Finite certificates for the lower edge of a weighted Weil operator

Scott Schubr
Luminary Oddities

October 26, 2026

Abstract

We characterize the reach of complete finite lower certificates for the original minimal weighted Weil operator. The infimum of their certified negative allowances equals the operator's actual negative spectral depth, and a computable constrained core supplies rigorous two-sided approximations to its lower edge at every prescribed positive accuracy. The proof combines logarithmic form-core density with compact domination of a monotone frequency limit. Exact derivative-null vectors allow finite deflation without changing the original form or discarding mixed terms. An executed certificate with eight such vectors proves

$$I + H \geq 0.999999978600004918685\, I,$$

with a retained positive potential on the null complement. All arithmetic, frequency, spatial, quadrature and finite-matrix errors are included; directed ball arithmetic verifies both signed matrices of order 4085. We also prove that, at fixed arithmetic cutoff and positive multiplication reference, boundedly many constraints cannot maintain the required cap as the regularization decreases, even with an arbitrary growing frequency band. The quantitative obstruction forces growing constraint dimension and identifies the continuation mechanism that the adaptive framework must control.

1 The lower-edge question

The Riemann hypothesis is equivalent to positivity of Weil's complete explicit form on its two-moment compact core. In a fixed weighted realization this becomes the coefficient-one endpoint $I + H \geq I$. A useful finite certificate must keep that realization, its constraints and the whole omitted complement. The problem is therefore not only to exhibit a small negative allowance, but to determine whether a finite certification architecture can approach the true lower edge and what prevents a natural continuation from succeeding.

This paper gives three connected answers. First, the complete signed known-reference certificates are spectrally exact: if $d = \max\{0, -\inf \operatorname{spec} H\}$, their infimal allowance is exactly d. A core parameterization and a dovetailed upper/lower search then approximate the actual spectral bottom to every positive error tolerance. Second, exact global null vectors can be incorporated into that architecture and lifted back to the entire original domain. With eight derivative-null vectors, the explicit application in Theorem 6.2 gives

$$q_H[h] \geq -\nu_7 \|h\|_2^2 + \nu_7 \|(I - P)h\|_2^2 + \lambda_7 \|sPh\|_2^2, \tag{1}$$
$$\nu_7 = \frac{4279999016263}{2 \cdot 10^{20}}, \qquad \lambda_7 = \frac{983737}{10^{13}},$$

where P is the orthogonal projection off the eight-dimensional exact-null span. The scalar allowance is about 72.57 times smaller than the published lower certificate [1]. The potential in (1) is explicitly projected. Third, Theorem 5.1 proves that a fixed arithmetic cutoff, positive reference parameter and bounded constraint dimension cannot maintain the needed cap for

small regularization, regardless of the frequency schedule. Its quantitative form requires constraint growth of order at least $\log\log(1/\alpha)$ in that fixed-cutoff architecture.

The numerical instance and the structural theorems form one argument. The exactness theorem prevents an architectural loss at a strict threshold; the executed deflation theorem demonstrates a whole-domain gain; the obstruction explains why simply continuing a fixed finite design cannot supply the endpoint. A Schur-complement receiver records an alternative way to pay the complete off-range covariance. The final appendix studies the cost of a fixed derivative-null prefix in an exact off-line-zero detector, retaining the original weighted norm and multiplicities.

The weighted realization and derivative-null identities have antecedents in van Hemert [6] and the quantitative null-compression theorem [2]. The preceding explicit lower certificate [1] used two original constraint rows and no exact-null deflation. Joint cutoff and frequency localization results [3, 4] concern complementary spectral information. We retain their distinct conclusions rather than identifying a high-frequency estimate with the actual low compression required here.

Finite/complement spectral certification, variational approximation, Schur complements and computable-spectrum theory are established tools [11, 12, 13]. Zhu's compact-window Weil certificates [14] and the fixed-window negative-certificate construction [15] are close methodological precedents. Our recipient is instead the complete full-line reciprocal-theta weighted realization, with exact global null rows and an adaptive lower-certificate characterization on its specified minimal domain. The signed finite reduction pays that recipient's actual tails, metric and mixed terms. Quasi-Riemann zero-free half-planes [20, 21] give a different arithmetic conclusion; their estimates are not inputs to the lower bound.

Related arithmetic cutoff and null-adaptation certificates in the local Nyman–Beurling setting [5] concern its Cauchy-weighted zeta observation. Its sharp capacity optimizes the leading tail variance; optimization of the full head, mass and remainder costs is a separate question. These are adjacent finite-certificate results, with a different observation norm from the reciprocal-theta norm of (1).

2 The original constrained realization

Inner products are conjugate-linear in the first variable. Set

$$\mathcal{M}f(z)=\int_{\mathbb{R}} f(u)e^{zu}\,du, \qquad \widehat{f}(\xi)=\int_{\mathbb{R}} f(u)e^{-iu\xi}\,du,$$

with Plancherel measure $d\xi/(2\pi)$. The compact two-moment core is

$$\mathcal{F}_0=\{f\in C_c^\infty(\mathbb{R};\mathbb{C}) : \mathcal{M}f(1/2)=\mathcal{M}f(-1/2)=0\}.$$

Write $\Psi_0=\Gamma'/\Gamma$, $c_n=\Lambda(n)/\sqrt{n}$, and $S_bf(u)=f(u+b)$. The complete physical form is

$$\begin{aligned} Q[f]=&\int_{\mathbb{R}}[\operatorname{Re}\Psi_0(1/4+i\xi/2)-\log\pi]|\widehat{f}(\xi)|^2\frac{d\xi}{2\pi}\\ &-\sum_{n\ge 2}c_n\big(\langle f,S_{\log n}f\rangle+\langle f,S_{-\log n}f\rangle\big). \end{aligned} \tag{2}$$

Use the normalized Riemann theta kernel

$$\psi(u)=2\sum_{k\ge 1}(2\pi^2k^4e^{9u/2}-3\pi k^2e^{5u/2})e^{-\pi k^2e^{2u}}, \tag{3}$$

$$c(u)=\cosh(u/2), \quad w(u)=\psi(u)c(u), \quad s(u)=\sqrt{\psi(u)/c(u)}.$$

The theta identity makes ψ positive and even, with $\mathcal{M}\psi(z) = \xi(1/2+z)$. Define

$$\mathcal{K} = \{h \in L^2(\mathbb{R}) : \langle \sqrt{w}, h\rangle = \langle \sqrt{w}\tanh(u/2), h\rangle = 0\},$$
$$a(\xi) = \operatorname{Re}\Psi_0(1/4 + i\xi/2) - \Psi_0(1/4), \qquad m_0 = \Psi_0(1/4) - \log\pi.$$

The positive series

$$a(\xi) = \sum_{j\geq 0} \frac{(\xi/2)^2}{(j+1/4)((j+1/4)^2 + (\xi/2)^2)} \tag{4}$$

shows that a is increasing on $[0,\infty)$, nonnegative and comparable to $\log(2+|\xi|)$ outside a bounded interval after adding 1. On $\mathcal{C} = C_c^\infty(\mathbb{R}) \cap \mathcal{K}$ take

$$q_A[h] = \int_{\mathbb{R}} a(\xi)|\widehat{sh}(\xi)|^2 \frac{d\xi}{2\pi}.$$

The domain $\mathcal{V}$ is its specified minimal closure in $\|h\|_2^2 + q_A[h]$. Two fixed compact bump corrections make $\mathcal{C}$ dense in $\mathcal{K}$: the ratio of the row profiles is the nonconstant function $\tanh(u/2)$, so bumps near two distinct points have an invertible row matrix. The restricted graph map $h \mapsto \sqrt{a}\,\widehat{sh}$ is closable.

With $P_{\mathcal{K}}$ denoting orthogonal projection onto $\mathcal{K}$, put

$$B = P_{\mathcal{K}}\left[m_0 M_s^2 - \sum_{n\geq 2} c_n M_s(S_{\log n} + S_{-\log n})M_s\right]\Big|_{\mathcal{K}}, \qquad H = A + B. \tag{5}$$

The norm-convergent arithmetic perturbation and its complete quantitative tails are established in [1]. In particular, if B_M retains every $n \leq M$ and the same m_0 term, a convenient computable full tail is

$$\|B - B_M\| \leq \delta_M^{\text{coarse}} := \frac{640(M+1)^2}{20^{M+1}} \longrightarrow 0. \tag{6}$$

The tighter paired-shift tail used for the executed certificate is derived below. No diagonal or shift channel is deleted from the target H.

For $h \in \mathcal{C}$, $f = sh \in \mathcal{F}_0$ and

$$q_H[h] = Q[f], \qquad \|h\|_2^2 = \int_{\mathbb{R}} |f(u)|^2 \frac{c(u)}{\psi(u)}\,du. \tag{7}$$

Conversely $f \in \mathcal{F}_0$ gives $h = f/s \in \mathcal{C}$. The row identities are $\langle\sqrt{w}, h\rangle = \int cf$ and $\langle\sqrt{w}\tanh(u/2), h\rangle = \int \sinh(u/2)f$. The constrained Weil criterion [7, 8, 1] therefore reads

$$\text{RH} \iff Q[f] \geq 0 \ (f \in \mathcal{F}_0) \iff H \geq 0 \iff I + H \geq I. \tag{8}$$

Here the constraints remove the two polar terms of the explicit formula. The reverse passage from the constrained to the unrestricted compact Weil criterion uses ψ and ψ' to repair the two moments: their transform values vanish at every centered zero and their polar moment matrix is invertible. Smooth cutoffs followed by two compact bump repairs pass this identity to the original compact tests; see [1, Section2]. This fixes the realization and normalization used throughout.

For a finite arithmetic cutoff set

$$H_M = A + B_M, \qquad m_M(\xi) = a(\xi) + m_0 - 2\sum_{2\leq n\leq M} c_n\cos(\xi\log n).$$

Given $\alpha, \mu > 0$ and a complete exterior floor $m_M \geq \mu$ on $|\xi| \geq L$, write

$$D_\alpha = \alpha + \mu s^2, \quad t_\alpha = s/\sqrt{D_\alpha}, \quad b_L = 1_{[-L,L]}(\mu - m_M), \quad K_{\alpha,L} = M_{t_\alpha} b_L(D) M_{t_\alpha}. \tag{9}$$

For finite L, this signed band operator is compact: the Fourier restriction composed with M_{t_α} is Hilbert–Schmidt, since $t_\alpha \in L^2$. For the actual complete arithmetic debit $\delta \geq \|B - B_M\|$,

$$q_H[h] + (\alpha + \delta)\|h\|_2^2 \geq \|D_\alpha^{1/2} h\|_2^2 - \langle D_\alpha^{1/2} h, K_{\alpha,L} D_\alpha^{1/2} h \rangle. \tag{10}$$

The inequality retains the entire omitted exterior as a nonnegative form. The original rows in $x = D_\alpha^{1/2} h$ are exactly $t_\alpha \cosh(u/2)$ and $t_\alpha \sinh(u/2)$. All subsequent penalties vanish on the stated exact row kernel.

3 Completeness of the lower certificates

The executed bound of Theorem 6.2 is one member of a family of finite lower certificates. We first determine what this family can approximate. All realizations in this section are the original ones of Section 2. In particular, the closure defining $\mathcal{V}$ is part of the problem.

Write

$$b_* := \inf \operatorname{spec} H, \qquad d = -b_*.$$

The bounded perturbation B makes b_* finite. The nonzero exact null vector $(\psi'' - \psi/4)/s$ gives $b_* \leq 0$, so $d \geq 0$. A strict finite certificate means the known-reference construction with complete arithmetic and operator approximation errors paid, together with a strictly positive finite signed Gram inequality. Its scalar deficit e is the resulting proved bound $H \geq -eI$. The finite data may include exact null constraints, but the completeness theorem below already holds using the two original polar rows.

