FINITE-PROJECTION HOLOMORPHIC STABILIZATION OF CONTRACTIBLE AFFINE MANIFOLDS

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Abstract. Let X be a smooth contractible affine manifold of dimension $n \geq 2$ with the holomorphic density property. We prove that $X \times \mathbb{C}^n \cong_{\mathrm{hol}} \mathbb{C}^{2n}$ whenever X admits a finite flat morphism to $\mathbb{C}^n$ with curvilinear scheme fibers. A global algebra generator is unnecessary: local primitive elements describe the fiber embedding spaces, and stratified Oka section existence supplies a fiberwise embedding into a Runge Euclidean basin. Entire interpolation and parametric Andersén–Lempert approximation then give compact capture and a Runge-ball exhaustion. Every smooth connected affine surface or threefold admits a finite flat linear projection with curvilinear fibers, so the stabilization conclusion holds for all contractible density-property examples in these dimensions. It also holds for affine hypersurfaces in every dimension $n \geq 2$, saving one stabilizing direction relative to Huang's proper-embedding construction. Consequently, for McLean's exotic varieties K_n, $n \geq 4$, a one-direction holomorphic cancellation counterexample occurs in some dimension $m \in \{n, \dots, 2n-1\}$, sharpening Huang's range $\{n, \dots, 2n\}$, and the Koras–Russell cubic satisfies $X_{\mathrm{KR}} \times \mathbb{C}^3 \cong_{\mathrm{hol}} \mathbb{C}^6$. Further applications are sufficient dimension reductions in specified nonlinearizable-action constructions, with disconnected cyclic fixed loci on $\mathbb{C}^6$ and a contractible non-Euclidean fixed surface on $\mathbb{C}^8$.

1. Introduction

Whole finite fibers and stabilization. Finite curvilinear projections replace the proper-embedding parameter by the dimension of the manifold. Their fibers can collide and have monodromy; retaining their finite algebras preserves the infinitesimal data required for interpolation. Curvilinear fibers admit local primitive elements, which suffice for the construction even when a single global generator is unavailable.

A Stein manifold has the *holomorphic density property* if the Lie algebra generated by its complete holomorphic vector fields is dense, in the compact-open topology, in all holomorphic vector fields. Here density property always means this ordinary holomorphic property, rather than its volume-preserving analogue. The notation $\cong_{\mathrm{hol}}$ denotes biholomorphic equivalence.

Huang proved that a contractible Stein n-manifold with the density property admitting a proper holomorphic embedding into $\mathbb{C}^k$ satisfies

$$X \times \mathbb{C}^k \cong_{\mathrm{hol}} \mathbb{C}^{n+k};$$

see [1, Theorem 1.2]. His proof combines exact parametric point standardization with a cylinder automorphism and Runge-ball exhaustion. For an affine hypersurface its given embedding supplies $k = n + 1$. A finite curvilinear projection gives the following criterion.

Theorem 1.1 (Stabilization from curvilinear finite fibers)**.** *Let X be a smooth contractible affine variety of dimension $n \geq 2$ with the holomorphic density property. Suppose an algebraic morphism $\pi : X \to \mathbb{C}^n$ is finite, flat and surjective, and every local algebra of every scheme fiber*

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is isomorphic to $\mathbb{C}[\tau]/(\tau^m)$. *Then* $X \times \mathbb{C}^n \cong_{\mathrm{hol}} \mathbb{C}^{2n}$. *The finite algebra* $\pi_*\mathcal{O}_X$ *may be generated locally on the base.*

The two concrete consequences below supply finite curvilinear projections.

Theorem 1.2. *Let* $n \geq 2$, *and let* $X \subset \mathbb{C}^{n+1}$ *be a nonempty smooth contractible affine hypersurface with the holomorphic density property. Then*

$$X \times \mathbb{C}^n \cong_{\mathrm{hol}} \mathbb{C}^{2n}.$$

Corollary 1.3. *Every smooth contractible affine manifold of dimension* $n = 2$ *or* 3 *with the holomorphic density property satisfies* $X \times \mathbb{C}^n \cong_{\mathrm{hol}} \mathbb{C}^{2n}$.

For a general contractible Stein density-property manifold of dimension n, Huang's theorem combined with the proper holomorphic embedding into $\mathbb{C}^{\lfloor 3n/2 \rfloor + 1}$ of Eliashberg–Gromov and Schürmann [11, 12] gives $k = \lfloor 3n/2 \rfloor + 1$ (see [1, Remark 1.3(2)]). For smooth contractible affine varieties of dimension two and three, Corollary 1.3 gives $k = n$, two directions fewer than this general bound; for hypersurfaces in every dimension the saving relative to their given embedding is one direction. Theorem 1.2 follows from a monic hypersurface projection. For Corollary 1.3, a generic linear projection is finite and has differential corank at most one: the corank-two incidence has codimension four. Smoothness then gives flatness, and its finite fibers are curvilinear. Section 5 supplies this argument.

