FINITE-PROJECTION STABILIZATION OF CONTRACTIBLE AFFINE HYPERSURFACES

SCOTT SCHUBR

ABSTRACT. Let $X \subset \mathbb{C}^{n+1}$, $n \geq 2$, be a smooth contractible affine hypersurface with the holomorphic density property. We prove that $X \times \mathbb{C}^n$ is biholomorphic to $\mathbb{C}^{2n}$. The proof replaces the proper-embedding parameter in Huang's stabilization construction by a finite monic projection, retaining colliding fibers as curvilinear finite schemes. Their relative embedding space is a holomorphic submersion with Oka and sufficiently connected fibers on root-multiplicity strata. Stratified Oka interpolation and parametric Andersén–Lempert approximation yield compact capture in a fixed Runge Euclidean basin. Applications include $K_n \times \mathbb{C}^n \cong_{\mathrm{hol}} \mathbb{C}^{2n}$ for McLean's varieties, $n \geq 4$, and $X_{\mathrm{KR}} \times \mathbb{C}^3 \cong_{\mathrm{hol}} \mathbb{C}^6$ for the Koras–Russell cubic. We also lower by one the sufficient ambient dimensions in Huang's specified nonlinearizable-action constructions, obtaining disconnected cyclic fixed loci on $\mathbb{C}^6$ and a fixed contractible non-Euclidean surface on $\mathbb{C}^8$.

1. INTRODUCTION

A Stein manifold has the *holomorphic density property* if the Lie algebra generated by its complete holomorphic vector fields is dense, in the compact-open topology, in all holomorphic vector fields. Here density property always means this ordinary holomorphic property, rather than its volume-preserving analogue. The notation $\cong_{\mathrm{hol}}$ denotes biholomorphic equivalence.

Huang proved that a contractible Stein n-manifold with the density property admitting a proper holomorphic embedding into $\mathbb{C}^k$ satisfies

$$X \times \mathbb{C}^k \cong_{\mathrm{hol}} \mathbb{C}^{n+k};$$

see [9, Theorem 1.2]. His proof combines exact parametric point standardization with a cylinder automorphism and Runge-ball exhaustion. For an affine hypersurface its given embedding supplies $k = n + 1$. We obtain one fewer stabilizing direction.

Theorem 1.1. *Let $n \geq 2$, and let $X \subset \mathbb{C}^{n+1}$ be a nonempty smooth contractible affine hypersurface with the holomorphic density property. Then*

$$X \times \mathbb{C}^n \cong_{\mathrm{hol}} \mathbb{C}^{2n}.$$

The additional ingredient is compact capture over a finite projection. After a linear coordinate change, the equation is monic in one coordinate t, so projection $\pi : X \to B = \mathbb{C}^n$ is finite and flat. Its scheme fibers are disjoint unions of $\operatorname{Spec} \mathbb{C}[\tau]/(\tau^m)$. Multiplicity and monodromy prevent direct use of a theorem about a fixed tuple of distinct points. We interpolate embeddings of the entire fiber algebras instead.

Fix a Runge domain $\Omega \subset X$ biholomorphic to $\mathbb{C}^n$. For every compact $K \subset X$, we construct a holomorphic family $A^v \in \mathrm{Aut}(X)$, $v \in B$, with holomorphic inverse family, such that

$$A^{\pi(x)}(x) \in \Omega \qquad (x \in K). \tag{1}$$

Date: October 23, 2026 (edition 2).

2020 *Mathematics Subject Classification.* 32M17, 32E30, 32Q28; 14R10.

Key words and phrases. Holomorphic cancellation, density property, Oka principle, affine hypersurface, finite scheme.

The family may depend on K, and its members need not preserve π on X. No global ordering of sheets is used. Capture only requires an image in an open basin, permitting approximation of an embedded finite scheme instead of exact collapse to one point.

We construct the scheme interpolation in Section 3, prove capture in Section 4, and apply Huang's cylinder mechanism in Section 5. The applications appear in Section 6.

