Fekete's lemma in Banach spaces needs unbounded pair ranges

Scott Schubr
Luminary Oddities

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Abstract

Kulikov and Shao proved that if vectors v_n in a uniformly convex Banach space satisfy $\|v_{n+m}\| \le \|v_n + v_m\|$ for all n and m, then v_n/n converges. In Question 3.2 of their 2024 paper they asked whether, for some function f, the inequality for the pairs $n \le m \le f(n)$ forces convergence. We answer this question in the negative for every uniformly convex space of dimension at least two. For every normed space of dimension at least two and every function f there is a sequence satisfying the inequality on these pairs, with $\|v_n\|/n$ tending to 1, whose terms v_n/n accumulate exactly on a nondegenerate arc of the unit sphere of a plane, and on the whole circle in a Euclidean plane. With their Proposition 3.1 for dimension one, the answer is positive exactly in real dimension at most one. The construction keeps the direction constant on long blocks, so that no admissible pair skips a block, and pays each small turn between consecutive blocks with a summable decrease of the norm ratio. Conversely, in uniformly convex Banach spaces the argument of Kulikov and Shao gives convergence from comparable pairs together with infinitely many indices whose partner sets are cofinite. Hence, in every uniformly convex Banach space of dimension at least two, for the corridor $n \le m \le f(n)$ with $f(n) \ge 2n$, enlarged by all pairs whose smaller index lies in a set S, the vector Fekete lemma holds exactly when S is infinite.

Keywords. Fekete lemma; subadditive sequences; uniformly convex Banach spaces; restricted pairs; de Bruijn–Erdős lemma; normed planes.
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1 Introduction

1.1 The question

Throughout, $\mathbb{N} = \{1, 2, 3, \dots\}$, and all normed spaces are real; a complex space is regarded as a real space of twice its dimension. Fekete's lemma [2] states that a real sequence with $a_{n+m} \le a_n + a_m$ for all $n, m \in \mathbb{N}$ satisfies $a_n/n \to \inf_k a_k/k$, where the limit may be $-\infty$. De Bruijn and Erdős [1] showed that the pairs with $\frac{1}{2}n \le m \le 2n$ suffice, and Füredi and Ruzsa [3, Theorem 2] showed that the pairs $N \le n \le m \le \kappa n$ suffice for any fixed $\kappa > 1$ and $N \in \mathbb{N}$.

Kulikov and Shao [4] studied the vector form of the hypothesis. A sequence (v_n) in a normed space $(X, \|\cdot\|)$ satisfies

$$\|v_{n+m}\| \le \|v_n + v_m\|. \tag{1}$$

By the triangle inequality, (1) makes $(\|v_n\|)$ subadditive on the same pairs, so the scalar results above control the norms. The question is whether the vectors v_n/n converge. Kulikov and Shao proved that they do when (1) holds for all $n, m \in \mathbb{N}$ and X is a uniformly convex Banach space [4, Theorem 1.1], following a MathOverflow discussion [7]. In dimension one the de Bruijn–Erdős

pairs suffice [4, Proposition 3.1]. In the Euclidean plane they do not: a sequence whose direction turns slowly and continuously in $\ln n$ satisfies (1) for $\frac{1}{2}n \le m \le 2n$ without a limit of v_n/n, and they indicate that the same method reaches the pairs $n \le m \le n\exp((\ln n)^{1-\varepsilon})$ for every $\varepsilon > 0$ [4, Section 3]. They then posed the following question.

Question 1.1 (Kulikov and Shao [4, Question 3.2])**.** *Let X be a uniformly convex Banach space. Does there exist a function $f\colon \mathbb{N} \to \mathbb{N}$ such that every sequence (v_n) in X satisfying* (1) *for all $n \le m \le f(n)$ has a limit $\lim_{n\to\infty} v_n/n$?*

No further hypothesis is placed on (v_n). Since (1) is symmetric in n and m, it is enough to describe the admissible pairs with $n \le m$. We use the following language.