The proof of Theorem 1.1 passes through the relative embedding space of the finite fibers. Fixed-type embedding spaces are bundles of regular curve jets over ordered configuration spaces. Their Oka property and connectivity provide both an endpoint in a Euclidean basin and an entire interpolation from the identity. Compact capture then feeds Huang's cylinder and exhaustion mechanism.

Set $B = \mathbb{C}^n$, and fix a Runge domain $\Omega \subset X$ biholomorphic to $\mathbb{C}^n$. For every compact $K \subset X$, we construct a holomorphic family $A^v \in \mathrm{Aut}(X)$, $v \in B$, with holomorphic inverse family, such that

$$A^{\pi(x)}(x) \in \Omega \qquad (x \in K). \tag{1}$$

The family may depend on K. It acts on the target X while retaining the independent base parameter v; local fiber labels allow arbitrary monodromy. Capture only requires an image in an open basin, permitting approximation of an embedded finite scheme instead of exact collapse to one point.

We construct the scheme interpolation in Section 3, prove capture in Section 4, and apply Huang's cylinder mechanism in Section 5. The applications appear in Section 6.

2. Analytic tools

A compact L in a Stein manifold M is $\mathcal{O}(M)$-convex if every point outside L is separated from L by a holomorphic function on M. A Stein domain $U \subset M$ is *Runge* if its holomorphic functions can be approximated on compact subsets by functions in $\mathcal{O}(M)$.

Proposition 2.1 (Stratified section existence)**.** *Let* B *be a finite-dimensional reduced Stein space and* $p : E \to B$ *a holomorphic submersion. Suppose* B *has a finite descending filtration by closed complex subvarieties with smooth strata* S, *and* $p^{-1}(S) \to S$ *is a holomorphic fiber bundle with path-connected Oka fiber* Y_S *satisfying*

$$\pi_q(Y_S) = 0 \qquad (1 \leq q \leq \dim_{\mathbb{C}} S - 1).$$

A holomorphic section prescribed on a closed complex subvariety of B *extends to a global holomorphic section.*

This is the interpolation part of [2, Theorem 7.1], with empty approximation compact. Connectivity supplies the extensions over Stein Morse handles; an initial global continuous section or homotopy is not an additional hypothesis in this existence statement. We use the equivalence of the Oka and convex approximation properties. We also use Oka bundle stability: a holomorphic fiber bundle with Oka base and Oka fiber has Oka total space [3, Theorem 2.11].

Proposition 2.2 (Parametric approximation)**.** *Let W be Stein, X a Stein manifold with the density property, and $U \subset W \times X$ open. Suppose $F_s : U \to W \times X$, $0 \le s \le 1$, is a smooth isotopy of injective holomorphic maps preserving W, with F_0 the inclusion. If a compact $Q \subset U$ has $\mathcal{O}(W \times X)$-convex images $F_s(Q)$ for every s, then F_1 can be approximated uniformly on Q by automorphisms of $W \times X$ preserving W.*

This is [4, Theorem 2.2]. It supplies entire parameter automorphisms and their holomorphic inverses. We will verify its moving-compact hypothesis explicitly.

Three further density-property consequences are used. A Stein density-property n-manifold contains an attracting basin biholomorphic to $\mathbb{C}^n$, by Varolin's basin construction with nonresonant jets [5, Corollary 4.1 and the following remark]. This basin is Runge: take a small $\mathcal{O}(X)$-convex compact neighborhood V of its fixed point in the basin, carry any basin compact into V by an iterate, apply Oka–Weil to the transported function near V, and compose the approximation with the iterate. Also $X \times \mathbb{C}^n$ has the density property: a Stein manifold with the density property is holomorphically flexible, $X \times \mathbb{C}$ then has the density property by [9, Theorem 3], and induction on the number of factors applies, exactly as in [1, proof of Theorem 1.2]. Finally we use Boc Thaler's criterion:

Proposition 2.3 (Runge-ball exhaustion)**.** *A Stein manifold M with the density property is biholomorphic to $\mathbb{C}^{\dim M}$ if it admits an exhaustion by Runge images of Euclidean balls.*

See [6, Theorem IV.15, pp. 62–64] and the accessible proof in [7, Theorem 2]; also [1, Theorem 2.3]. Our ball images will be nested and relatively compact, with holomorphically convex compact closures.

Proof dependencies. Table 1 records the interfaces used in the proof. Each row identifies an inherited theorem and the hypotheses verified here.

TABLE 1. Analytic inputs and their use.