2. Analytic tools

A compact L in a Stein manifold M is $\mathcal{O}(M)$-convex if every point outside L is separated from L by a holomorphic function on M. A Stein domain $U \subset M$ is *Runge* if its holomorphic functions can be approximated on compact subsets by functions in $\mathcal{O}(M)$.

Proposition 2.1 (Stratified section existence)**.** *Let B be a finite-dimensional reduced Stein space and $p : E \to B$ a holomorphic submersion. Suppose B has a finite descending filtration by closed complex subvarieties with smooth strata S, and $p^{-1}(S) \to S$ is a holomorphic fiber bundle with path-connected Oka fiber Y_S satisfying*

$$\pi_q(Y_S) = 0 \qquad (1 \le q \le \dim_{\mathbb{C}} S - 1).$$

A holomorphic section prescribed on a closed complex subvariety of B extends to a global holomorphic section.

This is the interpolation part of [6, Theorem 7.1], with empty approximation compact. Connectivity supplies the extensions over Stein Morse handles; an initial global continuous section or homotopy is not an additional hypothesis in this existence statement. We use the equivalence of the Oka and convex approximation properties. We also use Oka bundle stability: a holomorphic fiber bundle with Oka base and Oka fiber has Oka total space [7, Theorem 2.11].

Proposition 2.2 (Parametric approximation)**.** *Let W be Stein, X a Stein manifold with the density property, and $U \subset W \times X$ open. Suppose $F_s : U \to W \times X$, $0 \le s \le 1$, is a smooth isotopy of injective holomorphic maps preserving W, with F_0 the inclusion. If a compact $Q \subset U$ has $\mathcal{O}(W \times X)$-convex images $F_s(Q)$ for every s, then F_1 can be approximated uniformly on Q by automorphisms of $W \times X$ preserving W.*

This is [10, Theorem 2.2]. It supplies entire parameter automorphisms and their holomorphic inverses. We will verify its moving-compact hypothesis explicitly.

Three further density-property consequences are used. A Stein density-property n-manifold contains an attracting basin biholomorphic to $\mathbb{C}^n$, by Varolin's basin construction with nonresonant jets [15, Corollary 4.1 and the following remark]. This basin is Runge: take a small $\mathcal{O}(X)$-convex compact neighborhood V of its fixed point in the basin, carry any basin compact into V by an iterate, apply Oka–Weil to the transported function near V, and compose the approximation with the iterate. Also $X \times \mathbb{C}$ has the density property whenever X is Stein with that property [14, Theorem 3.7]. This iterates without assuming density property for $\mathbb{C}$ itself. Finally we use Boc Thaler's criterion:

Proposition 2.3 (Runge-ball exhaustion)**.** *A Stein manifold M with the density property is biholomorphic to $\mathbb{C}^{\dim M}$ if it admits an exhaustion by Runge images of Euclidean balls.*

See [5, Theorem IV.15, pp. 62–64], also [9, Theorem 2.3]. Our ball images will be nested and relatively compact, with holomorphically convex compact closures.

3. Embedding the finite fibers

3.1. **A monic projection.** Write $X = \{F = 0\}$, with F reduced of degree d. Choose e with $F_d(e) \neq 0$, where F_d is its highest homogeneous part. Make e the last coordinate direction and divide by $F_d(e)$. Then in coordinates (v, t), $P_v(T) = F(v, T)$ is monic of degree d. If $d = 1$, $X \cong \mathbb{C}^n$ already; assume $d \geq 2$.

Projection $\pi : X \to B = \mathbb{C}^n$ is finite, flat and onto, with free algebra basis $1, T, \ldots, T^{d-1}$ over $\mathbb{C}[v]$. The monic root bound over compact coefficient sets also proves analytic properness. Put

$$A_v = \mathbb{C}[T]/(P_v), \qquad Z_v = \operatorname{Spec} A_v.$$

For distinct support roots of multiplicities $\lambda = (m_1, \ldots, m_r)$,

$$A_v \cong A_\lambda := \prod_{i=1}^{r} \mathbb{C}[\tau_i]/(\tau_i^{m_i}). \tag{2}$$

All fibers below retain this full scheme structure.