Theorem 3.1 (Exactness at the lower edge)**.** *For every $\nu \geq 0$ such that $H \geq -\nu I$ and every $\varepsilon > \nu$, there is a strict finite known-reference certificate with deficit $e < \varepsilon$. Consequently*

$$d = \inf\{e : \ e \text{ is the deficit of a strict finite certificate}\}.$$

In particular, the coefficient-one endpoint is equivalent to the existence of such certificates below every positive tolerance.

The assertion includes the forward existence of finite witnesses. Classical core approximation and compact domination are important here: convergence of each individual Rayleigh quotient would not suffice.

3.1 The original logarithmic graph core

Lemma 3.2. *The specified minimal domain equals*

$$\mathcal{V}_{max} = \left\{ h \in \mathcal{K} : \int_{\mathbb{R}} a(\xi) |\widehat{sh}(\xi)|^2 \frac{d\xi}{2\pi} < \infty \right\}.$$

Proof. The positive digamma series and its logarithmic asymptotic give

$$1 + a(\xi) \asymp \log(2 + |\xi|).$$

These estimates also follow from the standard digamma formulas [16]. Put $v(\xi) = \sqrt{\log(2 + |\xi|)}$. For $f = sh$, the form topology is therefore equivalent to the joint norm consisting of $\|h\|_2$ and $\|v\widehat{f}\|_2$; the ordinary L^2 norm of f is controlled because s is bounded.

The inequality

$$\log(2+|\xi|) \le \log(2+|\eta|) + \log(2+|\xi-\eta|)$$

implies $v(\xi) \le v(\eta) + v(\xi-\eta)$. Thus frequency translation $T_\zeta F(\xi) = F(\xi-\zeta)$ satisfies

$$\|vT_\zeta F\|_2 \le \|vF\|_2 + v(\zeta)\|F\|_2.$$

Frequency translations are strongly continuous in this weighted space: first check compactly supported bounded Fourier functions, then use density and the displayed bound.

Choose $\chi \in C_c^\infty$, $0 \le \chi \le 1$, equal to one near zero, and set $\chi_R(u) = \chi(u/R)$. The Fourier convolution kernels $\widehat{\chi}_R/(2\pi)$ have mass one and uniformly bounded weighted L^1 norms. Their weighted absolute tails outside every fixed neighborhood of zero tend to zero: after scaling, this follows from Schwartz decay and $v(\zeta/R) \le v(\zeta)$ for $R \ge 1$. Weighted translation continuity and a near–far decomposition of the convolution therefore give

$$\chi_R f \longrightarrow f \quad \text{in the logarithmic Fourier graph norm.}$$

Independently $\chi_R h \to h$ in L^2, by dominated convergence.

At a fixed spatial cutoff, mollification of $\chi_R f$ converges in that same graph norm: its Fourier multiplier is uniformly bounded and tends pointwise to one. The mollified functions have support in one slightly larger compact interval. Dividing by s, whose minimum there is positive, also gives L^2 convergence to $\chi_R h$. We have obtained compact smooth approximants in both required norms.

Their two polar row errors tend to zero, since those row functionals are bounded on L^2. Choose two fixed compact smooth bumps with invertible row matrix. Such bumps exist because the two row profiles are linearly independent. Subtracting the corresponding small bump combinations imposes both rows exactly; the correction coefficients tend to zero and the bumps have finite graph norm. A diagonal choice of cutoffs and mollifications proves $\mathcal{V}_{max} \subset \mathcal{V}$. The reverse inclusion follows from closedness of $h \mapsto a(D)^{1/2}(sh)$ and the definition of the minimal closure. □

3.2 A compact domination lemma

Lemma 3.3. *Let K_j be decreasing bounded selfadjoint compact operators on a Hilbert space, and put*

$$q_j[x] = \|x\|^2 - \langle x, K_j x\rangle, \qquad q_\infty[x] = \sup_j q_j[x].$$

Suppose the limit domain is dense and $q_\infty[x] \ge g\|x\|^2$ there, where $0 < g < 1$. For every $0 < \gamma < g$, eventually $q_j \ge \gamma I$.

Proof. Let $C = (K_1)_+$, a positive compact operator. Since $K_j \le K_1 \le C$, the bounded form $q_j[x] + \langle x, Cx\rangle - \|x\|^2$ is nonnegative and weakly lower semicontinuous. If the conclusion failed, there would be an increasing subsequence of indices, still denoted by j, and unit vectors x_j with $q_j[x_j] < \gamma$. Pass to a weakly convergent subsequence $x_j \rightharpoonup x$. Compactness gives $\langle x_j, Cx_j\rangle \to \langle x, Cx\rangle$. For each fixed k, monotonicity gives $q_k[x_j] \le q_j[x_j]$ once $j \ge k$. Lower semicontinuity consequently implies

$$q_k[x] \le \|x\|^2 - 1 + \gamma.$$

Taking the supremum puts x in the limit domain and gives $(1-g)\|x\|^2 \ge 1-\gamma$. This contradicts $\|x\| \le 1$ and $\gamma < g$. No norm convergence of the increasingly negative parts of K_j has been used. □

3.3 From a strict threshold to a finite witness

Proof of Theorem 3.1. First take rational ν, ε with $\varepsilon > \nu$. Choose M so that

$$\delta := \delta_M^{\mathrm{coarse}} \le \frac{\varepsilon - \nu}{4}, \qquad \alpha = \frac{\varepsilon + \nu}{2}, \qquad \mu = 1.$$

Then $H_M \ge -(\nu + \delta)I$. Write

$$D_\alpha = \alpha + \mu s^2, \quad t_\alpha = s/\sqrt{D_\alpha}, \quad b_L = 1_{[-L,L]}(\mu - m_M), \quad K_L = M_{t_\alpha} b_L(D) M_{t_\alpha}.$$

Because $m_M(\xi) \to \infty$, choose L_0 with $m_M \ge \mu$ outside $[-L_0, L_0]$. Each K_L is compact: $Q_L M_{t_\alpha}$ is Hilbert–Schmidt, with squared Hilbert–Schmidt norm $L\|t_\alpha\|_2^2/\pi$, and the band multiplier is bounded. For $L \ge L_0$ the newly admitted multiplier is nonpositive, so K_L decreases in form order.

The bounded invertible change $x = D_\alpha^{1/2} h$ sends $\mathcal{K}$ to the common row kernel X of

$$\ell_0 = t_\alpha c, \qquad \ell_1 = t_\alpha \sinh(u/2).$$

Compress K_L to X and take, for example, the integer cutoffs above L_0. By Lemma 3.2, the limit form has domain $D_\alpha^{1/2}\mathcal{V}$ and is exactly

$$q_\infty[x] = q_{H_M}[h] + \alpha\|h\|^2.$$

Indeed $m_M - a$ is a bounded multiplier, so finiteness of the limiting high-frequency energy is precisely the logarithmic graph condition. The complete arithmetic debit has not been added to this identity. It gives instead the lower bound

$$q_\infty[x] \ge (\alpha - \nu - \delta)\|h\|^2 \ge g\|x\|^2, \qquad g = \frac{\alpha - \nu - \delta}{\alpha + \mu\|s\|_\infty^2} > 0.$$

Here $g < 1$. The limit domain is dense in X by the bounded invertible change of variables and core density. Lemma 3.3 now supplies a finite L with $I - K_L \ge \gamma I$ on X, where $\gamma = g/2$.

We next remove the restriction to X by an exact row penalty. Let $E = \mathrm{span}\{\ell_0, \ell_1\}$, let $r_* > 0$ be the least eigenvalue on E of $\ell_0\langle\ell_0, \cdot\rangle + \ell_1\langle\ell_1, \cdot\rangle$, and put $C_0 = \|I - K_L\|$. For $x = y + z$, $y \in X$, $z \in E$, Young's inequality gives

$$\langle x, (I - K_L)x\rangle \ge \frac{\gamma}{2}\|y\|^2 - \left(C_0 + \frac{2C_0^2}{\gamma}\right)\|z\|^2.$$

Choose a positive rational τ so large that

$$\tau r_* > C_0 + 2C_0^2/\gamma + \gamma/2.$$

Then

$$I - K_L + \tau \sum_{i=0}^{1} \ell_i\langle\ell_i, \cdot\rangle \ge \frac{\gamma}{2} I.$$

The penalty vanishes identically on X.

For completeness, this strict ambient gap has finite phase witnesses with effective errors. Let P_R be spatial restriction and $P_{R,p}$ the orthogonal projection onto the local polynomials of degree below p on equal cells of half-width r in $[-R, R]$. The real Taylor remainder of $e^{-iu\xi}$ about each cell center, followed by the best-approximation property of the projection, gives

$$\|Q_L(P_R - P_{R,p})\| \le \|Q_L(P_R - P_{R,p})\|_{\mathrm{HS}} \le \sqrt{\frac{2RL}{\pi}}\,\frac{(rL)^p}{p!}.$$

Consequently

$$\eta := \|Q_L M_{t_\alpha} - Q_L P_{R,p} M_{t_\alpha}\| \le \|1_{|u|>R} t_\alpha\|_\infty + \|t_\alpha\|_\infty \sqrt{\frac{2RL}{\pi}} \frac{(rL)^p}{p!}.$$

The complete theta envelope makes the first term tend to zero with R, and the second tends to zero with p at fixed R, L, r. Both maps have norm at most $j = \|t_\alpha\|_\infty$, so their signed products differ by at most $2\|b_L\|_\infty j\eta$. This elementary bound suffices for existence; the sharper Legendre budget used in Theorem 6.2 is more efficient numerically.

Let Y be an orthonormal column map for this local polynomial space, $C = Y^* b_L(D) Y$ and $W = [M_{t_\alpha} Y, \ell_0, \ell_1]$. The finite signed operator with coefficient $C_{\mathrm{aug}} = \mathrm{diag}(C, -\tau, -\tau)$ approximates the penalized K_L in whole operator norm. Choose its error $\Gamma < \gamma/8$. It then has an upper cap strictly below $1 - \gamma/4$, and $(1 - \gamma/4) + \Gamma < 1$.

Its Gram $G = W^* W$ is strictly positive. The local columns are independent because $t_\alpha > 0$; a nontrivial combination of the two global rows cannot vanish on an exterior open interval, since it would be t_α times a nontrivial combination of cosh and sinh. Thus there is a strictly positive finite matrix

$$\beta G - G C_{\mathrm{aug}} G > 0$$

for a rational β with $\beta + \Gamma < 1$. All its integrals have effective compact enclosures and complete theta tails. Sufficiently small entry radii and one-sided pads preserve the strict inequality. A dyadic matrix sufficiently near an inverse-Cholesky congruence makes its congruence strictly diagonally dominant, furnishing a finite rational acceptance witness. Symmetric cells and the exact integer parity fold may also be imposed; they preserve these strict inequalities and the physical parity ranges. These are the same signed-Gram and complete-error requirements used for the executed construction, with freely chosen finite parameters.

The resulting certificate proves

$$H \ge -(\alpha + \delta) I, \qquad \alpha + \delta \le \frac{3\varepsilon + \nu}{4} < \varepsilon.$$

For arbitrary real ν, ε, choose rationals $\nu < \nu' < \varepsilon' < \varepsilon$ and apply the rational case; if appropriate a rational endpoint $\nu' = \nu$ is also permitted. Soundness gives $e \ge d$ for every certificate, while the construction gives $e < \varepsilon$ for every $\varepsilon > d$. This proves the infimum identity. The endpoint assertion follows from $H \ge 0$ and the criterion identified in Section 2. □

4 Effective two-sided lower-edge approximation

The strict margins in the preceding proof allow a computability statement. Computable-spectral frameworks distinguish this information from general spectral computation and classification; see [12, 13]. Here the effective global tails and the specific original core supply the required additional data.

Theorem 4.1. *The number $b_* = \inf \operatorname{spec} H$ is a computable real. For every positive rational tolerance one can obtain a finite rigorous interval containing b_* with width below that tolerance. The certificate deficits have nonincreasing certified upper approximants converging to $d = -b_*$.*

Proof. We give both sides of the approximation. Put $\mathscr{P} = \partial_u^2 - 1/4$. Every compact smooth g gives $f = \mathscr{P} g$ with both polar moments zero, by integration by parts. Conversely, if f is compact smooth with these moments zero, then

$$g(u) = -\int_{\mathbb{R}} e^{-|u-v|/2} f(v)\, dv$$

is compact smooth and $\mathscr{P}g = f$. The derivative jump of $e^{-|u|/2}$ is -1. Outside the support of f the convolution is either $-e^{-u/2}\mathcal{M}f(1/2)$ or $-e^{u/2}\mathcal{M}f(-1/2)$, hence zero. The distributional equation then gives smoothness, including the endpoints.