Input	Hypotheses and proof interface
Stratified Oka sections	Proposition 2.1, from [2, Theorem 7.1]: the embedding map is a holomorphic submersion; fixed-multiplicity strata have locally trivial Oka fibers. Connectivity through $2n-2$ pays for the complex-time base of dimension $n+1$.
Configuration and jet fibers	[4, Theorem 3.1] and Oka bundle stability [3, Theorem 2.11]: support configurations are Oka; regular curve jets have punctured vector-space factors and affine higher coefficients.
Parametric approximation	Proposition 2.2, from [4, Theorem 2.2]: the Stein parameter is retained; the local flow is injective, and every moving graph compact is holomorphically convex.
Euclidean basin and exhaustion	[5, Corollary 4.1 and the following remark] supplies a nonresonant attracting basin. Its Runge property and the nested Runge-ball exhaustion are verified below; Proposition 2.3 then gives uniformization.

3. Embedding the finite fibers

3.1. **Finite curvilinear algebras.** Fix $\pi : X \to B = \mathbb{C}^n$ as in Theorem 1.1, and put $\mathcal{A} = \pi_*\mathcal{O}_X$, a locally free algebra of constant positive rank d. Its fiber A_v is the coordinate algebra of the finite scheme $Z_v = \pi^{-1}(v)$. For a partition $\lambda = (m_1, \dots, m_r)$ of d, write

$$A_\lambda = \prod_{i=1}^{r} \mathbb{C}[\tau_i]/(\tau_i^{m_i}). \tag{2}$$

By hypothesis every A_v has one of these types.

Each fiber algebra has a primitive element: choose distinct constants on its local factors and a nonzero linear term on each factor of length at least two. Lift it locally to a section a of $\mathcal{A}$. The determinant of $1, a, \dots, a^{d-1}$ is nonzero at the chosen point, and remains nonzero nearby. Hence locally $\mathcal{A} \cong \mathcal{O}_B[T]/(P_v)$, with P_v the monic characteristic polynomial of multiplication by a. This presentation is local on the base; the mapping space below is intrinsic.

For the hypersurface case, write $X = \{F = 0\} \subset \mathbb{C}^{n+1}$, with F reduced of degree d. Choose e such that its top homogeneous part satisfies $F_d(e) \neq 0$, make e the last coordinate direction t, and divide by $F_d(e)$. Then $P_v(T) = F(v, T)$ is monic, so the projection to $v \in \mathbb{C}^n$ is finite, flat and surjective, with free algebra basis $1, T, \dots, T^{d-1}$. The monic root bound proves analytic properness. Its local factors are exactly (2), including degree one.

3.2. **The relative embedding submersion.** Write

$$X = \operatorname{Spec} R_X, \qquad R_X = \mathbb{C}[z_1, \dots, z_N]/I,$$

with I finitely generated. The algebraic constructions below are analytified when taking holomorphic sections. In $\mathcal{A}^{\oplus N}$ impose those relations on $(q_1, \dots, q_N)$. The resulting relative affine scheme over B represents the relative mapping space $E \to B$, with fiber $\operatorname{Hom}(Z_v, X)$. Changes of local basis in $\mathcal{A}$ preserve the construction.

Lemma 3.1. *The space E is smooth and $E \to B$ is a holomorphic submersion of relative dimension nd. Its locus E_{emb} of closed scheme embeddings is Zariski open.*

Proof. A square-zero thickening of an affine test base induces a surjection of its finite algebras with square-zero kernel; finite flatness preserves the quotient. The smooth affine target algebra supplies a lift of every map to that quotient. Thus $E \to B$ is formally smooth and of finite presentation, hence smooth by the infinitesimal lifting criterion [20, Tag 02H6]. At a map q, its relative tangent space is $\operatorname{Der}_{\mathbb{C}}(R_X, A_v)$. The pullback cotangent module has rank n on the finite Artin scheme, so this space has dimension nd. The analytic map is therefore a holomorphic submersion with smooth total space.

A map embeds precisely when its coordinate images generate A_v. Words of degree at most $d-1$ suffice: their successive spans either increase strictly or stabilize permanently, since equality makes the span invariant under every generator. Beginning with $\mathbb{C} \cdot 1$, at most $d-1$ strict increases are possible. Generation is consequently the full-rank condition for one finite matrix of algebraic bundle maps. □

A section of $E_{\mathrm{emb}} \to B$ evaluates to a holomorphic map $f : X \to X$.

Lemma 3.2. *For such a section, $(\pi, f) : X \to B \times X$ is a closed embedding. For a section holomorphic also in $\zeta \in \mathbb{C}$,*

$$J(\zeta, x) = (\zeta, \pi(x), f_\zeta(x))$$

is a closed embedding of $\mathbb{C} \times X$.