3.2. **The relative embedding submersion.** Represent $n + 1$ elements $q = (q_1, \ldots, q_n, q_t)$ of A_v in the fixed monomial basis. In $B \times \mathbb{C}^{(n+1)d}$, impose the d remainder coefficients of

$$F(q_1, \ldots, q_n, q_t) = 0 \pmod{P_v}. \tag{3}$$

Call the resulting space E. Its fiber consists of maps $Z_v \to X$, with no requirement that they preserve π.

Lemma 3.1. *The space E is smooth and $E \to B$ is a holomorphic submersion of relative dimension nd. Its locus E_{emb} of closed scheme embeddings is Zariski open.*

Proof. At q, the vertical derivative is

$$A_v^{n+1} \longrightarrow A_v, \qquad (\delta q_j)_j \longmapsto \sum_j F_j(q)\delta q_j.$$

In each residue field the entries are the gradient at a smooth target point. They generate the unit ideal in A_v, so choose h_j with $\sum_j h_j F_j(q) = 1$. Then $a \mapsto (ah_j)_j$ is a right inverse. The complex rank is d, also at total collision. The relative implicit function theorem proves smoothness and submersivity.

A map embeds exactly when its coordinate images generate A_v. Words of degree at most $d - 1$ suffice: successive word spans either strictly increase or stabilize permanently, since equality makes the span invariant under every generator. Beginning with $\mathbb{C} \cdot 1$, at most $d - 1$ strict increases are possible. Generation is thus an open finite matrix full-rank condition. □

A section of $E_{\mathrm{emb}} \to B$ evaluates to a holomorphic map $f : X \to X$.

Lemma 3.2. *For such a section, $(\pi, f) : X \to B \times X$ is a closed embedding. For a section holomorphic also in $\zeta \in \mathbb{C}$,*

$$J(\zeta, x) = (\zeta, \pi(x), f_\zeta(x))$$

is a closed embedding of $\mathbb{C} \times X$.

Proof. Points with the same graph image lie in one fiber and are separated by its scheme embedding. A vector killed by the graph differential lies in $\ker d\pi$, the tangent space of the scheme fiber, where df is injective. Hence the graph is an injective immersion. The inverse image of an ambient compact is closed inside the compact inverse image under π of its base projection. The graph is proper and therefore a closed embedding. For J use the proper map $(\zeta, \pi) : \mathbb{C} \times X \to \mathbb{C} \times B$. □

3.3. **Fixed-type fibers.** Refine the finitely many algebraic constructible root-multiplicity loci into a finite smooth algebraic stratification with a descending closed algebraic filtration. Take smooth open pieces of irreducible closures and repeat on their boundaries and singular loci. Dimension drops, so the process is finite; grouping the pieces by dimension gives the required closed filtration.

Locally within a stratum, distinct support roots $\rho_i(v)$ of fixed multiplicities m_i are holomorphic. Isolate each cluster by a contour; the trace of multiplication by T on its cluster algebra, divided by m_i, is its support root. No cluster splits in this stratum neighborhood. Chinese remainders and $\tau_i = T - \rho_i(v)$ identify the algebras holomorphically with (2). Thus E_{emb} over each stratum is a holomorphic bundle with fiber

$$Y_\lambda = \mathrm{Emb}(\mathrm{Spec}\, A_\lambda, X).$$

The abstract source components are labeled. Global monodromy may permute them in transition maps; local triviality needs no global root labels.

Lemma 3.3. *If X is contractible of dimension $n \geq 2$ and has the density property, then Y_λ is Oka and $(2n-2)$-connected.*

Proof. Support evaluation is a holomorphic locally trivial bundle over

$$\mathrm{Conf}_r(X) = X^r \setminus \bigcup_{i<j} \{x_i = x_j\}.$$

Choose disjoint target coordinate neighborhoods. A length-one component has no jet data. For $m_i \geq 2$, a truncated curve embeds exactly when its first coefficient in $\mathbb{C}^n$ is nonzero; its $m_i - 2$ higher coefficients are arbitrary. Sufficiency follows by inverting the truncated series in a coordinate with nonzero linear term. Thus the fiber is

$$\prod_{m_i \geq 2} ((\mathbb{C}^n \setminus \{0\}) \times \mathbb{C}^{n(m_i-2)}). \tag{4}$$

Arbitrary coefficient sizes are allowed because finite jets are germs. Coordinate changes are finite Taylor substitutions, holomorphic and invertible on the whole coefficient spaces. There is no quotient by source reparametrizations.