Definition 1.2. A *pair set* is a set $P \subseteq \{(n,m) \in \mathbb{N}\times\mathbb{N} :\ n \le m\}$. The *partner set* of $n \in \mathbb{N}$ is $P(n) = \{m \in \mathbb{N} :\ (n,m) \in P\}$. A sequence $(v_n)_{n\ge1}$ in a normed space X is *P-subadditive* if (1) holds for every $(n,m) \in P$. *Fekete's lemma holds for P in X* if v_n/n converges in norm for every P-subadditive sequence (v_n) in X. For $f\colon \mathbb{N} \to \mathbb{N}$, the *corridor* of f is $P_f = \{(n,m) :\ n \le m \le f(n)\}$.

In this language, Question 1.1 asks whether Fekete's lemma holds for some corridor P_f in X. Every partner set of a corridor is finite, since $P_f(n) \subseteq [n, f(n)]$.

1.2 Main results

A *nondegenerate arc* of the unit sphere $S_Y = \{y \in Y :\ \|y\| = 1\}$ of a normed space Y is the image of a continuous injective map from a compact interval of positive length into S_Y.

Theorem 1.3. *Let X be a normed space of dimension at least two, and let P be a pair set such that only finitely many $n \in \mathbb{N}$ have an infinite partner set $P(n)$. Then there are a two-dimensional subspace $Y \subseteq X$, a nondegenerate arc $\Gamma \subseteq S_Y$ and a P-subadditive sequence (v_n) in Y with the following properties.*

(a) $\|v_{n+m}\| \le \|v_n\| + \|v_m\|$ for all $n, m \in \mathbb{N}$; the ratio $\|v_n\|/n$ is nonincreasing in n and tends to 1.

(b) The set of limit points of the sequence (v_n/n) is Γ.

In particular, Fekete's lemma fails for P in X.

Applied to corridors, Theorem 1.3 answers Question 1.1.

Corollary 1.4. *Let X be a uniformly convex Banach space of dimension at least two, and let $f\colon \mathbb{N} \to \mathbb{N}$ be any function. There is a sequence (v_n) in X satisfying* (1) *for all $n \le m \le f(n)$, with $\|v_n\|/n \to 1$, for which $\lim_{n\to\infty} v_n/n$ does not exist. Consequently Question 1.1 has a negative answer for every uniformly convex Banach space of dimension at least two, and, by [4, Proposition 3.1], a positive answer, with $f(n) = 2n$, for every space of dimension at most one.*

In a Euclidean plane the limit set can be the whole circle, and the sequence has a closed form. For $f(n) = n^2$, for instance, write $H_k = \sum_{j=1}^{k} 1/j$ for the harmonic numbers ($H_0 = 0$), $A_k = \prod_{j>k} \sec(1/j)$, and $k(n) = \lceil \log_2 \log_2 n \rceil$ for $n \ge 3$, $k(1) = k(2) = 0$.

Proposition 1.5. *The sequence $v_n = n\,A_{k(n)}(\cos H_{k(n)},\ \sin H_{k(n)})$ in the Euclidean plane satisfies $\|v_{n+m}\| \le \|v_n + v_m\|$ for all $n \le m \le n^2$, and every unit vector is a limit point of v_n/n.*

The direction of this sequence is constant on each block $2^{2^{k-1}} < n \le 2^{2^k}$, and its angle $H_{k(n)}$ equals $\ln\ln\ln n + O(1)$. The positive counterpart of Theorem 1.3 isolates what the all-pairs theorem of Kulikov and Shao uses.

Theorem 1.6. *Let X be a uniformly convex Banach space and let P be a pair set with the following two properties.*

(i) There are $\kappa > 1$ and $N_0 \in \mathbb{N}$ such that $(n, m) \in P$ whenever $N_0 \le n \le m \le \kappa n$.

(ii) Infinitely many $n \in \mathbb{N}$ have a cofinite partner set, that is, $P(n) \supseteq \{m : \ m \ge M_n\}$ for some $M_n \in \mathbb{N}$.

Then Fekete's lemma holds for P in X.

The two theorems meet along the following classes of pair sets.