Proof. Points with the same graph image lie in one fiber and are separated by its scheme embedding. A vector killed by the graph differential lies in $\ker d\pi$, the tangent space of the scheme fiber, where df is injective. Hence the graph is an injective immersion. The inverse image of an ambient compact is closed inside the compact inverse image under π of its base projection. The graph is proper and therefore a closed embedding. For J use the proper map $(\zeta, \pi) : \mathbb{C} \times X \to \mathbb{C} \times B$. □

3.3. **Fixed-type fibers.** There are finitely many partitions λ of d. Their algebra-type loci are constructible: isomorphisms from A_λ to a fiber algebra are described by finitely many multiplication identities and a nonzero determinant, so Chevalley's theorem applies to the isomorphism scheme. Refine these loci into a finite smooth algebraic stratification with a descending closed filtration, by induction on dimension.

Use a local primitive element and its monic polynomial P_v. Locally within a stratum, distinct support roots $\rho_i(v)$ of fixed multiplicities m_i are holomorphic. Isolate each cluster by a contour; the trace of multiplication by T on its cluster algebra, divided by m_i, is its support root. No cluster splits in this stratum neighborhood. Chinese remainders and $\tau_i = T - \rho_i(v)$ identify the algebras holomorphically with (2). Thus E_{emb} over each stratum is a holomorphic bundle with fiber

$$Y_\lambda = \mathrm{Emb}(\mathrm{Spec}\, A_\lambda, X).$$

The abstract source components are labeled. Global monodromy may permute them in transition maps; local triviality needs no global root labels.

Lemma 3.3. *If X is contractible of dimension $n \geq 2$ and has the density property, then Y_λ is Oka and $(2n-2)$-connected.*

Proof. Support evaluation is a holomorphic locally trivial bundle over

$$\mathrm{Conf}_r(X) = X^r \setminus \bigcup_{i<j} \{x_i = x_j\}.$$

Choose disjoint target coordinate neighborhoods. A length-one component has no jet data. For $m_i \geq 2$, a truncated curve embeds exactly when its first coefficient in $\mathbb{C}^n$ is nonzero; its $m_i - 2$ higher coefficients are arbitrary. Sufficiency follows by inverting the truncated series in a coordinate with nonzero linear term. Thus the fiber is

$$\prod_{m_i \geq 2} ((\mathbb{C}^n \setminus \{0\}) \times \mathbb{C}^{n(m_i-2)}). \tag{3}$$

Arbitrary coefficient sizes are allowed because finite jets are germs. Coordinate changes are finite Taylor substitutions, holomorphic and invertible on the whole coefficient spaces. There is no quotient by source reparametrizations.

The ordered configuration space is Oka by [4, Theorem 3.1]. Each factor of (3) is Oka, since punctured $\mathbb{C}^n$ is homogeneous. Oka bundle stability proves that Y_λ is Oka.

The space X^r is contractible and its pair diagonals have real codimension $2n$. A sphere map of dimension $q \leq 2n-2$ into their complement extends to a disk in X^r. Relative smooth approximation and transversality to the finite collection of diagonals make this disk avoid all of them, since its dimension is at most $2n-1$. Paths are treated similarly. Hence $\mathrm{Conf}_r(X)$ is $(2n-2)$-connected. The fiber (3) has the homotopy type of a product of spheres S^{2n-1}, or a point. The bundle homotopy sequence gives the same connectivity for Y_λ. □

3.4. **Entire interpolation.** Fix a Runge basin $\Omega \cong_{\mathrm{hol}} \mathbb{C}^n$ in X. Let $E_\Omega \subset E_{\mathrm{emb}}$ consist of embeddings whose support images all lie in Ω. This is open: the universal finite source family is proper over the mapping space, so projection of the closed set of support images outside Ω is closed. Thus $E_\Omega \to B$ is a holomorphic submersion. On each fixed-type stratum it is a bundle with fiber $\mathrm{Emb}(\operatorname{Spec} A_\lambda, \Omega)$. Lemma 3.3, applied to $\Omega \cong \mathbb{C}^n$, makes these fibers Oka and $(2n-2)$-connected. Since $2n-2 \geq n-1$, section existence in Proposition 2.1 supplies a holomorphic section. Evaluation gives

$$h : X \longrightarrow \Omega \quad \text{embedding every scheme fiber of } \pi. \tag{4}$$

For a monic hypersurface projection one can alternatively take $h(x) = \Psi(\epsilon t(x), 0, \dots, 0)$, where $\Psi : \mathbb{C}^n \to \Omega$ is a biholomorphism and $\epsilon \neq 0$. If H is any coordinate of $\Psi(\epsilon T, 0, \dots, 0)$ and $M(v)$ is multiplication by T in the monomial basis, the remainder coefficients are $H(M(v))1$. Entire functional calculus makes these coefficients holomorphic through collisions. The section construction above supplies the endpoint for a general curvilinear family.

