\mathcal{L}_1[C](x) + mc^2 x^{-2m-2}| \le B_1 x^{-2m-4} + B_2 x^{-2m-6}.$$

In particular $\mathcal{L}_1[C](x) < 0$ whenever

$$x > \max\{1, \sqrt{(B_1 + B_2)/(mc^2)}\}. \tag{6}$$

Proof. Integrate the three transform integrals respectively $m+2, m+3, m+4$ times. All earlier boundary terms vanish by the first-odd-jet hypothesis; the remaining boundary terms and integral remainders give

$$|C - cx^{-m}| \le A_0 x^{-m-2}, \quad |C' + mcx^{-m-1}| \le A_1 x^{-m-3}, \quad |C'' - m(m+1)cx^{-m-2}| \le A_2 x^{-m-4}.$$

Expanding the full quadratic expression gives the stated bound. For $x > 1$ the error is at most $(B_1 + B_2)x^{-2m-4}$, proving (6). All quantities are finite complete source seminorms; any certified upper bounds for them and positive lower bound for $|c|$ give a numerical sufficient onset. □

3 Exact positive arithmetic repair

Define real polynomials by

$$P_0(X) = 2X^2 - 3X, \qquad P_{j+1}(X) = (1-4X)P_j(X) + 4XP_j'(X). \tag{7}$$

Then $h_q^{(j)}(u) = e^u P_j(qe^{4u})e^{-qe^{4u}}$ and $\deg P_j = j+2$. Fix $K \ge 0$, put $d = K+2$, and use the $d+1$ anchor indices $i = 1, \dots, d+1$. Let

$$L_i(X) = \prod_{\substack{1 \le j \le d+1 \\ j \ne i}} \frac{X - j^2}{i^2 - j^2}, \qquad B_i = \prod_{j \ne i} |i^2 - j^2|. \tag{8}$$

For $N \ge d+1$ set

$$\Delta_i(N) = \sum_{n > N} e^{-\pi(n^2 - i^2)} L_i(n^2), \qquad c_i^{(N)} = 1 + \Delta_i(N), \tag{9}$$

and take $c_n^{(N)} = 1$ for $d+1 < n \le N$.

Lemma 3.1. *These weights match every Gaussian moment of degree at most d:*

$$\sum_{n \le N} c_n^{(N)} e^{-\pi n^2} (\pi n^2)^k = \sum_{n \ge 1} e^{-\pi n^2} (\pi n^2)^k, \quad 0 \le k \le d. \tag{10}$$

Moreover, with $\rho_N = \exp(2d/(N+1) - \pi(2N+3)) < 1$,

$$|\Delta_i(N)| \le \frac{e^{-\pi((N+1)^2 - i^2)}(N+1)^{2d}}{B_i(1-\rho_N)} \longrightarrow 0. \tag{11}$$

Thus all weights are positive for sufficiently large N.

Proof. Cardinal interpolation gives $\sum_i i^{2k} L_i(X) = X^k$ for $0 \le k \le d$. Substitute this identity into (9); the tail sums converge absolutely and reproduce exactly the omitted moments. For $n > N \ge d+1$, $|L_i(n^2)| \le n^{2d}/B_i$. The ratio of consecutive terms in $n^{2d}e^{-\pi n^2}$ is at most ρ_N, because $\log(1 + 1/n) \le 1/n$. Summing this geometric majorant gives (11). Its right side tends to zero, so eventually $|\Delta_i| < 1/2$ for all anchors. Each $h_{\pi n^2}$ is positive on the half-line, since $\pi n^2 e^{4u} > 3/2$. □