Corollary 1.7. *Let X be a uniformly convex Banach space of dimension at least two.*

(1) Let P satisfy property (i) of Theorem 1.6, and let every partner set $P(n)$ be finite or cofinite. Then Fekete's lemma holds for P in X if and only if infinitely many partner sets are cofinite.

(2) Let $f\colon \mathbb{N} \to \mathbb{N}$ satisfy $f(n) \ge \kappa n$ for some $\kappa > 1$ and all large n, let $S \subseteq \mathbb{N}$, and let $P = P_f \cup \{(n, m) : \ n \in S, \ n \le m\}$. Then Fekete's lemma holds for P in X if and only if S is infinite.

1.3 Relation to earlier work and route of the proof

For real sequences, restricted pairs are a matter of comparable sizes. The de Bruijn–Erdős and Füredi–Ruzsa theorems use only pairs with $m \le \kappa n$, and even the much thinner corridors $n \le m \le n + n\,w(\log n)$, whose nonincreasing relative width w has a divergent square integral, give convergence of a_n/n under eventual adjacent-pair legality [6, Theorem 5] or balanced legal splits [5, Theorem 1]. Such scalar theorems apply to $\|v_n\|$; none of them is used below. Theorem 1.3 shows that the directions behave differently: no pair set with finite partner sets forces them to converge, even when the norms are subadditive for all pairs, as in part (a). The obstruction is therefore the absence of indices with unbounded partner ranges, and Theorem 1.6 shows that, in uniformly convex Banach spaces, infinitely many such indices together with comparable pairs restore the conclusion.

Kulikov and Shao characterize convergence under the all-pairs hypothesis in convex Banach spaces: when $\lim_{n\to\infty}\|v_n\|/n > 0$, the limit of v_n/n exists exactly when the normalized vectors $v_n/\|v_n\|$ form a uniformly convex subset [4, Theorem 5.5]. In Proposition 1.5 these vectors lie on the Euclidean circle, so the divergence comes from the pair set and not from the geometry of X.

The examples of Kulikov and Shao turn the direction continuously in $\ln n$, and then every pair in the corridor costs angle. Our sequences keep the direction constant on long blocks of indices, chosen by Lemma 2.1 so that an admissible pair never skips a block. Only pairs from consecutive blocks see a turn, and a supporting functional at the later direction converts the turn into a loss of norm. The gluing Lemma 2.2 records this. In a Euclidean plane the loss for a turn by angle η is $1 - \cos\eta$, which is summable along $\eta_j = 1/j$ while the total turn diverges (Section 2). In a general normed plane the unit circle may have corners, and we parametrize an arc by a concave function. The loss between neighbouring points is then at most the product of their distance and the change of a one-sided slope (Lemma 3.1), so sweeping the arc back and forth with summable meshes gives a summable total loss (Section 3). The positive Theorem 1.6 follows the proof of [4, Theorem 1.1], which uses the comparable pairs and, for one index n at a time, the pairs $(n, m + rn)$ along an arithmetic progression (Section 4).

2 Blocks and the Euclidean plane

The first lemma chooses the blocks. For a pair set P let $E(P) = \{n \in \mathbb{N} :\ P(n) \text{ is infinite}\}$.

Lemma 2.1. *Let P be a pair set with $E = E(P)$ finite. There are integers $0 = N_0 < N_1 < N_2 < \cdots$ with $E \subseteq \{1, \dots, N_1\}$ such that, for every $k \geq 0$, every $n \notin E$ with $N_k < n \leq N_{k+1}$ and every $m \in P(n)$ satisfy $m \leq N_{k+2}$.*

Proof. Put $N_1 = \max(E \cup \{1\})$. Given N_{k+1}, the set $\bigcup\{P(n) :\ n \leq N_{k+1},\ n \notin E\}$ is a finite union of finite sets, so we may let N_{k+2} be the larger of $N_{k+1} + 1$ and its maximum (taken as 0 if the union is empty). □

For such a sequence (N_k), the *block index* $\beta(n)$ of $n \in \mathbb{N}$ is the unique $k \geq 0$ with $N_k < n \leq N_{k+1}$. It is nondecreasing in n, every block is nonempty, and Lemma 2.1 says that

$$(n, m) \in P,\ n \notin E \implies \beta(m) \in \{\beta(n), \beta(n) + 1\}, \tag{2}$$

because $n \leq m$ gives $\beta(m) \geq \beta(n)$.