Proposition 3.4. *There is a holomorphic $f : \mathbb{C} \times X \to X$ with $f_0 = \mathrm{id}_X$, $f_1 = h$, and each (π, f_ζ) a closed embedding.*

Proof. Pull $E_{\mathrm{emb}} \to B$ back to $B \times \mathbb{C}_\zeta$. Its product filtration has strata of dimension at most $n+1$. Their Oka fibers have vanishing homotopy groups through degree $2n-2$, and $2n-2 \geq n$ for every $n \geq 2$. Thus Proposition 2.1 applies, including the endpoint $n = 2$. Prescribe the identity over $B \times \{0\}$ and h over $B \times \{1\}$. These sections are holomorphic on disjoint neighborhoods of the two closed hyperplanes. Extend to an entire section, evaluate, and apply Lemma 3.2. □

4. From interpolation to compact capture

For Stein B, write $\mathrm{Aut}_B(X)$ for automorphisms of $B \times X$ of the form $(v, x) \mapsto (v, A^v(x))$.

Lemma 4.1. *Let X be Stein with the density property, and $\pi : X \to B$ proper with B Stein. Suppose $f : \mathbb{C} \times X \to X$ is holomorphic, $f_0 = \mathrm{id}_X$, and each (π, f_ζ) is a closed embedding. For compact $K \subset X$ and $\delta > 0$, there is $A \in \mathrm{Aut}_B(X)$ with*

$$d(A^{\pi(x)}(x), f_1(x)) < \delta \qquad (x \in K),$$

for any distance inducing the topology of X.

Proof. Choose an $\mathcal{O}(B)$-convex compact L containing $\pi(K)$. The compact $K^* = \pi^{-1}(L)$ is $\mathcal{O}(X)$-convex by pullback of separating base functions.

The joint graph $J(\zeta, x) = (\zeta, \pi(x), f_\zeta(x))$ is closed and embedded by the proof of Lemma 3.2. On it, $\partial_\zeta f$ is a holomorphic section of the ambient vertical bundle $\mathrm{pr}_X^* TX$ on $\mathbb{C} \times B \times X$. The restriction exact sequence and Cartan's theorem B extend it to the entire Stein product. Restriction to $\zeta = s \in [0, 1]$ gives smooth time dependence and zero B-component.

Write $j_s = (\pi, f_s)$. The known trajectories on the initial graph are $j_s(x)$, with compact track for $x \in K^*$. Local ODE dependence, continued over finitely many time intervals, gives one open neighborhood U of $j_0(K^*)$ on which the full flow F_s exists through time 1. It is injective by uniqueness, holomorphic, parameter preserving, and starts at inclusion. Global completeness is not required.

Each moving compact $F_s(j_0(K^*)) = j_s(K^*)$ is $\mathcal{O}(B \times X)$-convex. Outside $j_s(X)$, separate from the closed submanifold by a function in its coherent ideal, using Cartan's theorem A. On $j_s(X)$, identify with X, separate from K^*, and extend that function to $B \times X$ by Cartan's theorem B. Proposition 2.2 applied to $j_0(K^*)$ now gives the required time-one approximation. □

Theorem 4.2 (Compact capture)**.** *In Theorem 1.1, fix its finite curvilinear projection $\pi : X \to \mathbb{C}^n$ and a Runge basin $\Omega \cong_{\mathrm{hol}} \mathbb{C}^n$. Every compact $K \subset X$ admits $A \in \mathrm{Aut}_{\mathbb{C}^n}(X)$ satisfying*

$$A^{\pi(x)}(x) \in \Omega \quad (x \in K).$$

Proof. Use Proposition 3.4 and Lemma 4.1. The compact $h(K)$ lies in open Ω, so a sufficiently close approximation lies in Ω as well. □

Remark 4.3. Complex time supplies a jointly holomorphic velocity extension and hence a smoothly time-dependent local flow on the common graph neighborhood.

5. Stabilization

Proof of Theorem 1.1. For compact $K \subset X$, take a family A from capture. On $M = X \times \mathbb{C}^n$ the map

$$\Phi(x, w) = (A^{w+\pi(x)}(x), w + \pi(x)) \tag{5}$$

is an automorphism with inverse

$$(y, v) \longmapsto ((A^v)^{-1}(y),\ v - \pi((A^v)^{-1}(y))).$$

Capture gives $\Phi(K \times \{0\}) \subset \Omega \times \mathbb{C}^n$. Compactness supplies $r > 0$ with $\Phi(K \times \overline{\mathbb{B}}_r) \subset \Omega \times \mathbb{C}^n$. For arbitrary $\rho > 0$, let $D_{r/\rho}(x, w) = (x, (r/\rho)w)$. Then

$$(\Phi \circ D_{r/\rho})^{-1}(\Omega \times \mathbb{C}^n)$$

is a Runge Euclidean domain in M containing $K \times \overline{\mathbb{B}}_\rho$. Thus every compact in M lies in such a domain.