Proof of the matching and convergence assertions in Theorem 1.1. For $j \le K$, P_j has degree at most d. Thus (10) gives exact derivative matching. For any fixed derivative order j, the recurrence implies

$$\sup_{u\ge 0} |h^{(j)}_{\pi n^2}(u)| \le C_j n^{2j+4} e^{-\pi n^2}$$

for all sufficiently large n. To see this, bound the coefficients of P_j and write $e^u = (qe^{4u}/q)^{1/4}$; each resulting monomial is maximized at its left endpoint once $q = \pi n^2$ exceeds its fixed power. The Gaussian tail is summable, and the finitely many anchor corrections tend to zero. This proves convergence in every fixed C^j norm, even when $j > K$. Multiplication by $u^\ell e^{Yu}$ for fixed ℓ, Y gives the same integrable summable majorant, since powers of $\log(qe^{4u}/q)$ are dominated by an additional fixed power of qe^{4u}. Differentiating the cosine integral now proves local entire convergence with every fixed derivative. The signed compact-scale theorem applies to each nonzero positive finite mixture and gives the global failure and zero statements. □

3.1 The first missed odd jet

Put

$$W_d(X) = \prod_{i=1}^{d+1}(X - i^2), \quad S_d = \sum_{i=1}^{d+1} i^2 = \frac{(d+1)(d+2)(2d+3)}{6}, \quad A_j = 2(-4)^j.$$

Proposition 3.2. *Let j_* be the least odd integer greater than K. Then*

$$f_N^{(j_*)}(0) = -A_{d-1}\pi^{d+1} \sum_{n>N} e^{-\pi n^2} W_d(n^2) > 0, \qquad K \text{ even}, \tag{12}$$

$$f_N^{(j_*)}(0) = -A_d \pi^{d+2} \sum_{n>N} e^{-\pi n^2} W_d(n^2) \left[n^2 + S_d - \frac{(2d+3)(d+2)}{4\pi} \right] > 0, \qquad K \text{ odd}. \tag{13}$$

Consequently its first leakage exponent in Theorem 2.1 is exactly $m = j_* + 1$.

Proof. The recurrence gives leading coefficient A_j and next-coefficient ratio

$$\frac{[X^{j+1}]P_j}{A_j} = -\frac{(2j+3)(j+2)}{4}. \tag{14}$$

Indeed the leading coefficient is multiplied by -4, and if the ratio is r_j, then $r_{j+1} = r_j - (4j+9)/4$; its initial value is $-3/2$. This proves (14) by induction.

Let I_d denote interpolation at $1^2, \dots, (d+1)^2$. The full odd jet is zero, so

$$f_N^{(j_*)}(0) = \sum_{n>N} e^{-\pi n^2} \{ I_d[P_{j_*}(\pi X)](n^2) - P_{j_*}(\pi n^2) \}.$$

If K is even, $j_* = d-1$, and polynomial division gives $P_{d-1}(\pi X) - I_d[P_{d-1}(\pi X)] = A_{d-1}\pi^{d+1} W_d(X)$. If K is odd, $j_* = d$, and the same division, using (14) and $[X^d]W_d = -S_d$, gives the quotient

$$A_d \pi^{d+2} \left[X + S_d - \frac{(2d+3)(d+2)}{4\pi} \right].$$

These are exactly (12)–(13). Here j_* is odd, so $-A_{j_*} > 0$, and $W_d(n^2) > 0$ for $n > N$. For the second case, $S_d > (2d+3)(d+2)/(4\pi)$, already using $\pi > 3$. Every earlier odd jet vanishes by matching. Positivity is an exact infinite-tail identity, not a floating cancellation calculation. □

3.2 Any fixed finite collection of linear data

Proposition 3.3. *Let $\ell_1, \dots, \ell_s$ be real linear functionals on a vector space containing the atoms and their sum Φ. Suppose explicitly that they are series-compatible and absolutely summable:*

$$\ell_j(\Phi) = \sum_{n\geq 1} \ell_j(h_{\pi n^2}), \qquad \sum_{n\geq 1} \|v_n\| < \infty, \quad v_n = (\ell_j(h_{\pi n^2}))_{j=1}^{s}.$$

Then positive finite mixtures on the actual arithmetic scales match all ℓ_j exactly, converge as above, and have the global failure of Theorem 2.1.