The ordered configuration space is Oka by [10, Theorem 3.1]. Each factor of (4) is Oka, since punctured $\mathbb{C}^n$ is homogeneous. Oka bundle stability proves that Y_λ is Oka.

The space X^r is contractible and its pair diagonals have real codimension $2n$. A sphere map of dimension $q \leq 2n-2$ into their complement extends to a disk in X^r. Relative smooth approximation and transversality to the finite collection of diagonals make this disk avoid all of them, since its dimension is at most $2n-1$. Paths are treated similarly. Hence $\mathrm{Conf}_r(X)$ is $(2n-2)$-connected. The fiber (4) has the homotopy type of a product of spheres S^{2n-1}, or a point. The bundle homotopy sequence gives the same connectivity for Y_λ. □

3.4. **Entire interpolation.** Fix a biholomorphism $\Psi : \mathbb{C}^n \to \Omega$ to a Runge Euclidean basin and $\epsilon \neq 0$, and set

$$h(x) = \Psi(\epsilon t(x), 0, \ldots, 0). \tag{5}$$

It embeds each scheme fiber because t generates A_v. It also defines a holomorphic coefficient section. Each coordinate of $\Psi(\epsilon T, 0, \ldots, 0)$ is an entire function $H(T)$. If $M(v)$ is multiplication by T in the monomial basis of A_v, then $H(M(v))1$ gives its remainder coefficients. Entire functional calculus is holomorphic in v, through all collisions. Algebraicity of Ψ is unnecessary.

Proposition 3.4. *There is a holomorphic $f : \mathbb{C} \times X \to X$ with $f_0 = \mathrm{id}_X$, $f_1 = h$, and each (π, f_ζ) a closed embedding.*

Proof. Pull $E_{\mathrm{emb}} \to B$ back to $B \times \mathbb{C}_\zeta$. Its product filtration has strata of dimension at most $n+1$. Their Oka fibers have vanishing homotopy groups through degree $2n-2$, and $2n-2 \geq n$ for every $n \geq 2$. Thus Proposition 2.1 applies, including the endpoint $n = 2$. Prescribe the identity over $B \times \{0\}$ and h over $B \times \{1\}$. These sections are holomorphic on disjoint neighborhoods of the two closed hyperplanes. Extend to an entire section, evaluate, and apply Lemma 3.2. □

4. From interpolation to compact capture

For Stein B, write $\mathrm{Aut}_B(X)$ for automorphisms of $B \times X$ of the form $(v, x) \mapsto (v, A^v(x))$.

Lemma 4.1. *Let X be Stein with the density property, and $\pi : X \to B$ proper with B Stein. Suppose $f : \mathbb{C} \times X \to X$ is holomorphic, $f_0 = \mathrm{id}_X$, and each (π, f_ζ) is a closed embedding. For compact $K \subset X$ and $\delta > 0$, there is $A \in \mathrm{Aut}_B(X)$ with*

$$d(A^{\pi(x)}(x), f_1(x)) < \delta \qquad (x \in K),$$

for any distance inducing the topology of X.

Proof. Choose an $\mathcal{O}(B)$-convex compact L containing $\pi(K)$. The compact $K^* = \pi^{-1}(L)$ is $\mathcal{O}(X)$-convex by pullback of separating base functions.

The joint graph $J(\zeta, x) = (\zeta, \pi(x), f_\zeta(x))$ is closed and embedded by the proof of Lemma 3.2. On it, $\partial_\zeta f$ is a holomorphic section of the ambient vertical bundle $\mathrm{pr}_X^* TX$ on $\mathbb{C} \times B \times X$. The restriction exact sequence and Cartan's theorem B extend it to the entire Stein product. Restriction to $\zeta = s \in [0, 1]$ gives smooth time dependence and zero B-component.