Choose the fixed computable flat bump

$$\eta(u) = \begin{cases} \exp[-1/(1-u^2)], & |u| < 1, \\ 0, & |u| \geq 1. \end{cases}$$

Enumerate its finite rational linear combinations under rational translations and positive rational dilations. They are dense among compact smooth functions in H^3, with supports in a common slightly larger compact interval when approximating a fixed target: mollify with a rescaled bump, approximate the convolution by finite Riemann sums, then approximate their coefficients rationally. Applying $\mathscr{P}$ gives H^1 convergence of f. The positive series gives $a(\xi) \leq 17\xi^2$, so this controls the required form energy. Division by s, with positive minimum on the common compact support, controls the original L^2 norm of $h = f/s$. The resulting list is therefore dense in the original form core and hence in $\mathcal{V}$. Real coefficients suffice, since H commutes with conjugation and its quadratic form on $u + iv$ is the sum of the two real forms.

Each admitted nonzero f has a computable positive denominator

$$N(f) = \int_{\mathbb{R}} |f|^2/s^2 \, du$$

and computable numerator $Q[f] = q_H[f/s]$. For the denominator, compact interval integration uses the complete theta series and a computable positive lower bound for s^2 on that interval. For a support in $[-R, R]$, every arithmetic shift overlap vanishes when $\log n > 2R$. Choose a computable integer upper bound for e^{2R} and retain every prime power through it; no equality decision at the endpoint is needed. Repeated integration by parts gives

$$|\widehat{f}(\xi)| \leq \|f^{(k)}\|_1 |\xi|^{-k}.$$

Together with the logarithmic symbol bound, this pays the entire archimedean Fourier tail. The explicit bump derivatives, finite integrals and finite prime coefficients admit outward enclosures of arbitrary accuracy. Refine denominators in parallel and admit a quotient only after its denominator lower endpoint is positive. Zero or canceling bump combinations never block the enumeration.

By form-core density and semibounded form spectral calculus, the infimum of these Rayleigh quotients is b_*. Their outward upper endpoints, together with the exact upper bound zero, therefore give decreasing upper approximants.

Independently enumerate the rational parameters and finite witnesses of the known-reference construction: arithmetic and frequency cutoffs, positive regularizations, cell partitions, degrees, penalties, integration precisions, error bounds and dyadic congruences. The exterior floor has finite witnesses: for example $m_M \geq a - d_M$, where $d_M = -m_0 + 2\sum_{2 \leq n \leq M} c_n$, and monotonicity of a supplies a half-line floor after a finite scalar check. Every other analytic tail bound and finite integral used above is likewise effective. Each fixed precision acceptance test is finite. Strict positivity is semidecidable by refinement; equality need never be accepted. Dovetail all these computations so that no failing, singular or slow candidate blocks another.

Every accepted witness yields a sound lower bound. By Theorem 3.1, their running maximum converges to b_*. For a given positive tolerance, continue the joint enumeration until the best upper and lower bounds differ by less than that tolerance. They converge to the same finite number, so this rule terminates. Negating the running lower bounds gives the stated approximants to d. □

5 A fixed arithmetic cutoff cannot reach the endpoint

The flexibility in Theorem 3.1 is essential. We now allow the frequency cutoff to grow arbitrarily, while retaining a fixed arithmetic set. These estimates use only the exact symbol, complete theta bounds and classical packet and dimension arguments.

Fix finite M, $\mu > 0$, and set

$$d_M = -m_0 + 2\sum_{2\le n\le M} c_n > 0, \quad B_M^+ = \mu + d_M, \quad C_M = 17 + \sum_{2\le n\le M} c_n(\log n)^2.$$

Choose L_0 so $m_M \ge \mu$ outside $[-L_0, L_0]$. For $L \in [L_0, \infty]$ write

$$b_L = 1_{[-L,L]}(\mu - m_M), \quad \mathsf{K}_{\alpha,L} = M_{t_\alpha} b_L(D) M_{t_\alpha}, \qquad t_\alpha = \frac{s}{\sqrt{\alpha + \mu s^2}}.$$

For $L = \infty$ omit the indicator and use its upper-semibounded quadratic form on $\{x : a(D)^{1/2}(t_\alpha x) \in L^2\}$. For a closed finite-codimensional subspace S let $\lambda_{\alpha,L}(S)$ be its variational upper edge, over the form domain when $L = \infty$.

Theorem 5.1 (Uniform-band obstruction)**.** *Let $S_\alpha \subset L^2(\mathbb{R})$ be arbitrary closed subspaces with $\operatorname{codim} S_\alpha \le D$, for one fixed integer D. For every choice $L(\alpha) \in [L_0, \infty]$,*

$$\lim_{\alpha\downarrow 0} \lambda_{\alpha,L(\alpha)}(S_\alpha) = 1 + \frac{d_M}{\mu} > 1.$$

Compact smooth tests already force the lower limit. This includes the two transformed polar rows and any uniformly bounded number of additional exact-null rows.

Proof. The full loss symbol satisfies

$$\begin{aligned} p_M(\xi) &:= B_M^+ - (\mu - m_M(\xi)) \\ &= a(\xi) + 2\sum_{2\le n\le M} c_n(1 - \cos(\xi \log n)), \\ 0 \le p_M(\xi) &\le C_M \xi^2. \end{aligned}$$

Thus for $f \in H^1$,

$$\langle f, b_\infty(D) f\rangle \ge B_M^+ \|f\|^2 - C_M \|f'\|^2.$$

For every $L \ge L_0$, the removed full symbol is nonpositive. Consequently $b_L \ge b_\infty$ and $b_L \le B_M^+$, including the full-band case. In particular

$$\lambda_{\alpha,L}(S) \le B_M^+/\mu.$$

Fix a unit $\eta \in C_c^\infty(-1, 1)$ and put $A_\eta = \|\eta'\|$. The $D + 1$ disjoint packets

$$v_j(u) = R^{-1/2}\eta((u - 3Rj)/R), \quad 0 \le j \le D,$$

span an orthonormal space X_R with $\|v'\| = (A_\eta/R)\|v\|$ on its whole span. Its compact support has $\sigma = \min s > 0$ and $S_1 = \max |s'| < \infty$. Direct differentiation gives

$$t_\alpha^2 \ge \frac{1}{\mu + \alpha/\sigma^2}, \qquad \|t_\alpha\|_\infty \le \mu^{-1/2}, \qquad \|t_\alpha'\| \le \frac{\alpha S_1}{\mu^{3/2}\sigma^3}$$

on this support. Every unit $v \in X_R$ therefore satisfies, uniformly over all the permitted L,

$$\langle v, \mathsf{K}_{\alpha,L} v\rangle \ge \frac{B_M^+}{\mu + \alpha/\sigma^2} - C_M\left(\frac{A_\eta}{R\sqrt{\mu}} + \frac{\alpha S_1}{\mu^{3/2}\sigma^3}\right)^2.$$

Dimension supplies a unit vector in $X_R \cap S_\alpha$ for every α. Take $\alpha \downarrow 0$ at fixed R, then $R \to \infty$, and combine with the universal upper bound.

For the actual transformed rows, $h = D_\alpha^{-1/2} v$ is compact smooth. Orthogonality to $t_\alpha c, t_\alpha \sinh(u/2)$ is exactly membership in the original $\mathcal{K}$, and an extra row $D_\alpha^{-1/2} z$ is exactly $\langle z, h \rangle = 0$. Thus the failed cap is witnessed on the original compact core. Every penalty on those exact rows vanishes on the same test. □

5.1 Necessary growth of the constraint dimension

Lemma 5.2. *The complete theta kernel satisfies*

$$\left| \frac{s'(u)}{s(u)} \right| \le 41 e^{2|u|}, \qquad s(u)^2 \ge \exp[-4e^{2|u|}].$$

Proof. For $u \ge 0$ put $r = \pi e^{2u}$ and $r_k = rk^2$. The kth theta atom is $e^{u/2}(4r_k^2 - 6r_k)e^{-r_k}$. Its derivative polynomial is $-8r_k^3 + 30r_k^2 - 15r_k$, of absolute value below $20r_k^3$ for $r_k \ge \pi > 3$. The first positive atom gives $\psi \ge 2r^2 e^{u/2 - r}$. With $z = e^{-3\pi} < 1/100$, $k^2 - 1 \ge 3(k-1)$ and

$$\sum_{k \ge 1} k^6 z^{k-1} = \frac{1 + 57z + 302z^2 + 302z^3 + 57z^4 + z^5}{(1-z)^7} < 2,$$

the complete differentiated series gives $|\psi'/\psi| \le 20r < 80e^{2u}$. Since $|c'/c| \le 1/2$, this implies $|s'/s| \le 40e^{2u} + 1/4 < 41e^{2u}$. Also the first atom and $c(u) \le e^{u/2}$ give $s^2 \ge 2r^2 e^{-r} \ge \exp[-4e^{2u}]$, using $\pi < 4$. Evenness proves both assertions on the other half-line. □

Proposition 5.3 (Uniform-band row-growth bound)**.** *Keep M, μ fixed and put $q = \log(1/\alpha) \ge 64$ and $R_q = \frac{1}{2}\log(q/8)$. If S_α has finite codimension r_α, then for every $L \ge L_0$, including infinity,*

$$\lambda_{\alpha,L}(S_\alpha) \ge B_M^+ \left(\frac{1}{\mu} - \frac{e^{-q/2}}{\mu^2} \right) - C_M \left(\frac{\pi(r_\alpha + 1)}{2R_q\sqrt{\mu}} + \frac{41 q e^{-q/2}}{8\mu^{3/2}} \right)^2.$$

Consequently, if $\lambda_{\alpha,L(\alpha)}(S_\alpha) \le 1$ for all sufficiently small α, then

$$\liminf_{\alpha \downarrow 0} \frac{r_\alpha + 1}{R_q} \ge \frac{2}{\pi} \sqrt{\frac{d_M}{C_M}}.$$

Proof. Lemma 5.2 gives $s^2 \ge e^{-q/2}$ on $[-R_q, R_q]$. The exact derivative of t_α therefore yields

$$t_\alpha^2 \ge \mu^{-1} - \mu^{-2} e^{-q/2}, \qquad |t_\alpha'| \le \frac{41 q e^{-q/2}}{8\mu^{3/2}}$$

there. Use the first $n = r_\alpha + 1$ Dirichlet sine modes on that interval, extended by zero. Their whole span obeys $\|v'\| \le \pi n \|v\|/(2R_q)$ and meets S_α nontrivially. The full-symbol inequality from Theorem 5.1 proves the displayed bound. These compact H^1 tests are in the required form domain. For the actual constraints, division by the fixed positive smooth $D_\alpha^{1/2}$ and compact smoothing with polar row repairs puts their pullbacks in the original $\mathcal{V}$, as also follows from Lemma 3.2.

If the edge is at most one, the bound implies, once the radicand is positive,

$$\frac{r_\alpha + 1}{R_q} \ge \frac{2}{\pi} \left[\sqrt{\frac{d_M - B_M^+ e^{-q/2}/\mu}{C_M}} - \frac{41 q e^{-q/2}}{8\mu} \right].$$

Take the lower limit. Since $R_q \sim \frac{1}{2}\log\log(1/\alpha)$, the necessary codimension grows at least on that scale. □

Corollary 5.4. *For every fixed finite M, the paired truncation H_M has infinite negative form index on the original constrained space.*

Proof. Choose R with $C_M A_\eta^2/R^2 < d_M/2$. Arbitrarily many disjoint packets of this fixed width span spaces on which

$$\int m_M(\xi)|\widehat{f}(\xi)|^2\frac{d\xi}{2\pi} \le -d_M\|f\|^2 + C_M\|f'\|^2 < -\frac{d_M}{2}\|f\|^2.$$

The pullback $h = f/s$ is compact smooth. Imposing the two original rows removes at most two dimensions, so negative subspaces of arbitrarily large dimension remain. Any additional fixed finite number of rows can be handled by initially taking more packets. □

Here H_M retains the complete positive reference A and truncates only its bounded paired arithmetic perturbation. The negative directions need not have a uniform negative Rayleigh quotient in the h norm. Thus this conclusion is consistent with the norm-small complete arithmetic remainder, and concerns a different operator from raw positive-jump deletion. Together with the uniform-band theorem it explains why adaptive arithmetic, as well as full complement control, enters the lower-edge search.