Proof. Choose a basis $v_{i_1}, \dots, v_{i_t}$ for the span of all v_n. Once N contains these anchors, the tail $\sum_{n>N} v_n$ lies in that span and tends to zero. Its coordinates $\beta_j(N)$ in this fixed basis tend to zero. Set the corresponding anchor weights to $1 + \beta_j(N)$ and all other retained weights to one. Exact matching, eventual positivity and convergence follow. If the span is zero, no corrections are needed. The compact-scale result still applies. □

Continuity in a topology in which the source series converges is one sufficient series-compatibility condition. Summability of the scalar values alone does not suffice for an arbitrary discontinuous functional. The hypotheses hold for fixed endpoint jets, polynomial source moments and finitely many transform derivatives at real or complex frequencies; complex observations are split into real and imaginary parts. For a general effective construction, the selected basis needs a certified nonzero minor. The explicit cardinal construction avoids that issue for jets. The proposition does not match arbitrary nonlinear Gram entries.

4 Complete infinite-tail compensation

For real entire H put

$$D_y(H)(x) = \frac{\partial_y |H(x+iy)|^2}{8y} \quad (y>0), \qquad D_0(H)(x) = \mathcal{L}_1[H](x)/4. \tag{15}$$

The value at zero is the continuous limit: expand $|H(x+iy)|^2 = H(x)^2 + y^2\mathcal{L}_1[H](x) + O(y^4)$.

Theorem 4.1. *Let C be any nonzero compact-scale transform from Theorem 2.1, let $F = \mathcal{C}\Phi$, and put $E = F - C$. Define the complete cross term*

$$B_y(C,E) = D_y(C+E) - D_y(C) - D_y(E), \qquad a_* = mc^2/4 > 0.$$

For each fixed finite $Y \geq 0$, uniformly in $0 \leq y \leq Y$,

$$D_y(C)(x) = -a_* x^{-2m-2} + O_Y(x^{-2m-4}), \tag{16}$$

$$D_y(E)(x) = -a_* x^{-2m-2} + O_Y(x^{-2m-4}), \tag{17}$$

$$B_y(C,E)(x) = 2a_* x^{-2m-2} + O_Y(x^{-2m-4}), \tag{18}$$

$$D_y(F)(x) = O_{Y,A}(x^{-A}) \quad \textit{for each fixed } A>0. \tag{19}$$

Also $E(x+iy)/C(x+iy) \to -1$ uniformly on that strip. Thus the cross/diagonal ratios tend respectively to -2 and 1:

$$\frac{B_y(C,E)}{D_y(C)} \longrightarrow -2, \quad \frac{D_y(E)}{D_y(C)} \longrightarrow 1, \quad \frac{D_y(F)-D_y(C)}{|D_y(C)|} \longrightarrow 1.$$

All denominators are nonzero for sufficiently large x, uniformly on the fixed strip. For repaired $C = \mathcal{C}f_N$, these are fixed-N limits, not interchanged limits in N and x.

Proof. The even extension of Φ and all its derivatives, multiplied by $e^{Y|u|}$ and any fixed polynomial in u, are integrable. Integrate its full-line transform by parts arbitrarily often. There is no boundary at zero, so F and each fixed complex derivative decrease faster than any power on each horizontal strip. This proves (19), including $y = 0$ by the continuous limiting formula.

The compact-scale integrations by parts also hold on horizontal strips: the source seminorms pay e^{Yu}. With real b they give

$$C(z) = cz^{-m} + bz^{-m-2} + R(z), \qquad R^{(j)}(x+iy) = O_Y(x^{-m-4}), \quad j = 0, 1, 2. \tag{20}$$

The derivative remainder bounds follow either from the differentiated integrals with additional integrations by parts, or from Cauchy estimates after bounding the analytic remainder on the larger strip $|\Im z| \le Y + 1$. They are not derivatives of an uncontrolled O term. Since F decreases rapidly, E has the same form with leading coefficients $-c, -b$.