Write $j_s = (\pi, f_s)$. The known trajectories on the initial graph are $j_s(x)$, with compact track for $x \in K^*$. Local ODE dependence, continued over finitely many time intervals, gives one open neighborhood U of $j_0(K^*)$ on which the full flow F_s exists through time 1. It is injective by uniqueness, holomorphic, parameter preserving, and starts at inclusion. Global completeness is not required.

Each moving compact $F_s(j_0(K^*)) = j_s(K^*)$ is $\mathcal{O}(B \times X)$-convex. Outside $j_s(X)$, separate from the closed submanifold by a function in its coherent ideal, using Cartan's theorem A. On $j_s(X)$, identify with X, separate from K^*, and extend that function to $B \times X$ by Cartan's theorem B. Proposition 2.2 applied to $j_0(K^*)$ now gives the required time-one approximation. □

Theorem 4.2 (Compact capture)**.** *In Theorem 1.1, fix a monic finite projection $\pi : X \to \mathbb{C}^n$ and a Runge basin $\Omega \cong_{\mathrm{hol}} \mathbb{C}^n$. Every compact $K \subset X$ admits $A \in \mathrm{Aut}_{\mathbb{C}^n}(X)$ satisfying*

$$A^{\pi(x)}(x) \in \Omega \quad (x \in K).$$

Proof. Use Proposition 3.4 and Lemma 4.1. The compact $h(K)$ lies in open Ω, so a sufficiently close approximation lies in Ω as well. □

Remark 4.3. Complex time supplies one jointly holomorphic velocity extension; separate real-time extensions would not alone justify smooth dependence. No Oka property of the infinite-dimensional group $\mathrm{Aut}(X)$ is used.

5. Stabilization

Proof of Theorem 1.1. For compact $K \subset X$, take a family A from capture. On $M = X \times \mathbb{C}^n$ the map

$$\Phi(x, w) = \big(A^{w+\pi(x)}(x), w + \pi(x)\big) \tag{6}$$

is an automorphism with inverse

$$(y, v) \longmapsto \big((A^v)^{-1}(y),\, v - \pi((A^v)^{-1}(y))\big).$$

Capture gives $\Phi(K \times \{0\}) \subset \Omega \times \mathbb{C}^n$. Compactness supplies $r > 0$ with $\Phi(K \times \overline{\mathbb{B}}_r) \subset \Omega \times \mathbb{C}^n$. For arbitrary $\rho > 0$, let $D_{r/\rho}(x, w) = (x, (r/\rho)w)$. Then

$$(\Phi \circ D_{r/\rho})^{-1}(\Omega \times \mathbb{C}^n)$$

is a Runge Euclidean domain in M containing $K \times \overline{\mathbb{B}}_\rho$. Thus every compact in M lies in such a domain.

For a compact exhaustion C_j of M, inductively take a Runge Euclidean domain containing $C_j \cup \overline{U_{j-1}}$, with $U_0 = \varnothing$. In its global Euclidean coordinates choose a large ball with image U_j containing that union. Then $\overline{U_{j-1}} \subset U_j \Subset M$, and the ball images exhaust M. Runge transitivity makes them Runge in M; their compact closures are $\mathcal{O}(M)$-convex by the same approximation argument. Iterated product density gives DP on M. Proposition 2.3 completes the proof. □

6. Applications

6.1. **McLean's varieties.** Let K_4 be the complement of the strict transform of $x^7 + y^2 + z^2 + w^2 = 0$ in the blow-up of $\mathbb{C}^4$ at $(0, 0, 1, i)$. McLean's construction gives smooth contractible affine varieties K_n, $n \geq 4$, not biholomorphic to $\mathbb{C}^n$; see [13] and [9, Proposition 3.2] for the biholomorphic distinction. Huang gives each K_n a smooth affine hypersurface presentation and its density property [9, Theorem 1.5 and Corollary 1.6]. These object-specific facts are inputs here.