A linear functional φ on a normed space Y is *norming* at a unit vector e if $\varphi(e) = 1$ and $\varphi(y) \leq \|y\|$ for all $y \in Y$.

Lemma 2.2. *Let P be a pair set with $E = E(P)$ finite, let $0 = N_0 < N_1 < \cdots$ be integers satisfying the conclusion of Lemma 2.1, let β be the corresponding block index, and let Y be a normed space. Let $(e_k)_{k \geq 0}$ be unit vectors in Y, let φ_k be norming at e_k for $k \geq 1$, and let $A_0 \geq A_1 \geq A_2 \geq \cdots > 0$ satisfy*

$$A_k\, \varphi_{k+1}(e_k) \geq A_{k+1} \qquad (k \geq 0), \tag{3}$$

$$A_0\, \varphi_k(e_0) \geq A_k \qquad (k \geq 1). \tag{4}$$

Then the sequence $v_n = n\, A_{\beta(n)}\, e_{\beta(n)}$ is P-subadditive, and $\|v_{n+m}\| \leq \|v_n\| + \|v_m\|$ for all $n, m \in \mathbb{N}$. When $E = \emptyset$, condition (4) may be omitted.

Proof. Since β is nondecreasing and (A_k) is nonincreasing, $\|v_{n+m}\| = (n+m)A_{\beta(n+m)}$ is at most $(n+m)A_l$ for every $l \leq \beta(n+m)$, and in particular for $l = \max(\beta(n), \beta(m))$. This gives $\|v_{n+m}\| \leq nA_{\beta(n)} + mA_{\beta(m)} = \|v_n\| + \|v_m\|$.

Let $(n, m) \in P$, put $k = \beta(n)$ and $l = \beta(m) \geq k$; then $\|v_{n+m}\| \leq (n+m)A_l$ by the first paragraph. If $l = k$, then $v_n + v_m = (n+m)A_k e_k$ has norm $(n+m)A_k \geq (n+m)A_l$. If $l \geq k+1$, then the norming property of φ_l gives

$$\|v_n + v_m\| \ \geq\ \varphi_l(v_n + v_m) \ =\ nA_k\, \varphi_l(e_k) + mA_l.$$

When $n \notin E$, (2) gives $l = k+1$, and (3) gives $nA_k\varphi_{k+1}(e_k) \geq nA_{k+1}$. When $n \in E$, Lemma 2.1 gives $k = 0$, and (4) gives $nA_0\varphi_l(e_0) \geq nA_l$. In both cases $\|v_n + v_m\| \geq (n+m)A_l \geq \|v_{n+m}\|$. When $E = \emptyset$, the case $n \in E$ does not occur. □

In a Euclidean plane the inner product supplies the norming functionals.

Proposition 2.3. *Let $(\eta_j)_{j \geq 1}$ be numbers in $(0, 1]$ with $\sum_j \eta_j = \infty$ and $\sum_j \eta_j^2 < \infty$. Put $\theta_k = \sum_{j=1}^{k} \eta_j$, $e_k = (\cos\theta_k, \sin\theta_k) \in \mathbb{R}^2$ and $A_k = \prod_{j>k} \sec\eta_j$ for $k \geq 0$. Let P be a pair set in which every partner set is finite, and let β be the block index of integers $0 = N_0 < N_1 < \cdots$ satisfying the conclusion of Lemma 2.1 for P. Then $v_n = nA_{\beta(n)}e_{\beta(n)}$ is a P-subadditive sequence in the Euclidean plane, it satisfies Theorem 1.3(a), and the set of limit points of (v_n/n) is the unit circle.*