6 An executed certificate with eight exact null rows

We now give a strict certificate in the family studied above. The reference operator, signed arithmetic multiplier and two original constraints are those of Section 2. We add actual null vectors, rather than approximate small eigenvectors. This distinction permits a return to every vector in the original domain without losing the proved scalar lower bound.

Lemma 6.1 (Exact derivative nulls)**.** *Put*

$$q_j = (\partial_u^2 - 1/4)\partial_u^j\psi, \qquad z_j = q_j/s \quad (0 \le j \le 7), \qquad \mathcal{N} = \operatorname{span}\{z_0,\dots,z_7\}.$$

These eight vectors are independent elements of $\operatorname{Dom} H \cap \mathcal{K}$*, and* $Hz_j = 0$*. The orthogonal projection $P : \mathcal{K} \to \mathcal{N}^\perp$ preserves $\mathcal{V}$, and $q_H[h] = q_H[Ph]$ for every $h \in \mathcal{V}$.*

Proof. Here are the domain details needed for this particular realization; compare the full known-null construction in [2, 6]. The theta series and its derivatives decrease faster than every exponential on both real tails, and $|\psi^{(k)}(u)| \le C_k\psi(u)e^{C_k|u|}$. Choose even smooth cutoffs χ_R, equal to one on $[-R, R]$, with uniformly bounded derivatives, and set $f_{j,R} = (\partial_u^2 - 1/4)(\chi_R\partial_u^j\psi)$. Integration by parts gives $\mathcal{M}f_{j,R}(\pm 1/2) = 0$ exactly. Thus $h_{j,R} = f_{j,R}/s$ belongs to the original compact core. The displayed derivative bounds give $h_{j,R} \to z_j$ in L^2 and $f_{j,R} \to q_j$ in $H^1(\mathbb{R})$. The positive series (4) gives $a(\xi) \le 18\xi^2$: indeed $\sum_{k\ge 0}(k+1/4)^{-3} \le 64 + \int_{1/4}^{\infty} x^{-3}\,dx = 72$. Consequently the cutoffs are Cauchy in the minimal form norm, proving $z_j \in \mathcal{V}$ rather than only in a maximal weighted domain.

The Mellin transform is

$$\mathcal{M}q_j(z) = (z^2 - 1/4)(-z)^j\xi(1/2 + z).$$

It vanishes at both polar points and at every centered zero. The mixed explicit formula therefore gives $Q(q_j, f) = 0$ for every compact smooth constrained test f. The complete zero sum is legitimate here: repeated integration by parts on this exponentially decreasing smooth class, together with the usual $O(T\log T)$ zero count, gives absolute convergence. Continuity

of the closed form extends the mixed identity from its core to $\mathcal{V}$, so the representation theorem gives $z_j \in \operatorname{Dom} H$ and $Hz_j = 0$. Finally, a linear dependence would give a polynomial times $(z^2 - 1/4)\xi(1/2 + z)$ identically zero; hence that polynomial is zero. The finite-dimensional projection onto these form-domain null vectors preserves $\mathcal{V}$, and all its mixed form terms vanish. This proves the last assertion. □

Theorem 6.2 (The executed lower bound)**.** *For the original minimal operator of Section 2, set*

$$\nu = \frac{4279999016263}{2 \cdot 10^{20}}, \qquad \lambda = \frac{983737}{10^{13}}.$$

For every $h \in \mathcal{V}$,

$$q_H[h] \geq -\nu\|h\|_2^2 + \nu\|(I - P)h\|_2^2 + \lambda\|sPh\|_2^2. \tag{11}$$

In particular

$$I + H \geq 0.999999978600004918685\, I.$$

The potential in (11) is evaluated on Ph; this is the potential term proved by the null-space return argument below.

6.1 Parameters and the finite signed recipient

All arithmetic channels through $M = 7$ are retained; their nonzero indices are $2, 3, 4, 5, 7$. We use

$$\alpha = 10^{-8}, \quad \mu = 1/5, \quad L = 2400, \quad R = 17/10, \quad r = 1/80, \quad p = 60, \quad \tau = 1000. \tag{12}$$

There are 136 physical cells of half-width r covering $[-R, R]$. Let $Y : \mathbb{C}^{8160} \to L^2(\mathbb{R})$ have orthonormal columns

$$\phi_{aj}(u) = 1_{|u-c_a|<r}\sqrt{\frac{2j+1}{2r}}P_j((u-c_a)/r), \quad c_a = -R+(2a+1)r, \quad 0 \leq a < 136, \quad 0 \leq j < 60. \tag{13}$$

Write $t = t_\alpha$ and $b = b_L$. Define two raw five-row lists

$$r_+(u) = (c, cq_0/\psi, cq_2/\psi, cq_4/\psi, cq_6/\psi), \qquad r_-(u) = (\sinh(u/2), cq_1/\psi, cq_3/\psi, cq_5/\psi, cq_7/\psi).$$

The actual row columns are $\ell_\sigma = tr_\sigma A_\sigma$, where A_σ are the fixed 5 by 5 dyadic matrices specified by `MIX` in `code/verify_certificate.py`, also recorded in binary64 hexadecimal form in `data/certificate_summary.json`. They are exact finite witnesses with $\max|A_{ij}| < 2$. All their columns annihilate $x = D_\alpha^{1/2}h$ when $h \in \mathcal{K} \cap \mathcal{N}^\perp$.

Set

$$C = Y^*b(D)Y, \qquad W = [M_tY, \ell_+, \ell_-], \qquad C_{\mathrm{aug}} = \operatorname{diag}(C, -\tau I_{10}), \quad G = W^*W. \tag{14}$$

The rows in W are complete functions on $\mathbb{R}$. Only the phase columns are compactly supported. In particular, all phase–row and row–row cross terms are part of G.

The appendix proves the complete estimates

$$\|B - B_7\| < \delta := \frac{114}{10^{10}}, \quad m_7 \geq \mu \quad (|\xi| \geq L), \quad \|K_{\alpha,L} - M_tYY^*b(D)YY^*M_t\| \leq \Gamma := \frac{16263}{2 \cdot 10^{12}}. \tag{15}$$

The first is a full paired arithmetic remainder, the second a complete exterior floor, and the third includes all spatial and all-degree phase tails. Directed finite arithmetic proves

$$WC_{\mathrm{aug}}W^* \leq \beta I, \qquad \beta = \frac{9999995}{10^7}. \tag{16}$$

We describe exactly what that finite assertion means next.

6.2 Parity, normalization and directed acceptance

Reflection sends phase index (a, j) to $(135 - a, j)$ with sign $(-1)^j$. It acts by $+1$ on the five even rows and -1 on the five odd rows. For either parity choose the 4080 integer paired phase columns and the five matching row coordinates, and call the resulting inclusion E_σ. Then

$$E_\sigma^* E_\sigma = D := \mathrm{diag}(2I_{4080}, I_5), \qquad U_\sigma = E_\sigma D^{-1/2}$$

is an isometry. The two output ranges WU_+ and WU_- are orthogonal. Consequently their upper bounds combine by the maximum, not by addition. Use integer-fold matrices $C_\sigma = E_\sigma^* C_{\mathrm{aug}} E_\sigma$, $G_\sigma = E_\sigma^* G E_\sigma$ of order 4085.

The program reconstructs exact symmetric dyadic centers $C_{0\sigma}$, $G_{0\sigma}$ and verifies, including analytic quadrature remainders,

$$\|D^{-1/2}(C_\sigma - C_{0\sigma})D^{-1/2}\| < e, \qquad \|D^{-1/2}(G_\sigma - G_{0\sigma})D^{-1/2}\| < e, \quad e = 10^{-8}. \tag{17}$$

Put $\bar{C} = C_{0\sigma} + eD$, $\bar{G} = G_{0\sigma} + 2eD$. Every entry of these targets is exact rational. The accepted inequality is

$$F_\sigma = \beta\bar{G} - \bar{G}D^{-1}\bar{C}D^{-1}\bar{G} > 0. \tag{18}$$

For a proposed dyadic square matrix V_σ, the program encloses $V_\sigma^* F_\sigma V_\sigma$ in 192-bit ball arithmetic and proves that each diagonal lower bound exceeds the sum of all off-diagonal absolute upper bounds. Thus the congruence is positive definite; it also forces V_σ to be nonsingular, so proves (18).

This implication retains the sign of C. Indeed, in normalized coordinates $\bar{G}_n \geq G_n + eI > 0$ and $\bar{C}_n \geq C_n$. Equation (18) gives $\bar{C}_n \leq \beta\bar{G}_n^{-1}$. With $W_n = WU_\sigma$,

$$W_n C_n W_n^* \leq \beta W_n \bar{G}_n^{-1} W_n^* \leq \beta I,$$

since $W_n^* W_n = G_n \leq \bar{G}_n$. This proves (16) even though C_n is indefinite.

Quantity	even	odd
Normalized C source error	$< 1.557 \cdot 10^{-10}$	$< 1.557 \cdot 10^{-10}$
Normalized G source error	$< 6.889 \cdot 10^{-11}$	$< 6.847 \cdot 10^{-11}$
Analytic C allowance	10^{-12}	10^{-12}
Analytic G allowance	10^{-20}	10^{-20}
Minimum congruence row margin	> 0.99998	> 0.99997

These are directed bounds, rather than eigenvalue estimates of a floating matrix. The portable verifier regenerates every primitive, every quadrature sum and both congruences; no stored matrix arrays are required as inputs. NumPy proposes dyadic centers and witnesses only. The complete sums and final inequalities are re-enclosed with Arb, so no accuracy hypothesis about floating matrix multiplication is used.

Proof of Theorem 6.2. Apply (10) to $v = Ph$ and put $x = D_\alpha^{1/2} v$. Its ten exact row values vanish. The penalty in (14) therefore vanishes on x. Equations (15) and (16) imply

$$q_H[v] + (\alpha + \delta)\|v\|_2^2 \geq \kappa \|D_\alpha^{1/2} v\|_2^2, \qquad \kappa = 1 - \beta - \Gamma = \frac{983737}{2 \cdot 10^{12}} > 0.$$

Hence $q_H[v] \geq -\nu\|v\|_2^2 + \lambda\|sv\|_2^2$, with $\nu = \alpha + \delta - \alpha\kappa$ and $\lambda = \mu\kappa$ exactly as stated. Lemma 6.1 and the orthogonal identity $\|h\|_2^2 = \|Ph\|_2^2 + \|(I - P)h\|_2^2$ finish the proof. □

7 A quadratic complementary-space receiver

The certificate above bounds its complete phase-approximation error linearly. There is a different sufficient finite test in which the complementary-space payment is quadratic. This is the classical Schur-complement mechanism; the point here is to identify the actual matrices and whole-output bounds needed for its use with the known multiplication reference. We do not use this alternative to obtain the numerical constant of the executed certificate.