For a compact exhaustion C_j of M, inductively take a Runge Euclidean domain containing $C_j \cup \overline{U_{j-1}}$, with $U_0 = \varnothing$. In its global Euclidean coordinates choose a large ball with image U_j containing that union. Then $\overline{U_{j-1}} \subset U_j \Subset M$, and the ball images exhaust M. Runge transitivity makes them Runge in M; their compact closures are $\mathcal{O}(M)$-convex by the same approximation argument. The product $M = X \times \mathbb{C}^n$ has the density property, as recalled in Section 2. Proposition 2.3 completes the proof. □

Lemma 5.1 (Curvilinear linear projections in dimensions two and three)**.** *Let $X \subset \mathbb{C}^N$ be a smooth connected closed affine variety of dimension $n \in \{2, 3\}$. A nonempty Zariski open set of linear maps $L : \mathbb{C}^N \to \mathbb{C}^n$ restricts to finite flat surjective morphisms on X with curvilinear scheme fibers.*

Proof. If $N = n$, then $X = \mathbb{C}^n$ and invertible linear maps suffice. Assume $N \geq n + 1$. Let $\mathcal{P} = \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^n)$. For every $x \in X$, restriction $\mathcal{P} \to \mathrm{Hom}(T_x X, \mathbb{C}^n)$ is a surjective linear map. The locus of square n-by-n matrices of rank at most $n - 2$ has codimension four. Hence the closed algebraic incidence

$$\mathcal{I} = \{(x, L) : \mathrm{rank}(L|_{T_x X}) \leq n - 2\} \subset X \times \mathcal{P}$$

has dimension $\dim \mathcal{P} + n - 4 < \dim \mathcal{P}$. The Zariski closure of its projection to $\mathcal{P}$ is proper. Outside that closure $d(L|_X)$ has corank at most one everywhere.

There is also a nonempty Zariski open set of finite linear projections. For completeness, let $\overline{X} \subset \mathbb{P}^N$ be the projective closure. Its infinity section has dimension $n - 1$. A generic center $\mathbb{P}(\ker L)$ in the infinity hyperplane has dimension $N - n - 1$ and misses that section: the corresponding incidence has codimension at least one, since the sum of their dimensions is $N - 2$ whereas the infinity hyperplane has dimension $N - 1$. Such centers exist and avoidance is open. The induced projective map $\overline{X} \to \mathbb{P}^n$ is finite. Indeed a fiber lies in a projective linear space whose hyperplane is the center; any positive-dimensional projective fiber would meet that

hyperplane, contradicting avoidance. Properness and zero-dimensional fibers imply finiteness. The inverse image of $\mathbb{C}^n \subset \mathbb{P}^n$ is exactly X, so $L|_X$ is finite. Its image is all of $\mathbb{C}^n$, because it is closed and has dimension n. Intersect this open set with the corank open set.

At closed points the source local ring is regular, hence Cohen–Macaulay, of dimension n; the target local ring is regular of dimension n and the fiber has dimension zero. Miracle flatness [20, Tag 00R4] therefore applies. Flatness at all closed points gives flatness everywhere for this finite morphism (equivalently check its finite module locally at maximal ideals). Thus the finite algebra is locally free, of constant positive rank.

The tangent space of a scheme fiber at x is $\ker d(L|_X)_x$, of dimension at most one. A finite local $\mathbb{C}$-algebra with embedding dimension zero is $\mathbb{C}$; with embedding dimension one its maximal ideal is generated by one element by Nakayama's lemma, and it is $\mathbb{C}[\tau]/(\tau^m)$. Hence all local scheme fibers are curvilinear, including collision points. This proves the lemma. □

Proof of Theorem 1.2 and Corollary 1.3. The monic hypersurface projection constructed in Section 3 satisfies Theorem 1.1. For affine surfaces and threefolds, choose a finite curvilinear projection by Lemma 5.1 and apply the same theorem. □

6. Applications

6.1. **McLean's varieties.** Let K_4 be the complement of the strict transform of $x^7 + y^2 + z^2 + w^2 = 0$ in the blow-up of $\mathbb{C}^4$ at $(0, 0, 1, i)$. McLean's construction gives smooth contractible affine varieties K_n, $n \geq 4$, not biholomorphic to $\mathbb{C}^n$; see [8] and [1, Proposition 3.2] for the biholomorphic distinction. Huang gives each K_n a smooth affine hypersurface presentation and its density property [1, Theorem 1.5 and Corollary 1.6]. Corollary 6.1 uses these two facts from [1].