Corollary 6.1. *For every $n \geq 4$, $K_n \times \mathbb{C}^n \cong_{\mathrm{hol}} \mathbb{C}^{2n}$. If $s(K_n)$ is the least nonnegative j with $K_n \times \mathbb{C}^j \cong_{\mathrm{hol}} \mathbb{C}^{n+j}$, then $1 \leq s(K_n) \leq n$. A one-coordinate holomorphic cancellation counterexample occurs in some dimension $m \in \{n, \ldots, 2n-1\}$.*

Proof. Apply Theorem 1.1. Non-Euclideanity gives the lower bound. For $j = s(K_n)$, $K_n \times \mathbb{C}^{j-1}$ is not Euclidean but its product with $\mathbb{C}$ is, and its dimension is $n + j - 1$. □

Huang's embedding method gives $K_n \times \mathbb{C}^{n+1} \cong_{\mathrm{hol}} \mathbb{C}^{2n+1}$. The corollary removes one direction without identifying the minimum.

For K_4 a finite projection is explicit. Set

$$r = x, \quad A = (z - iw)/2, \quad B = 2(z + iw), \quad D = y.$$

Then $P = r^7 + AB + D^2$ and the center is $(0, 1, 0, 0)$. The modification coordinates $r = ta,\ A = 1 + tb,\ B = tc,\ D = td$ give

$$t^6 a^7 + (1 + tb)c + td^2 - 1 = 0. \tag{7}$$

Its coordinate ring maps onto $\mathbb{C}[r, A, B, D][r/P, (A-1)/P, B/P, D/P]$, with $t = P$. The polynomial, linear in c, is primitive and irreducible: $1 + tb$ is coprime to $t^6 a^7 + td^2 - 1$. The target is integral of dimension four with the original fraction field, so the surjection has zero kernel. This includes $t = 0, c = 1$. Smoothness follows from the gradient: simultaneous vanishing of $1 + tb, tc, 2td, 7t^6 a^6$ would make the defining polynomial equal to -1.

With $u = a - t$, projection to (u, b, c, d) has monic equation

$$T^6(u + T)^7 + (1 + Tb)c + Td^2 - 1 = 0$$

of degree 13. On $u = b = d = 0$ it is $T^{13} = 1 - c$, with a length-13 collision at $c = 1$ and cyclic monodromy on the punctured line. No sheet ordering is needed. The modification model is previously studied [9, 8]; Huang's suspension [9, Remark 3.8] also gives a simpler degree-two finite projection. Degree 13 illustrates collision handling, not a necessity of the method.

6.2. **The Koras–Russell cubic.** The hypersurface

$$X_{\mathrm{KR}} = \{x + x^2y + z^2 + t^3 = 0\} \subset \mathbb{C}^4$$

is smooth and contractible. Leuenberger proved its holomorphic density property [11]; see also [12, 9].

Corollary 6.2. *The Koras–Russell cubic satisfies*

$$X_{\mathrm{KR}} \times \mathbb{C}^3 \cong_{\mathrm{hol}} \mathbb{C}^6.$$

Proof. Apply Theorem 1.1 with $n = 3$. □

This replaces the four-direction cylinder in [9, Theorem 1.8] by a three-direction cylinder. It does not determine whether X_{KR} is biholomorphic to $\mathbb{C}^3$, or whether its one-direction cylinder is Euclidean. Algebraic inequivalence is a different assertion.

6.3. **Nonlinearizable holomorphic actions.** We apply the sharper stabilization bound to the actions in [9, Section 4]. These are improvements of the sufficient dimensions in those particular constructions. Nonlinearizable holomorphic $\mathbb{C}^*$-actions on $\mathbb{C}^N$, $N \geq 4$, were already constructed in [1, Proposition 4.3]. All actions below are transported by biholomorphisms.

Corollary 6.3. *For every odd $\beta \geq 3$ and $N \geq 6$, there is an effective holomorphic $\mathbb{C}^*$-action on $\mathbb{C}^N$ whose fixed locus for the subgroup $\mu_{2\beta}$ of 2β-th roots of unity is*

$$\mathbb{C}^{N-2} \sqcup (\mathbb{C}^* \times \mathbb{C}^{N-3}).$$

Neither the full action nor this cyclic restriction is topologically conjugate to a linear action. For fixed N, distinct odd β give inequivalent holomorphic actions, even allowing inversion of the group parameter.