Fix a finite exact null space $Z \subset \ker H$, and impose its orthogonality conditions together with the two polar conditions. Write

$$D_0 = \alpha + \mu s^2, \qquad t = sD_0^{-1/2}, \qquad b(\xi) = \mathbf{1}_{[-L,L]}(\xi)(\mu - m_M(\xi)), \qquad K_0 = M_t b(D) M_t.$$

Thus K_0 is bounded and selfadjoint on $L^2(\mathbb{R})$. The complete arithmetic tail bound δ and the exterior floor $m_M \geq \mu$ give, with $\epsilon = \alpha + \delta$ and $x = D_0^{1/2} h$,

$$q_H[h] + \epsilon \|h\|^2 \geq \|x\|^2 - \langle x, K_0 x \rangle. \tag{19}$$

The rows defining the allowed x are $t\cosh(u/2)$, $t\sinh(u/2)$ and $D_0^{-1/2} z$, $z \in Z$. All are complete physical rows, including their exterior tails. Let $W : \mathbb{C}^n \longrightarrow L^2(\mathbb{R})$ be injective and contain these rows in its range. If the column map of the constraint rows is WJ, choose any positive semidefinite matrix Ω and set

$$\Lambda = J\Omega J^*, \qquad K = K_0 - W\Lambda W^*, \qquad G = W^*W > 0.$$

The penalty vanishes on every allowed x. In particular $K \leq I$ proves the desired original-domain lower bound after exact null lifting.

Proposition 7.1 (Whole-covariance receiver)**.** *Suppose F is selfadjoint,* $\operatorname{ran} F \subseteq \operatorname{ran} W$*, and $\|K_0 - F\| \leq \eta < 1$. Define the actual compression $S = W^* K_0 W$. For any matrix $V \in \mathbb{C}^{n \times n}$ put*

$$Z_V = K_0 W - WV.$$

If a Hermitian matrix $\overline{R}$ satisfies $\overline{R} \succeq Z_V^* Z_V$ on the complete $L^2(\mathbb{R})$ output, then

$$G - S + G\Lambda G - \frac{\overline{R}}{1-\eta} \succeq 0 \quad \Longrightarrow \quad K \leq I. \tag{20}$$

Proof. Put $U = WG^{-1/2}$, $\mathsf{P} = UU^*$ and $\mathsf{Q} = I - \mathsf{P}$. These are the isometry onto the receiving space and its orthogonal projections; P is distinct from the original null-complement projection P. Since $\mathsf{Q}F = 0$,

$$\mathsf{Q}K\mathsf{Q} = \mathsf{Q}K_0\mathsf{Q} \leq \eta\mathsf{Q}, \qquad A := U^*KU = G^{-1/2}(S - G\Lambda G)G^{-1/2}.$$

For the mixed block $E = \mathsf{Q}KU = \mathsf{Q}K_0WG^{-1/2}$ we have

$$E^*E \preceq G^{-1/2}\overline{R}G^{-1/2},$$

because $\mathsf{Q}Z_V = \mathsf{Q}K_0W$ and Q is a contraction. The complementary block of $I - K$ is at least $(1-\eta)I$, and hence is boundedly invertible. Completing the square in these two orthogonal spaces shows that $I - K \succeq 0$ follows from

$$I - A - E^*(I - \mathsf{Q}K\mathsf{Q})^{-1}E \succeq 0.$$

The inverse is bounded above by $(1-\eta)^{-1}I$; congruence by $G^{1/2}$ now gives (20). This retains the full mixed covariance and every complementary direction. □

Corollary 7.2. *Under the same approximation hypotheses, the sufficient test*

$$\left(1 - \frac{\eta^2}{1-\eta}\right) G - S + G\Lambda G \succeq 0 \tag{21}$$

does not require computing K_0^2.

Proof. Directly,

$$(\mathsf{Q}K_0W)^*(\mathsf{Q}K_0W) = (\mathsf{Q}(K_0 - F)W)^*(\mathsf{Q}(K_0 - F)W) \preceq \eta^2 G.$$

Use this upper bound for the mixed covariance in the preceding proof. It is a bound on the projected residual; it need not bound $Z_V^* Z_V$ for an arbitrary inaccurate V. □

All quantities in the proposition have explicit integral representations. If W_j is column j, put $r_j = tW_j$ and

$$v_j(u) = \frac{1}{2\pi} \int_{-L}^{L} b(\xi) \widehat{r}_j(\xi) e^{iu\xi} \, d\xi.$$

Then $K_0 W_j = tv_j$, and

$$\begin{aligned} G_{ij} &= \int_{\mathbb{R}} \overline{W_i(u)} W_j(u) \, du, \\ S_{ij} &= \frac{1}{2\pi} \int_{-L}^{L} b(\xi) \overline{\widehat{r}_i(\xi)} \widehat{r}_j(\xi) \, d\xi, \\ (Z_V^* Z_V)_{ij} &= \int_{\mathbb{R}} \overline{\left(tv_i - \sum_k W_k V_{ki}\right)} \left(tv_j - \sum_k W_k V_{kj}\right) \, du. \end{aligned}$$

There is no unknown resolvent in these formulas. Cauchy–Schwarz and Plancherel give

$$|v_j(u)| \le \|b\|_\infty \sqrt{L/\pi} \|r_j\|_2,$$

so the omitted squared norm of tv_j outside $[-R, R]$ is at most

$$\|b\|_\infty^2 (L/\pi) \|r_j\|_2^2 \int_{|u|>R} t(u)^2 \, du.$$

Here $t^2 \le s^2/\alpha$ has an explicit double-exponential tail. The derivative-null columns used here have derivative-dependent tails of the same integrable type. Other supplied exact-null rows require their own complete tail estimates. A complete covariance enclosure must also include the WV tail and all mixed terms, for example through a norm bound on the entire residual column map. Dropping signed exterior entries separately is not a Loewner upper bound.

An exact low-block margin γG can absorb a complementary debit $\eta^2 G/(1-\eta)$, allowing η of order $\sqrt{\gamma}$. This improvement is conditional on enclosing the *actual* S to the required margin. Replacing it by W^*FW with only the error ηG restores a linear debit. Neither compact approximation nor the receiver supplies the sign of its finite test uniformly as ϵ tends to zero.

Reproducibility

The supplementary package contains the editable manuscript and portable verification programs. The programs reconstruct the scalar bounds, weighted features, complete quadrature enclosures and two directed congruences from the formulas. Floating-point calculations propose centers and witnesses; acceptance uses complete directed enclosures. The reader instructions distinguish the executed finite certificate from the arbitrary-precision existence/search theorem.

Acknowledgment

AI assistance was used in mathematical development, literature comparison, manuscript preparation and computational verification.

A Complete bounds for the executed certificate

This appendix supplies the analytic inputs to the finite computation. The two supplementary programs have different tasks: `verify_bounds.py` evaluates the scalar endpoint inequalities below; `verify_certificate.py` reconstructs the complete finite signed matrices and their directed acceptance witnesses. Neither program substitutes for the original-domain identities proved in the text.

A.1 Theta envelopes and the complete paired arithmetic remainder

For $u \geq 0$ write $r_1 = \pi e^{2u}$. Directly from the theta series,

$$0 < \psi(u) \leq 8\pi^2 e^{9u/2-\pi e^{2u}}, \qquad s(u)^2 \leq 16\pi^2 e^{4u-\pi e^{2u}}. \tag{22}$$

For completeness, after removing the first exponential the required series is bounded by $\sum_{k\geq 1} k^4 q^{k-1} = (1+11q+11q^2+q^3)/(1-q)^5 < 2$, where $q = e^{-3\pi} < 1/100$; use $k^2-1 \geq 3(k-1)$. Evenness treats the other tail. The known log-concavity of the Riemann theta kernel [9, Theorems 4.2 and 4.5], with the scaling in (3), and convexity of $\log c$, make s even and log-concave.

If $U = M_s S_b M_s$, $b > 0$, set $S = s(0)^2$, $P_b = s(b/2)^2$ and $Q_b = s(b)^2$. Midpoint concavity gives $\|U\| = P_b$. The logarithm of $s(u)s(u+b)^2 s(u+2b)$ is symmetric and concave about $-b$, so $\|U^2\| = SQ_b$. Moreover $UU^* + U^*U$ multiplies by $s(u)^2(s(u+b)^2 + s(u-b)^2)$. On $u \geq 0$ the first summand is at most SQ_b and the second at most P_b^2; reflection treats $u \leq 0$. Expanding $(U+U^*)^2$ therefore proves the full paired estimate

$$\|M_s(S_b + S_{-b})M_s\| \leq \sqrt{P_b^2 + 3SQ_b}. \tag{23}$$

No member of a pair is dropped. Compression to $\mathcal{K}$ can only lower its norm.

For a coarse complete tail, $c_n \leq 1$ and (22) give a paired triangle bound $32\pi^2 n^2 e^{-\pi n}$. For $k \geq 2$, the exact geometric sum yields $\sum_{n\geq k} n^2 e^{-\pi n} < 2k^2 20^{-k}$. Since $32\pi^2 < 320$, this proves (6), including all unselected prime powers. For the sharper $M = 7$ tail, retain the remaining nonzero indices through 16 explicitly:

$$\delta_7 \leq \frac{640 \cdot 17^2}{20^{17}} + \sum_{(n,p)\in\{(8,2),(9,3),(11,11),(13,13),(16,2)\}} \frac{\log p}{\sqrt{n}} \sqrt{s(\tfrac{1}{2}\log n)^4 + 3s(0)^2 s(\log n)^2} < \frac{114}{10^{10}}. \tag{24}$$

The last inequality is evaluated with directed balls. Each theta value is the sum of its first eight atoms plus an absolute error 2^{-250}. To justify that error uniformly on the real half-line, the omitted atoms are bounded by a constant times $k^4 e^{-3k^2}$; monotonicity of $r^2 e^{-(k^2-1)r}$ for $r \geq 3$, $k \geq 9$, gives, more explicitly, $20 \cdot 243^2 (3/8)^{240}$ for the scaled tail introduced below. Multiplying back by $e^{u/2-\pi e^{2u}} \leq 1$ makes this less than 2^{-250}. Thus (24) is a complete infinite-tail bound, not a finite prime sum.

A.2 The whole exterior-frequency floor and the band norm

The symbol used in the calculation is exactly

$$m_7(\xi) = \Re\Psi_0(1/4 + i\xi/2) - \log\pi - 2 \sum_{n\in\{2,3,4,5,7\}} c_n \cos(\xi \log n).$$

The scalar checker covers $[2400, 3000]$ by the 60000 closed intervals $[2400 + k/100, 2400 + (k+1)/100]$. On each whole interval Arb evaluates the complex digamma and the cosine terms and proves $m_7 > 1/5$. The minimum certified lower endpoint exceeds 0.23053. With $d_7 = |m_0| + 2\sum_{n\le 7} c_n < 12$, it also proves $a(3000) - d_7 > 0.31602 > 1/5$. Monotonicity of a then treats the entire half-line $[3000, \infty)$; evenness treats the negative half-line. This pays the exterior floor used in (10).

An elementary upper bound from (4) is

$$a(L) \le 4 + \tfrac{1}{2}\log(1 + 4L^2) < 13 \quad (L = 2400). \tag{25}$$

Indeed the $j = 0$ term is at most 4; the remaining decreasing terms are bounded by $\int_{1/2}^{\infty} y^2/[x(x^2+y^2)]\,dx$ with $y = L/2$, which is already no greater than the logarithmic allowance displayed. The checker verifies the final logarithmic inequality by a positive partial sum for e^{18}. It follows that the complete signed band symbol satisfies

$$\|b\|_\infty < \tfrac{1}{5} + 13 + 12 < 26. \tag{26}$$

A.3 All-degree local approximation and spatial tails

The Fourier transform of (13) is

$$\widehat{\phi}_{aj}(\xi) = e^{-ic_a\xi}(-i)^j\sqrt{2r(2j+1)}\,\mathrm{j}_j(r\xi),$$

where j_j is the spherical Bessel function. Rodrigues' formula followed by j integrations by parts and the beta integral gives, for real x, $|\mathrm{j}_j(x)| \le |x|^j/(2j+1)!!$. Let F_L be Fourier restriction to $[-L, L]$, with its Plancherel measure, and $\Pi = YY^*$. Summing the squared Fourier tails over every cell gives

$$e_N^2 := \|F_L(1_{[-R,R]} - \Pi)\|_{\mathrm{HS}}^2 \le \frac{2RL}{\pi}\frac{(rL)^{2p}}{((2p+1)!!)^2(1-q)}, \qquad q = \frac{(rL)^2}{(2p+1)(2p+3)} = \frac{300}{4961} < 1. \tag{27}$$

The factor $2j+1$ in the column normalization cancels the integral of ξ^{2j}; subsequent terms have ratio at most q. Thus this estimate pays all degrees $j \ge 60$, not just a chosen numerical tail. Here $rL = 30$, and rational evaluation with $\pi > 3$ proves $e_N < 3 \cdot 10^{-11}$.