For $H_0(z) = cz^{-m}$ the exact density is

$$D_y(H_0)(x) = -\frac{mc^2}{4}(x^2 + y^2)^{-m-1} = -a_* x^{-2m-2} + O_Y(x^{-2m-4}).$$

The second y derivative of the modulus cross term between cz^{-m} and bz^{-m-2} is $O_Y(x^{-2m-4})$, by differentiating the rational powers. The b^2 term has a smaller bound. Terms involving R have the same second-derivative bound by (20). These error expressions are even in y, since their coefficients and the entire function are real. Their first derivative vanishes at zero, so the fundamental theorem of calculus bounds the first derivative divided by y by the supremum of the second derivative, uniformly through $y = 0$. This proves (16); apply the same argument to E for (17). The exact quadratic identity defining B_y and (19) give (18). Finally $C(x+iy) = cx^{-m}(1+O_Y(x^{-1}))$ uniformly and $E = -C + F$, proving the ratios. □

Corollary 4.2. *For each fixed $y \ge 0$ and integer $k \ge 0$, the absolute even moments of either separate density $D_y(C)$ or $D_y(E)$ are finite exactly when $k \le m$:*

$$\int_{\mathbb{R}} |x|^{2k} |D_y(C)(x)| \, \mathrm{d}x < \infty \quad \Longleftrightarrow \quad k \le m,$$

and likewise for E. Their complete combination has all such moments. No estimate $|D_y(F) - D_y(C)| \le \theta |D_y(C)|$ with fixed $\theta < 1$ can hold for all sufficiently large x.

Proof. Each density is continuous on finite intervals and even in x. Its nonzero absolute tail is asymptotic to $a_*|x|^{-2m-2}$. Thus its weighted tail is integrable precisely when $2k - 2m - 2 < -1$, equivalently $k \le m$ for integer k. The complete combination is (19); the relative estimate is excluded by the last ratio in the theorem. □

The failure of all moments in the separate pieces is relevant when applying moment-based positivity criteria: it is not repaired by matching any finite number of endpoint jets. Cancellation in the complete arithmetic sum restores rapid decay.

Corollary 4.3 (Finite-modification rigidity)**.** *Suppose an arithmetic theta series has weights equal to one except at finitely many indices. If all its odd endpoint jets vanish, every weight equals one.*

Proof. Its difference from Φ is a finite signed atom mixture g. All odd jets of g vanish, so g is even entire. If its scale measure were nonzero, Theorem 2.1 would contradict this. Distinct integer-square scales then give zero modification. □

5 Rigorous small examples and use of the construction

The accompanying standard-library program `verify.py` checks cardinal moment identities, the derivative recurrence and the missed-jet polynomial divisions by exact rational arithmetic. Its small examples also have rational sufficient bounds for positivity. For $N \geq d+1$, $\rho_N < 1/2$; using $\pi > 3$ and $e > 2$ in (11) gives

$$|\Delta_i(N)| < U_i := \frac{2(N+1)^{2d}}{B_i\, 2^{3((N+1)^2-i^2)}}. \tag{21}$$

Thus $\max_i U_i < 1/2$ is a rigorous sufficient check for weights in $(1/2, 3/2)$. The following cases pass this exact rational test:

K	0	1	2	3	5	7
N	4	5	6	7	9	12
j_*	1	3	3	5	7	9
m	2	4	4	6	8	10

The missed jet is strictly positive in each example, with a rational lower bound obtained by retaining $n = N+1$ in (12) or (13). For completeness, use $\pi > 3$, $\pi < 4$, $e < 3$, so $e^{-\pi n^2} > 3^{-4n^2}$, and replace the second-case bracket by $n^2 + S_d - (2d+3)(d+2)/12 > 0$. The verifier records these positive rational lower bounds. It also checks the leading quadratic compensation for $m = 2, 4, \ldots, 100$. These finite exact identities supplement the proofs; they neither sample a remote negative Fourier interval nor evaluate the complete original density's sign.

The construction separates exact finite validation from a global Laguerre sign inside the actual theta atoms. It excludes positive finite-scale repair as a globally sign-preserving surrogate, even with arbitrary fixed linear observations and local entire convergence. It does not exclude approximants outside this atom class, such as even cosh-modified kernels [7], or expanding finite-frequency arguments with a complete original infinite-tail estimate. The exact cross compensation explains why that infinite remainder must remain in the argument before a high-frequency sign limit is taken.

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