Proof. The first-kind Koras–Russell threefold $X_\beta = \{x^2y + x + z^2 + t^\beta = 0\}$ is smooth and contractible; see [2, Theorem 1.1] and its complex topological realization. Its ordinary density property follows from [11, Main Theorem, condition (A)]: in that notation take $a(z,t) = -z^2 - t^\beta$, $b = -1$, and $k = 0$. The common zeros of a, a_z have $z = t = 0$, where $b \neq 0$, and $a_z = -2z$ is not identically zero on the z-line. Thus Theorem 1.1 gives $X_\beta \times \mathbb{C}^{N-3} \cong_{\mathrm{hol}} \mathbb{C}^N$ for $N \geq 6$. Transport Huang's action

$$\lambda \cdot (x, y, z, t, w) = (\lambda^{2\beta}x, \lambda^{-2\beta}y, \lambda^\beta z, \lambda^2 t, w).$$

It is effective since $\gcd(\beta, 2) = 1$. A primitive 2β-th root fixes precisely $z = t = 0$; the equation becomes $x(xy + 1) = 0$, giving the stated two components. A linear fixed space is connected, and a conjugating homeomorphism preserves fixed sets. The full $\mathbb{C}^*$-fixed locus is $\mathbb{C}^{N-3}$, with normal weights $(-2\beta, \beta, 2)$: the tangent equation on X_β at the origin is $dx = 0$. These weight multisets distinguish the actions; inversion changes one negative weight into two. Huang's bound for this construction is $N \geq 7$. □

Corollary 6.4. *For $n \geq 4$ and $N \geq 2n$, there is an effective holomorphic $\mathbb{C}^*$-action on $\mathbb{C}^N$ whose full fixed locus, and whose fixed locus for every μ_m, $m \geq 2$, is biholomorphic to K_n. The full action and all these restrictions are holomorphically nonlinearizable, although they are simultaneously smoothly conjugate to linear actions.*

Proof. On $K_n \times \mathbb{C}^{N-n} \cong_{\mathrm{hol}} \mathbb{C}^N$, let $\lambda \cdot (p, w) = (p, \lambda w)$. Every stated fixed locus is $K_n \times \{0\}$. A holomorphic linearization would identify K_n with a linear subspace of complex dimension n, contrary to its exoticness. A real diffeomorphism $K_n \to \mathbb{C}^n$, supplied by McLean's construction, gives the smooth conjugacy by taking its product with the identity. Huang's corresponding sufficient bound is $N \geq 2n + 1$. □

Corollary 6.5. *For every $N \geq 8$, there is an effective holomorphic $\mathbb{C}^*$-action on $\mathbb{C}^N$ whose fixed locus for the full group and for every μ_m, $m \geq 2$, is the tom Dieck–Petrie surface $S_{7,2}$. These actions are not topologically conjugate to linear actions.*

Proof. Use Huang's presentation [9, Remark 3.8 and Theorem 4.2]

$$K_4 \cong \{zuv = p_{7,2}(x, y, z)\}, \qquad p_{7,2} = \frac{(1 + xz)^7 - (1 + yz)^2 - z}{z},$$

where the quotient denotes its polynomial continuation at $z = 0$. Let $S_{7,2} = \{p_{7,2} = 0\}$. On $K_4 \times \mathbb{C}^{N-4} \cong_{\mathrm{hol}} \mathbb{C}^N$, use

$$\lambda \cdot (x, y, z, u, v, w) = (x, y, z, \lambda u, \lambda^{-1} v, \lambda w).$$