We have $\|t\|_\infty \le \mu^{-1/2} = \sqrt{5} < 9/4 =: j$. On $|u| \ge R$, (22) and $\pi > 3$ imply

$$j_R^2 := \|1_{|u|>R}t\|_\infty^2 < \frac{169}{\alpha}e^{4R-3e^{2R}} < \frac{169}{\alpha}(3/8)^{80} < (2\cdot 10^{-12})^2.$$

The checker proves $e^{17/5} > 29$ by a finite positive Taylor sum; this makes $4R - 3e^{2R} < -80$. The two maps $T = F_L M_t$ and $T_0 = F_L \Pi M_t$ therefore satisfy

$$\|T\|, \|T_0\| \le j, \qquad \|T - T_0\| \le j_R + je_N < \eta := \frac{139}{2\cdot 10^{12}}.$$

Since $K = T^* M_b T$ and its finite phase approximation is $T_0^* M_b T_0$, expanding the difference, without replacing b by its positive part, gives

$$\|K - T_0^* M_b T_0\| \le 2\|b\|_\infty j\eta < \frac{16263}{2\cdot 10^{12}} = \Gamma.$$

A.4 An explicit Gaussian quadrature estimate

If an integrand extends holomorphically to the Bernstein ellipse of parameter $\rho > 1$ over a cell of half-width r, and has modulus at most M there, its n-point Gauss–Legendre error is bounded by

$$E(M, n, r, \rho) = \frac{8rM\rho^{-2n}}{1 - \rho^{-1}}. \tag{28}$$

Indeed truncate the Chebyshev series at degree $2n-1$. Its coefficient bound $2M\rho^{-k}$ gives uniform remainder at most $2M\rho^{-2n}/(1-\rho^{-1})$. Exactness on that polynomial and positivity of the weights, whose sum is 2, bound the integral and rule errors by four times that remainder; rescaling gives (28).

A.5 Frequency quadrature for every phase block

Use 24 Gauss points on each of the 16 panels of width $1/4$ covering $[0,4]$ and the 1198 panels of width 2 covering $[4,2400]$. There are 29136 positive-frequency nodes. The multiplier is continued holomorphically as

$$b(z)=\mu+\log\pi-\tfrac{1}{2}\{\Psi_0(1/4+iz/2)+\Psi_0(1/4-iz/2)\}+2\sum_{n\le 7}c_n\cos(z\log n).$$

The real part operation and the band indicator are not continued. The latter is dealt with by integrating only the panels. For ellipse parameter $\rho=4$, the small panels have $|\Im z|\le 15/64$; both digamma arguments have positive real part. On the large panels $\Re z\ge 23/8$, $|\Im z|\le 15/8$ and $|z|<2403$. One digamma recurrence moves each argument to positive real part; the series estimate $|\Psi_0(w)|\le 1+1/\Re w+2|w|$ there, together with the recurrence term (whose argument stays away from zero), gives $|a(z)|<2500$. Since $c_n<1$ and $|\cos(z\log n)|\le e^{(15/8)\log 7}<49$, we have $|b(z)|<4000$.

Let $A_j(z)=\sqrt{2r(2j+1)}\,\mathrm{j}_j(rz)$. Its Legendre integral representation and $|P_j|\le 1$ on $[-1,1]$ give $|A_j(z)|\le\sqrt{r(2j+1)/2}\,2e^{r|\Im z|}<3$. Every cell lag $|d|\le 135$ has phase factor bounded by $e^{2r|d||\Im z|}<e^7<2187$. Consequently each scalar block integrand is bounded by $4000\cdot 9\cdot 2187<10^8$ on all these ellipses. Summing (28) over the panels, conservatively omitting the beneficial factor $1/\pi$, gives the per-entry error

$$E_f=\frac{16}{3}\,2400\cdot 10^8\,4^{-48}.$$

The normalized integer parity fold has row error at most $2\cdot 4080E_f<10^{-12}$. This is the entire analytic C allowance in (17).

A.6 A zero-free complex theta strip and derivative quotients

The physical Gram calculation requires complete global rows, including the derivative quotients, not only point values of ψ. Here are uniform complex bounds giving its quadrature remainder. For $z=x+iy$, $x\ge 0$, $|y|\le 1/100$, put $r_1=\pi e^{2z}$ and $\psi_1(z)=e^{z/2}(4r_1^2-6r_1)e^{-r_1}$. Then $\Re r_1>2.99$, and direct comparison of the theta atoms gives

$$|\psi_k(z)/\psi_1(z)|\le 3k^4e^{-\Re r_1(k^2-1)},\qquad \sum_{k\ge 2}|\psi_k/\psi_1|<1/10.$$

The second inequality follows by separating $k=2$ and summing the geometrically decreasing remainder; it is uniform for $\Re r_1\ge 2.99$. In particular ψ is zero-free on this strip and

$$|\psi(z)|\ge 1.8\pi^2e^{(9/2)x-\pi e^{2x}\cos(2y)},\qquad |\psi(z)|\le 120e^{(9/2)x-\pi e^{2x}\cos(2y)}. \tag{29}$$

Evenness treats negative x.

On the smaller strip $|y|\le h=1/200$ the denominator $d(z)=\alpha\cosh(z/2)+\mu\psi(z)$ stays in a right half-plane. For $0\le x\le 1.2$, the unwrapped phase of the first atom is bounded by $(13/2)h+2\pi e^{2.4}h<1$. Thus $\Re\psi_1>\frac{1}{2}|\psi_1|$, and the higher atoms have combined modulus below $|\psi_1|/10$, giving $\Re\psi>0$. For $x\ge 1.2$, the upper envelope in (29) is decreasing and is

below $120 \cdot 729(3/8)^{32} < 2.047 \cdot 10^{-9}$: use $e^{2.4} > 11$, $e^{5.4} < 729$ and $\pi e^{2.4}\cos(1/100) > 32$. Since $\Re\cosh(z/2) \geq \cos(1/400)$ and $\alpha = 10^{-8}$, both cases give

$$\Re d(z) > \alpha/2, \qquad |W_t(z)| := |\psi(z)/d(z)| < 4 \cdot 10^9. \tag{30}$$

For the last estimate the same envelope has maximum less than 17.

We need Cauchy estimates uniformly up to $|\Re z| \leq 2.2$. For $|\Im z| \leq 1/200$ and $|\zeta| \leq 1/200$, division of the upper and lower envelopes yields

$$\left|\frac{\psi(z+\zeta)}{\psi(z)}\right| \leq \frac{120}{1.8\pi^2} e^{9/400} \exp\{\pi e^{4.4}[1 - e^{-1/100}\cos(1/50)]\} < 512.$$

This coarse bound also holds if the circle crosses $\Re z = 0$, by evenness. For example $e^{4.4} < 82$, $e^{-1/100} \geq 99/100$, $\cos(1/50) \geq 4999/5000$ bound the last exponent by 3; the remaining prefactor is less than 16, and $e^3 < 27$ suffices. Cauchy's inequality now gives

$$|q_j(z)/\psi(z)| \leq K_j := 1024(j+2)!200^{j+2} < 10^{30}, \qquad 0 \leq j \leq 7. \tag{31}$$

The scalar checker evaluates the numerical endpoints in these deliberately loose bounds with directed balls or exact rational arithmetic.

A.7 Physical Gram quadrature, with every mixed tail

On each physical cell use 400 Gauss points and ellipse parameter $\rho = 7/5$. Its imaginary radius is $3/700 < 1/200$ and its real extension keeps the row integration region inside $|\Re z| < 2.2$. The local Legendre bound $|P_j(z)| \leq \rho^j$ on an ellipse gives $|\phi_{aj}| \leq \sqrt{4760}\rho^{59} < 10^{12}$. Also $|c|, |\sinh(z/2)| < 3$. Equations (30)–(31) show that *every raw integrand* for the phase–phase, phase–polar, phase–null, polar–polar, polar–null and null–null Gram entries has modulus below 10^{80}. For example the last has the form $W_t c^2(q_i/\psi)(q_j/\psi)$. Thus (28) gives a per-cell, per-entry error

$$E_g = \frac{8r\,10^{80}(7/5)^{-800}}{1 - 5/7} < 10^{-36}.$$

Including all cells, folds and row sums costs less than $10^6 E_g < 10^{-30}$ in the raw normalized operator norm. The actual dyadic row changes have $\|A_\sigma\|^2 \leq \|A_\sigma\|_{\mathrm{F}}^2 < 100$. Applying the block change of coordinates to this error multiplies it by at most 100; mixed entries are included.

For row–row entries only, integrate additionally over $1.7 < |u| \leq 2.1$, using 16 more cells on each side with the same rule. These cells add no phase columns. We next bound the remaining complete row tails; compactly truncating the rows would change the recipient. Define polynomials

$$P_0(v) = 4v^2 - 6v, \qquad \mathcal{L}P = (1/2 - 2v)P + 2vP', \qquad Q_j = \mathcal{L}^{j+2}P_0 - \tfrac{1}{4}\mathcal{L}^j P_0.$$

Then $\deg Q_j = j + 4$ and $q_j(u) = e^{u/2}\sum_{k\geq 1} Q_j(k^2 r_1)e^{-k^2 r_1}$ for $u \geq 0$. If $Q_j(v) = \sum_l d_{jl}v^l$, put $A_j = \sum_l |d_{jl}|200^{l-j-4}$, an exact positive rational. For $u \geq 2.1$, $r_1 = \pi e^{2u} > 200$. Since $\sum_{k\geq 1} k^{2j+8}e^{-r_1(k^2-1)} < 2$ and $\psi(u) \geq 2e^{u/2}r_1^2 e^{-r_1}$, we obtain

$$\frac{|q_j(u)|^2}{\psi(u)} \leq 2A_j^2 e^{u/2} r_1^{2j+6} e^{-r_1}.$$

The squared null row is $q_j^2/(s^2 D_\alpha)$, bounded by $q_j^2 c/(\alpha\psi)$. Changing variables $r_1 = \pi e^{2u}$, using $c(u) \leq e^{u/2}$ and adding both tails gives the convenient bound

$$T_j := \int_{|u|>2.1} \left|\frac{q_j}{s\sqrt{D_\alpha}}\right|^2 du \leq \frac{3A_j^2}{\alpha}\Gamma(2j+7, 200). \tag{32}$$

Both polar tails are at most $T_{\mathrm{pol}} := (8/\alpha)\Gamma(3, 200)$. The integer incomplete gamma values are evaluated by the exact identity

$$\Gamma(n+1, a) = n!e^{-a}\sum_{k=0}^{n}\frac{a^k}{k!}.$$

The largest raw null bound, at $j = 7$, is less than $3.547 \cdot 10^{-26}$; $T_{\mathrm{pol}} < 4.473 \cdot 10^{-74}$.