Corollary 6.1. *For every $n \geq 4$, $K_n \times \mathbb{C}^n \cong_{\mathrm{hol}} \mathbb{C}^{2n}$. If $s(K_n)$ is the least nonnegative j with $K_n \times \mathbb{C}^j \cong_{\mathrm{hol}} \mathbb{C}^{n+j}$, then $1 \leq s(K_n) \leq n$. A one-coordinate holomorphic cancellation counterexample occurs in some dimension $m \in \{n, \ldots, 2n-1\}$.*

Proof. Apply Theorem 1.2. Non-Euclideanity gives the lower bound. For $j = s(K_n)$, $K_n \times \mathbb{C}^{j-1}$ is not Euclidean but its product with $\mathbb{C}$ is, and its dimension is $n + j - 1$. □

Huang's embedding method gives $K_n \times \mathbb{C}^{n+1} \cong_{\mathrm{hol}} \mathbb{C}^{2n+1}$. Thus n stabilizing directions suffice in place of $n + 1$.

For K_4 a finite projection is explicit. Set

$$r = x, \quad A = (z - iw)/2, \quad B = 2(z + iw), \quad D = y.$$

Then $P = r^7 + AB + D^2$ and the center is $(0, 1, 0, 0)$. The modification coordinates $r = ta,\ A = 1 + tb,\ B = tc,\ D = td$ give

$$t^6 a^7 + (1 + tb)c + td^2 - 1 = 0. \tag{6}$$

Its coordinate ring maps onto $\mathbb{C}[r, A, B, D][r/P, (A-1)/P, B/P, D/P]$, with $t = P$. The polynomial, linear in c, is primitive and irreducible: $1 + tb$ is coprime to $t^6 a^7 + td^2 - 1$. The target is integral of dimension four with the original fraction field, so the surjection has zero kernel. This includes $t = 0, c = 1$. Smoothness follows from the gradient: simultaneous vanishing of $1 + tb, tc, 2td, 7t^6 a^6$ would make the defining polynomial equal to -1.

With $u = a - t$, projection to (u, b, c, d) has monic equation

$$T^6(u + T)^7 + (1 + Tb)c + Td^2 - 1 = 0$$

of degree 13. On $u = b = d = 0$ it is $T^{13} = 1 - c$, with a length-13 collision at $c = 1$ and cyclic monodromy on the punctured line. No sheet ordering is needed. The modification model is

previously studied [1, 10]; Huang's suspension [1, Remark 3.8] also gives a simpler degree-two finite projection. The degree-13 example displays both collision and monodromy explicitly.

6.2. **The Koras–Russell cubic.** The hypersurface

$$X_{\mathrm{KR}} = \{x + x^2y + z^2 + t^3 = 0\} \subset \mathbb{C}^4$$

is smooth and contractible. Leuenberger proved its holomorphic density property [14]; see also [15, 1].

Corollary 6.2. *The Koras–Russell cubic satisfies*

$$X_{\mathrm{KR}} \times \mathbb{C}^3 \cong_{\mathrm{hol}} \mathbb{C}^6.$$

Proof. Apply Theorem 1.2 with $n = 3$. □

This replaces the four-direction cylinder in [1, Theorem 1.8] by a three-direction cylinder. Likewise, Gorai's contractible density-property hypersurfaces $X_{d-1,d} \subset \mathbb{C}^5$ [10] satisfy $X_{d-1,d} \times \mathbb{C}^4 \cong_{\mathrm{hol}} \mathbb{C}^8$ by Theorem 1.2. Explicit compact-capture criteria on an adapted model of the Koras–Russell cylinder, including Runge capture domains and protected capture of compact bridges, are given in [13].

6.3. **Nonlinearizable holomorphic actions.** We apply the sharper stabilization bound to the actions in [1, Section 4]. The bounds below lower the dimensions of the constructions in [1, Section 4]. Nonlinearizable holomorphic $\mathbb{C}^*$-actions on $\mathbb{C}^N$, $N \geq 4$, were already constructed by Derksen and Kutzschebauch [16]. All actions below are transported by biholomorphisms.

Corollary 6.3. *For every odd $\beta \geq 3$ and $N \geq 6$, there is an effective holomorphic $\mathbb{C}^*$-action on $\mathbb{C}^N$ whose fixed locus for the subgroup $\mu_{2\beta}$ of 2β-th roots of unity is*

$$\mathbb{C}^{N-2} \sqcup (\mathbb{C}^* \times \mathbb{C}^{N-3}).$$

Neither the full action nor this cyclic restriction is topologically conjugate to a linear action. For fixed N, distinct odd β give inequivalent holomorphic actions, even allowing inversion of the group parameter.