For each stated group the fixed equations are $u = v = w = 0$, leaving exactly $S_{7,2}$, with no additional affine factor. This surface is smooth, contractible and has logarithmic Kodaira dimension one [3]. If it were homeomorphic to $\mathbb{R}^4$, it would be simply connected at infinity; Ramanujam's characterization [4] would make it algebraically $\mathbb{C}^2$, contradicting its logarithmic Kodaira dimension. Its real dimension excludes other Euclidean spaces. A linear fixed set is a vector subspace, so no topological linearization exists. Huang obtains this construction for $N \geq 9$; Corollary 6.1 lowers the bound to eight. □

7. Scope

The hypersurface hypothesis is concrete: the finite projection has one algebra generator, with curvilinear local fibers. Their embedding spaces have the regular-curve-jet fibers (4) and sufficient connectivity. Higher-codimension finite projections can have nonmonogenic local algebras; their topology and Oka property are not supplied by this argument.

The proof is qualitative, yielding compact-dependent automorphisms and uniformization by exhaustion rather than explicit coordinate formulas. Further reduction of stabilization needs additional input. The parameter estimate is explicit: the complex-time base has dimension $n + 1$, while the fiber connectivity is $2n - 2$, sufficient also at $n = 2$.

Disclosure. AI-assisted tools were used for literature searches, drafting, and proof checking.

References

[1] H. Derksen and F. Kutzschebauch, Nonlinearizable holomorphic group actions, *Math. Ann.* **311** (1998), 41–53.

[2] A. Dubouloz and J. Fasel, Families of $\mathbb{A}^1$-contractible affine threefolds, *Algebr. Geom.* **5** (2018), no. 1, 1–14. `https://content.algebraicgeometry.nl/2018-1/2018-1-001.pdf`.

[3] T. tom Dieck and T. Petrie, Contractible affine surfaces of Kodaira dimension one, *Japan. J. Math. (N.S.)* **16** (1990), no. 1, 147–169. doi:10.4099/math1924.16.147.

[4] C. P. Ramanujam, A topological characterisation of the affine plane as an algebraic variety, *Ann. of Math. (2)* **94** (1971), no. 1, 69–88.

[5] L. Boc Thaler, *Fatou components*, doctoral dissertation, University of Ljubljana, 2016, Theorem IV.15, pp. 62–64. `https://repozitorij.uni-lj.si/Dokument.php?id=106392&lang=slv`.

[6] F. Forstnerič, The Oka principle for sections of stratified fiber bundles, *Pure Appl. Math. Q.* **6** (2010), no. 3, 843–874. arXiv:0705.0591v5.

[7] F. Forstnerič, Oka manifolds: From Oka to Stein and back, *Ann. Fac. Sci. Toulouse Math. (6)* **22** (2013), no. 4, 747–809. arXiv:1211.6383v5.

[8] S. Gorai, *A new class of smooth algebraic varieties with the density property and a counterexample to the Tóth–Varolin conjecture*, preprint, 2026. arXiv:2610.27660v1.

[9] G. Huang, *A counterexample to the holomorphic cancellation problem*, preprint, 2026. arXiv:2610.12581v2.

[10] F. Kutzschebauch and A. Ramos-Peon, An Oka principle for a parametric infinite transitivity property, *J. Geom. Anal.* **27** (2017), no. 3, 2018–2043. arXiv:1401.0093v4.

[11] M. Leuenberger, (Volume) density property of a family of complex manifolds including the Koras–Russell cubic threefold, *Proc. Amer. Math. Soc.* **144** (2016), 3887–3902. arXiv:1507.03842.
[12] L. Makar-Limanov, On the hypersurface $x + x^2y + z^2 + t^3 = 0$ in $\mathbb{C}^4$ or a $\mathbb{C}^3$-like threefold which is not $\mathbb{C}^3$, *Israel J. Math.* **96** (1996), 419–429.
[13] M. McLean, Lefschetz fibrations and symplectic homology, *Geom. Topol.* **13** (2009), no. 4, 1877–1944. arXiv:0709.1639.
[14] D. Varolin, The density property for complex manifolds and geometric structures, *J. Geom. Anal.* **11** (2001), no. 1, 135–160.
[15] D. Varolin, The density property for complex manifolds and geometric structures. II, *Internat. J. Math.* **11** (2000), no. 6, 837–847.