The omitted global-row Gram is positive semidefinite. Its trace before mixing is bounded by $\sum_{j=0}^{7} T_j + 2T_{\mathrm{pol}}$. After the actual row change its norm is at most $100(\sum_j T_j + 2T_{\mathrm{pol}}) < 3.547 \cdot 10^{-24}$. This pays all mixed tail entries by positivity, rather than dropping off-diagonal covariances. Combining it with quadrature gives

$$E_G < 100 \cdot 10^6 E_g + 100\left(\sum_j T_j + 2T_{\mathrm{pol}}\right) < 10^{-20}. \tag{33}$$

A.8 Scaled theta evaluation and complete finite arithmetic

At real physical nodes it is important not to divide two absolute balls whose centers are both exponentially tiny. The program instead evaluates the complete scaled functions

$$T(u) = e^{r_1 - u/2}\psi(u) = \sum_{k\geq 1} P_0(k^2 r_1)e^{-(k^2-1)r_1}, \qquad Q_j^{\mathrm{sc}}(u) = \sum_{k\geq 1} Q_j(k^2 r_1)e^{-(k^2-1)r_1}.$$

For a polynomial of degree $d \leq 11$ with coefficient absolute sum A, $r_1 \geq 3$ and $k \geq 9$, the function $r_1^d e^{-(k^2-1)r_1}$ is decreasing. Thus its omitted term is at most $A(3k^2)^d e^{-3(k^2-1)}$. Starting at $k = 9$, consecutive majorants have ratio at most $(10/9)^{22}e^{-57} < 1/2$. Consequently the tail after eight atoms is bounded by

$$2A\,243^d(3/8)^{240}. \tag{34}$$

For P_0 use $A = 10, d = 2$. For Q_j the coefficients are reconstructed by the exact polynomial recurrence above. The code verifies $T > 0$, recovers $\psi = e^{u/2-r_1}T$ and $q_j/\psi = Q_j^{\mathrm{sc}}/T$, and reflects odd quotients with the factor $(-1)^j$.

Each real physical feature is multiplied by $\sqrt{r w_k W_t(u_k)}$ and by the actual dyadic row mixture before conversion to binary64. This avoids large unscaled derivative units in the enclosing matrices. The square root is used only at real positive nodes; the analytic quadrature proof concerns the products displayed above and needs no holomorphic square-root choice.

For clarity we give the exact phase signs and center conventions. Let $B_d(j, k)$ denote the full real Fourier integral block with cell lag d. If $\omega_j = (-i)^j$, it is the sum over positive-frequency nodes of

$$\frac{w_i b(\xi_i)}{\pi}A_j(\xi_i)A_k(\xi_i)\{\Re(\overline{\omega}_j\omega_k)\cos(2rd\xi_i) - \Im(\overline{\omega}_j\omega_k)\sin(2rd\xi_i)\}.$$

Use $B_{-d} = B_d^T$. The exact integer parity-fold center is

$$C_{0\sigma}(aj, bk) = 2\{B_{a-b}(j, k) + \sigma(-1)^k B_{a+b-135}(j, k)\}, \quad 0 \leq a, b < 68. \tag{35}$$

Here the B_d entries in the center are exact saved binary64 dyadics; the displayed addition and multiplication are exact in the final target. They satisfy $JB_d^T J = B_d$, $J_{jj} = (-1)^j$, exactly, so the folded center is symmetric. The row coefficient block is exactly $-1000I_5$. A rounded full folded matrix is used only to propose the congruence witness, never as the final accepted target.

Arb at 192 bits encloses the Gaussian nodes, weights, all special functions and all complete scaled tails. Each conversion is individually checked against radius 2^{-52} for frequency amplitudes, weighted symbols, cosines and sines, and 2^{-51} for weighted physical features. The respective conversion counts are 1748160, 29136, 3962496, 3962496, and 1968000: in total 11670288 gates. The program then re-encloses *every quadrature sum* directly from these balls. It compares those sums with the exact dyadic centers, forms absolute row and column errors for each Toeplitz block, and rounds stored positive error bounds upward with a further Arb check. The normalized parity-fold source error combines the two relevant block errors in (35); its factor 2 cancels the phase metric normalization. For G, phase–phase errors divide by 2, phase–row errors by $\sqrt{2}$, and row–row errors have unit normalization. The complete additional row annuli are included before these comparisons. For symmetric normalized errors, the maximum absolute row sum bounds the operator norm. Adding (33) and the full frequency allowance proves (17).

Finally the program constructs the two exact rational targets (18). NumPy proposes a Cholesky-based matrix V_σ; its entries are interpreted as exact dyadics. Arb forms the entire congruence and checks all 4085 strict diagonal-dominance rows in each parity. The implication uses the certified residual and exact acceptance inequalities. The accepted minima exceed 0.99998 and 0.99997, respectively. The source errors are less than $1.557 \cdot 10^{-10}$ for C and $6.889 \cdot 10^{-11}$ for G; together with their analytic allowances they are strictly below the common 10^{-8} padding budget. This completes the finite acceptance proof and hence the full-domain inequality of Theorem 6.2.

A.9 Reconstruction scope

The public verifier is the same mathematical program used for the completed fresh executions in ordinary and optimized Python modes. Its exception-based gates remain active in both modes. The supplementary analytic checker evaluates 60012 scalar gates and 30 final global-row and frequency gates. The larger verifier reconstructs its own primitive arrays, dyadic centers and congruence witnesses in a user-selected output directory. Resume files are checked by source and array hashes; omitting `-resume` performs a fresh reconstruction. The small files in `data/` specify mathematical parameters and accepted inequalities. Generated witnesses may depend on the floating backend; their validity is decided solely by the directed inequalities, not by matching an expected floating eigenvalue or array hash.

B Exact zero witnesses and the cost of a fixed null prefix

This appendix compares a concrete zero detector with the original weighted norm. Physical division and qualitative negative detection are established ingredients of the zero-detection construction of van Hemert [6]. We give the weighted-domain argument and the fixed-prefix estimate required here. Constants are uniform in height for fixed multiplicity and fixed prefix length; they are not uniform when either grows.

Write $D = d/\,du$, $\mathcal{D} = D^2 - 1/4$, and use the bilateral Laplace transform

$$\mathcal{M}f(z) = \int_{\mathbb{R}} f(u)e^{zu}\,du, \qquad \mathcal{M}(Df)(z) = -z\mathcal{M}f(z).$$

Let $\Xi_c(z) = \xi(1/2 + z)$, $\mathcal{M}\psi = \Xi_c$, and

$$\mathscr{L}(z) = (z^2 - 1/4)\Xi_c(z), \qquad q_k = \mathcal{D}D^k\psi, \qquad \phi_k = q_k/s.$$

Thus $\mathcal{M}q_k(z) = (-z)^k\mathscr{L}(z)$. The exact null-domain argument gives linearly independent $\phi_k \in \operatorname{Dom} H$ with $H\phi_k = 0$. For $N \geq 0$ put

$$Z_N = \operatorname{span}\{\phi_0, \dots, \phi_{N-1}\}, \qquad \Pi_N = \text{orthogonal projection onto } Z_N.$$

The empty prefix means $Z_0 = \{0\}$. No completeness of this null family is assumed.

B.1 Canceled physical division and the minimal domain

Let σ be a centered nontrivial zero of exact multiplicity m. For a function g satisfying $\mathcal{M}g(\sigma) = 0$, define

$$R_\sigma g(u) = e^{-\sigma u} \int_u^\infty e^{\sigma v} g(v)\, dv = -e^{-\sigma u} \int_{-\infty}^u e^{\sigma v} g(v)\, dv. \tag{36}$$

The equality follows from the canceled moment. Direct differentiation and integration by parts give

$$(-D - \sigma) R_\sigma g = g, \qquad \mathcal{M}(R_\sigma g)(z) = \frac{\mathcal{M}g(z)}{z - \sigma}.$$

The sign is fixed by our Laplace convention. At every division before the mth, the remaining transform still vanishes at σ. Therefore

$$R_\sigma^m g(u) = \int_u^\infty \frac{(v-u)^{m-1}}{(m-1)!} e^{\sigma(v-u)} g(v)\, dv. \tag{37}$$

The left-tail representation has coefficient $(-1)^m$ and kernel $(u-v)^{m-1}/(m-1)!$; equality uses the first m canceled transform jets. Put

$$\eta_\sigma = R_\sigma^m \psi, \qquad f_\sigma = \mathcal{D}\eta_\sigma, \qquad h_\sigma = f_\sigma / s.$$

Commutation with derivatives is justified by the same canceled integral formulas. Hence

$$\mathcal{M}f_\sigma(z) = \frac{\mathscr{L}(z)}{(z-\sigma)^m}, \qquad A_\sigma := \mathcal{M}f_\sigma(\sigma) = (\sigma^2 - 1/4)\frac{\xi^{(m)}(1/2+\sigma)}{m!} \neq 0. \tag{38}$$

The two polar moments vanish. At every other centered zero the transform value is zero, with no truncation of the zero set.

We record the norm control behind this construction. For every fixed k, differentiation of the complete theta series gives computable constants $C_k, c > 0$ such that, for $u \geq 0$ and $x = \pi e^{2u}$,

$$|q_k(u)| \leq C_k e^{u/2} x^{k+4} e^{-x}, \qquad \psi(u) \geq c e^{u/2} x^2 e^{-x}. \tag{39}$$

For clarity, each theta atom is $e^{u/2}(4a^2 - 6a)e^{-a}$ with $a = \pi n^2 e^{2u}$. Each differentiation sends a polynomial p to $(1/2)p + 2ap' - 2ap$, so the degree after applying $\mathcal{D}D^k$ is at most $k+4$. Absolute coefficient sums and the convergent series $\sum_{n\geq 1} n^{2k+8} e^{-(n^2-1)\pi}$ give the first constant. The positive first atom, $4x^2 - 6x \geq (4 - 6/\pi)x^2$, gives the second. Reflection gives the corresponding bounds at $u \leq 0$.

If $r \leq m$, $|\Re\sigma| \leq 1/2$ and the necessary r cancellations hold for q_k, use the upper-tail primitive for $u \geq 0$. For $v = u + w$, $e^{2w} \geq 1 + 2w$ implies that its absolute integrand, apart from $w^{r-1}/(r-1)!$, is bounded by

$$C_k e^{u/2} x^{k+4} e^{-x} \exp\{-(2x - 2k - 9)w\}.$$

For $x \geq 2k + 9$ this is bounded by the displayed prefactor times e^{-xw}. Integration supplies an additional factor x^{-r}. The canceled left-tail formula gives the reflected estimate. On a fixed compact interval $[-U, U]$, an upper bound independent of $\Im\sigma$ is

$$\frac{e^{U/2}}{(r-1)!} \int_{\mathbb{R}} (|v| + U)^{r-1} e^{|v|/2} |q_k(v)|\, dv.$$

The positive minimum of s there is nonzero. Squaring the tail bounds, dividing by $s^2 = \psi/\cosh(u/2)$ and changing variable to x leaves an incomplete-Gamma integral with a fixed polynomial weight. Consequently

$$\|R_\sigma^r q_k/s\|_2 \le C_{k,r} \tag{40}$$

with a finite computable constant, uniform in height at the zeros where the stated cancellations hold. The same proof applies to any fixed number of derivatives. In particular $h_\sigma \in L^2(\mathbb{R})$.

To obtain membership in the *minimal* form domain, take a smooth cutoff χ_R equal to one on $[-R, R]$, supported in $[-R-1, R+1]$, with uniformly bounded derivatives, and set $f_R = \mathcal{D}(\chi_R \eta_\sigma)$. Both polar moments of f_R vanish exactly. The preceding bounds show that $f_R/s \to h_\sigma$ in L^2 and that $f_R \to f_\sigma$ in every exponentially weighted derivative seminorm. The complete Weil formula is continuous for this convergence: the Archimedean singularity is bounded by a second derivative seminorm, and its remaining integrals and prime shifts by the exponential seminorms. Equivalently on the zero side, integration by parts gives arbitrary power decay uniformly in the critical strip, which is summable against the zero-count bound. On the physical side, exponential seminorms of order A bound a shift correlation by a constant times $e^{-A|\log n|/2}$; choosing $A > 2$ makes the complete prime sum absolutely summable. Finally, for $h_R = f_R/s$, the core identity gives

$$q_A[h_R - h_S] = Q[f_R - f_S] - \langle h_R - h_S, B(h_R - h_S)\rangle \longrightarrow 0,$$

because B is bounded. This proves convergence in the original form norm. Thus h_σ belongs to the same minimal domain and $q_H[h_\sigma] = Q[f_\sigma]$. Mixed pairings follow by polarization.