Proof. The first-kind Koras–Russell threefold $X_\beta = \{x^2y + x + z^2 + t^\beta = 0\}$ is smooth and contractible; see [17, Theorem 1.1] and its complex topological realization. Its ordinary density property follows from [14, Main Theorem, condition (A)]: in that notation take $a(z,t) = -z^2 - t^\beta$, $b = -1$, and $k = 0$. The common zeros of a, a_z have $z = t = 0$, where $b \neq 0$, and $a_z = -2z$ is not identically zero on the z-line. Thus Theorem 1.2 gives $X_\beta \times \mathbb{C}^{N-3} \cong_{\mathrm{hol}} \mathbb{C}^N$ for $N \geq 6$. Transport Huang's action

$$\lambda \cdot (x, y, z, t, w) = (\lambda^{2\beta}x, \lambda^{-2\beta}y, \lambda^\beta z, \lambda^2 t, w).$$

It is effective since $\gcd(\beta, 2) = 1$. A primitive 2β-th root fixes precisely $z = t = 0$; the equation becomes $x(xy + 1) = 0$, giving the stated two components. A linear fixed space is connected, and a conjugating homeomorphism preserves fixed sets. The full $\mathbb{C}^*$-fixed locus is $\mathbb{C}^{N-3}$, with normal weights $(-2\beta, \beta, 2)$: the tangent equation on X_β at the origin is $dx = 0$. These weight multisets distinguish the actions; inversion changes one negative weight into two. Huang's bound for this construction is $N \geq 7$. □

Corollary 6.4. *For $n \geq 4$ and $N \geq 2n$, there is an effective holomorphic $\mathbb{C}^*$-action on $\mathbb{C}^N$ whose full fixed locus, and whose fixed locus for every μ_m, $m \geq 2$, is biholomorphic to K_n. The full action and all these restrictions are holomorphically nonlinearizable, although they are simultaneously smoothly conjugate to linear actions.*

Proof. On $K_n \times \mathbb{C}^{N-n} \cong_{\mathrm{hol}} \mathbb{C}^N$, let $\lambda \cdot (p, w) = (p, \lambda w)$. Every stated fixed locus is $K_n \times \{0\}$. A holomorphic linearization would identify K_n with a linear subspace of complex dimension n, contrary to its exoticness. A real diffeomorphism $K_n \to \mathbb{C}^n$, supplied by McLean's construction, gives the smooth conjugacy by taking its product with the identity. Huang's corresponding sufficient bound is $N \geq 2n + 1$. □

Corollary 6.5. *For every $N \geq 8$, there is an effective holomorphic $\mathbb{C}^*$-action on $\mathbb{C}^N$ whose fixed locus for the full group and for every μ_m, $m \geq 2$, is the tom Dieck–Petrie surface $S_{7,2}$. These actions are not topologically conjugate to linear actions.*

Proof. Use Huang's presentation [1, Remark 3.8 and Proposition 4.6]

$$K_4 \cong \{zuv = p_{7,2}(x, y, z)\}, \qquad p_{7,2} = \frac{(1 + xz)^7 - (1 + yz)^2 - z}{z},$$

where the quotient denotes its polynomial continuation at $z = 0$. Let $S_{7,2} = \{p_{7,2} = 0\}$. On $K_4 \times \mathbb{C}^{N-4} \cong_{\mathrm{hol}} \mathbb{C}^N$, use

$$\lambda \cdot (x, y, z, u, v, w) = (x, y, z, \lambda u, \lambda^{-1} v, \lambda w).$$

For each stated group the fixed equations are $u = v = w = 0$, leaving exactly $S_{7,2}$, with no additional affine factor. This surface is smooth, contractible and has logarithmic Kodaira dimension one [18]. If it were homeomorphic to $\mathbb{R}^4$, it would be simply connected at infinity; Ramanujam's characterization [19] would make it algebraically $\mathbb{C}^2$, contradicting its logarithmic Kodaira dimension. Its real dimension excludes other Euclidean spaces. A linear fixed set is a vector subspace, so no topological linearization exists. Huang obtains this construction for $N \geq 9$; Corollary 6.1 lowers the bound to eight. □

7. The role of the finite projection

Curvilinearity supplies local primitive elements, finitely many algebra types, and regular-curve-jet embedding fibers. Their connectivity is $2n - 2$, sufficient for section existence over the complex-time base of dimension $n + 1$, including $n = 2$. The finite map also makes every interpolating graph proper, which supplies the moving holomorphic convexity required for approximation. Compact-dependent capture and Runge-ball exhaustion then produce the qualitative uniformization.

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