B.2 The exact negative pair

Suppose $\sigma = \delta + iT$ with $\delta > 0$, and put $\tau = -\overline{\sigma}$. The functional equation and conjugation give a distinct zero τ of the same multiplicity m. On the pole-null tests above the full sesquilinear Weil formula is

$$Q(f, g) = \sum_w m_w \mathcal{M}g(w)\overline{\mathcal{M}f(-\overline{w})}. \tag{41}$$

The series is absolutely convergent by the derivative bounds just proved. Let $e_\sigma = f_\sigma/A_\sigma$ and $e_\tau = f_\tau/A_\tau$. Their only nonzero zero-evaluation coordinates are respectively one at σ and one at τ. Thus

$$Q[e_\sigma] = Q[e_\tau] = 0, \qquad Q(e_\sigma, e_\tau) = m, \qquad Q[e_\sigma - e_\tau] = -2m. \tag{42}$$

Multiplicity contributes the scalar weight m, not m independent evaluation coordinates. Since $\mathscr{L}$ is real and even, $A_\tau = (-1)^m \overline{A_\sigma}$. In the unnormalized pair the off-diagonal entry has magnitude $m|A_\sigma|^2$.

Every vector in Z_N is an exact form null vector. The least original Hilbert norm after subtracting this prefix is therefore attained by

$$r_\sigma = (I - \Pi_N)h_\sigma, \qquad r_\tau = (I - \Pi_N)h_\tau.$$

For instance the negative source in (42) has optimal squared cost $\|r_\sigma/A_\sigma - r_\tau/A_\tau\|_2^2$. An unweighted physical-source norm cannot replace this quotient norm.

B.3 A lower bound on the quotient Gram matrix

Lemma B.1. *Fix integers $m \ge 1$, $N \ge 0$ and $\delta_* > 0$. There exist computable $b_{m,N} > 0$ and T_0 such that, whenever $\sigma = \delta + iT$ is a zero of exact multiplicity m, $\delta_* \le \delta < 1/2$ and $T \ge T_0$,*

$$\mathrm{Gram}(r_\sigma, r_\tau) \succeq b_{m,N}\delta^2 T^{-2m-2N-4} I_2. \tag{43}$$

The constants are for fixed m, N.

Proof. Repeated integration by parts in (37) gives an exact finite identity. For $J = m + K$,

$$R_\sigma^m q_0 = \sum_{k=0}^{K} c_{m,k} \frac{q_k}{\sigma^{m+k}} + \frac{(-1)^J}{\sigma^J} \sum_{a=0}^{m-1} \binom{J}{a} R_\sigma^{m-a} q_{J-a}, \tag{44}$$

$$c_{m,k} = (-1)^{m+k} \binom{m+k-1}{k}.$$

For example, this follows by differentiating the kernel $w^{m-1} q_0(u+w)/(m-1)!$ J times and accounting for its endpoint jets. Each q_{J-a} has the required transform cancellations. Dividing by s and using (40) proves a Hilbert-norm expansion, not merely a pointwise asymptotic:

$$h_\sigma = \sum_{k=0}^{K-1} c_{m,k} \phi_k \sigma^{-m-k} + O_{m,K}(|\sigma|^{-m-K}).$$

The constants are uniform for $|\Re\sigma| \le 1/2$ among roots of multiplicity m. Set $v_k = (I - \Pi_N)\phi_k$ and use $K = N + 4$ to obtain

$$r_\sigma = \sum_{k=N}^{N+3} c_{m,k} v_k \sigma^{-m-k} + O_{m,N}(|\sigma|^{-m-N-4}). \tag{45}$$

The four vectors $v_N, \dots, v_{N+3}$ are linearly independent, since a dependence would give one among $\phi_0, \dots, \phi_{N+3}$. Their positive Gram matrix gives bounded functionals ℓ_0, ℓ_1 which select respectively v_N, v_{N+1} and annihilate the other members of this four-vector list. Their norms can be bounded by enclosing that finite positive Gram matrix.

Put $a = m + N$. Applying (ℓ_0, ℓ_1) to the two columns in (45) gives

$$M_T = \begin{pmatrix} c_{m,N}\sigma^{-a} & c_{m,N}\tau^{-a} \\ c_{m,N+1}\sigma^{-a-1} & c_{m,N+1}\tau^{-a-1} \end{pmatrix} + O_{m,N}(|\sigma|^{-a-4}).$$

The determinant of the displayed matrix is exactly

$$c_{m,N} c_{m,N+1} \frac{\sigma - \tau}{\sigma^{a+1}\tau^{a+1}}.$$

Since $|\sigma| = |\tau|$ and $\sigma - \tau = 2\delta$, its magnitude is $2\delta |c_{m,N} c_{m,N+1}| |\sigma|^{-2a-2}$, while the largest singular value is at most $C_{m,N}|\sigma|^{-a}$. The least singular value is consequently at least $d_{m,N}\delta|\sigma|^{-a-2}$. For $\delta \ge \delta_*$ the error term is smaller than half this bound when T exceeds an effective threshold. Divide by the norm of the extracting map (ℓ_0, ℓ_1) and square. Since $|\sigma|$ and T are uniformly comparable for $T \ge 2$, this proves (43). The proof uses canceled primitives only at zeros; it does not assume weighted analyticity in σ between the two zeros. □

B.4 The fixed-prefix height barrier

Proposition B.2. *Under the fixed-m, N, δ_* hypotheses of Lemma B.1, every negative Rayleigh quotient in the two-detector span, after the best subtraction of Z_N, has magnitude at most*

$$C_{m,N,\delta_*}\, T^{2m+2N+14} e^{-\pi T/2}. \tag{46}$$

In particular this magnitude tends to zero as $T \longrightarrow \infty$.

Proof. At a zero $\rho = \beta + iT$ of exact multiplicity m, $\xi^{(m)}(\rho) = C(\rho)\zeta^{(m)}(\rho)$, where

$$C(z) = \tfrac{1}{2} z(z-1)\pi^{-z/2}\Gamma(z/2).$$

For $\beta > 0$ the continued Euler integral is

$$\zeta(z) = \frac{z}{z-1} - z\int_1^\infty \{x\}x^{-z-1}\,dx.$$

Differentiating m times under the absolutely convergent integral, and using

$$\int_1^\infty (\log x)^j x^{-\beta-1}\,dx = \frac{j!}{\beta^{j+1}},$$

gives, for $T \geq 2$,

$$\frac{|\zeta^{(m)}(\beta+iT)|}{m!} \leq T^{-m-1} + |\beta+iT|\beta^{-m-1} + \beta^{-m}.$$

For our right-hand zero $\beta = 1/2+\delta \in [1/2, 1]$, this is at most $C_m T$. The uniform vertical-strip Stirling bound [16, §5.11] yields

$$|\Gamma(\beta/2+iT/2)| \leq CT^{\beta/2-1/2}e^{-\pi T/4} \leq Ce^{-\pi T/4}.$$

The completed-ξ factor contributes $O(T^2)$ and the extra annihilator factor $\sigma^2-1/4$ contributes another $O(T^2)$. Consequently

$$|A_\sigma| \leq C_m T^5 e^{-\pi T/4}, \qquad |A_\tau| = |A_\sigma|.$$

The exact unnormalized Weil matrix on the pair has operator norm $m|A_\sigma|^2$. Its quotient norm matrix is bounded below by (43). For any coefficient vector, divide the absolute quadratic-form upper by this norm lower. Absorbing $\delta^{-2} \leq \delta_*^{-2}$ gives (46). □

Thus this particular two-detector construction, with any fixed derivative prefix, supplies no positive height-uniform negative depth against which a fixed deficit could be contradicted. A growing-prefix argument requires uniform bounds on its derivative Gram matrices and primitive constants. The estimate for fixed m, N cannot be used with $N = N(T)$ or $m = m(T)$ without those bounds.

References

[1] S. Schubr, *An explicit global lower certificate for a weighted Weil operator*, Luminary Research Archive, edition 1, 2026. https://research.luminaryoddities.com/explicit-global-lower-certificate-weighted-weil-operator/v/1.

[2] S. Schubr, *Quantitative Null-Space Compression for a Weighted Weil Operator*, Luminary Research Archive, edition 1, 2026. https://research.luminaryoddities.com/quantitative-null-space-compression-weighted-weil-operator/v/1.

[3] S. Schubr, *Joint Cutoffs and Negative-Spectrum Decay for Weighted Weil Operators*, Luminary Research Archive, edition 2, 2026. https://research.luminaryoddities.com/joint-cutoffs-negative-spectrum-decay-weighted-weil-operators/v/2.

[4] S. Schubr, *Frequency Localization for Weil Operator Certificates*, Luminary Research Archive, edition 2, 2026. https://research.luminaryoddities.com/frequency-localization-weil-operator-certificates/v/2.

[5] S. Schubr, *Arithmetic cutoff bounds and sharp tail capacity for local Nyman–Beurling certificates*, Luminary Research Archive, edition 1, 2026. https://research.luminaryoddities.com/arithmetic-cutoff-bounds-sharp-tail-capacity-local-nyman-beurling-certificates/v/1.

[6] L. van Hemert, *Exact Zero Detection and Compact Arithmetic Realizations for the Riemann Xi Function*, author manuscript, October 11, 2026. https://github.com/LeonardSEO/riemann-xi-zero-detection.

[7] A. Weil, *Sur les "formules explicites" de la théorie des nombres premiers*, Comm. Sém. Math. Univ. Lund, Tome supplémentaire (1952),252–265.

[8] A. Connes and C. Consani, *Weil positivity and Trace formula, the archimedean place*, manuscript, July 4, 2021, Appendix C. https://alainconnes.org/wp-content/uploads/Selecta.pdf.

[9] G. Csordas, *Fourier transforms of positive definite kernels and the Riemann ξ-function*, arXiv:1309.0055v2 (2014). https://arxiv.org/abs/1309.0055v2.

[10] T. Kato, *Perturbation Theory for Linear Operators*, second edition, Springer, 1980.

[11] G. Dusson, I. Sigal and B. Stamm, *The Feshbach–Schur map and perturbation theory*, arXiv:2105.02058 (2021). https://arxiv.org/abs/2105.02058.

[12] M. J. Colbrook and A. C. Hansen, *The foundations of spectral computations via the Solvability Complexity Index hierarchy*, arXiv:1908.09592v4 (2022). https://arxiv.org/abs/1908.09592v4.

[13] V. Brattka and R. Dillhage, *Computability of the Spectrum of Self-Adjoint Operators*, J. Universal Computer Science 11 (2005),1884–1900. https://www.jucs.org/jucs_11_12/computability_of_the_spectrum/jucs_11_12_1884_1900_brattka.pdf.

[14] X. Zhu, *Weil positivity in compact windows: a finite reduction, certified two-sided bounds, and a Landau–Widom decay law*, arXiv:2608.24827v2 (2026). https://arxiv.org/abs/2608.24827v2.

[15] AMRAL, *RH-W-04: Finite-Dimensional Completeness and Rational Negative Certificate*, W Engineering Package v0.1, July 23, 2026. https://amral.evemisslab.com/en/riemann/autonomous/p/w04-v0.1/.

[16] F. W. J. Olver et al., editors, *NIST Digital Library of Mathematical Functions*, Chapters 5, 10, 14 and 18. https://dlmf.nist.gov/.

[17] L. N. Trefethen, *Approximation Theory and Approximation Practice*, SIAM, 2013.

[18] F. Johansson, *Arb: efficient arbitrary-precision midpoint-radius interval arithmetic*, IEEE Trans. Comput. 66 (2017),1281–1292. https://fredrikj.net/arb/.

[19] The python-flint developers, *python-flint 0.9.0 documentation*. https://python-flint.readthedocs.io/.

[20] OpenAI, *The Quasi-Riemann Hypothesis: A Zero-Free Half-Plane* $\operatorname{Re} s > 7/8$, September 30, 2026. https://github.com/openai/math/blob/main/preprints/The-Quasi-Riemann-Hypothesis-September-30-2026/paper.pdf.

[21] B. Liu, *Improved zero-free half-planes for the quasi-Riemann hypothesis*, arXiv:2610.12234v2 (2026). https://arxiv.org/abs/2610.12234v2.