Proof. For $0 < x \le 1$ we have $\cos x \ge 1 - x^2/2$ and $-\ln(1-u) \le 2u$ for $0 \le u \le \frac{1}{2}$, so $0 \le \ln\sec x \le x^2$. Hence the product defining A_k converges, $A_k \ge 1$, (A_k) is nonincreasing, and $\ln A_k \le \sum_{j>k} \eta_j^2 \to 0$. With $\varphi_k(y) = \langle y, e_k\rangle$, which is norming at e_k, we have $\varphi_{k+1}(e_k) = \cos\eta_{k+1}$ and therefore $A_k\varphi_{k+1}(e_k) = A_{k+1}$, which is (3). Since $E(P) = \emptyset$, Lemma 2.2 gives P-subadditivity and Theorem 1.3(a); the ratio $\|v_n\|/n = A_{\beta(n)}$ is nonincreasing and tends to 1.

The steps η_k tend to 0 because $\sum \eta_k^2 < \infty$, and $\theta_k \to \infty$. Given an angle α and $\varepsilon > 0$, choose K with $\eta_k < \varepsilon$ for $k > K$. For every integer q with $\alpha + 2\pi q > \theta_K$, some $k > K$ has $|\theta_k - \alpha - 2\pi q| < \varepsilon$, since the increasing sequence $(\theta_k)_{k\ge K}$ passes $\alpha + 2\pi q$ with steps below ε. So $(\cos\alpha, \sin\alpha)$ is a limit point of (e_k). Every block is nonempty and $v_n/n = A_{\beta(n)}e_{\beta(n)}$ with $A_k \to 1$, so the limit points of (v_n/n) are exactly the limit points of (e_k), namely all unit vectors. □

Proof of Proposition 1.5. Take $\eta_j = 1/j$ in Proposition 2.3, so that $\theta_k = H_k$, and $P = P_f$ with $f(n) = n^2$. The integers $N_0 = 0$ and $N_k = 2^{2^{k-1}}$ $(k \ge 1)$ satisfy the conclusion of Lemma 2.1: $E(P_f) = \emptyset$, and if $n \le N_{k+1}$ then every partner $m \le n^2 \le N_{k+1}^2 = N_{k+2}$. The block index of $n \ge 3$ is the integer k with $2^{k-1} < \log_2 n \le 2^k$, that is $k = \lceil \log_2 \log_2 n \rceil$, and the block index of $n \in \{1, 2\}$ is 0. So $\beta = k(\cdot)$, and Proposition 2.3 applies. □

3 Normed planes

In a general normed plane the unit circle can have corners, and a norming functional need not be unique. We work on an arc that is the graph of a concave function. Let Y be a normed space of dimension two. Fix a linear isomorphism $Y \cong \mathbb{R}^2$, write $|\cdot|$ for the Euclidean norm of $\mathbb{R}^2$, and let $B = \{y : \|y\| \le 1\}$. It is compact, convex and symmetric, and there are $0 < \rho \le R$ with $\{|y| \le \rho\} \subseteq B \subseteq \{|y| \le R\}$. Put $s_* = \max\{x : (x, y) \in B\}$, so that $s_* \ge \rho$, and

$$h(x) = \max\{y : (x, y) \in B\} \qquad (|x| \le s_*).$$

Lemma 3.1. *The function h is concave on $[-s_*, s_*]$, and for $|s| < s_*$ the right derivative $d(s) = h'_+(s)$ exists and is finite. For $|s|, |t| < s_*$ the following hold.*

(a) $p(s) = (s, h(s))$ is a unit vector, and $s \mapsto p(s)$ is continuous and injective.

(b) d is nonincreasing, and $h(x) \le h(s) + d(s)(x - s)$ for all $|x| \le s_*$.

(c) The number $c(s) = h(s) - d(s)s$ satisfies $c(s) \ge \rho$, and the functional $\varphi_s(x, y) = (y - d(s)x)/c(s)$ is norming at $p(s)$.

(d) $0 \le 1 - \varphi_s(p(t)) \le |d(s) - d(t)|\,|s - t|/\rho$.

Proof. For $(x_1, y_1), (x_2, y_2) \in B$ and $\lambda \in [0, 1]$, convexity of B gives $(\lambda x_1 + (1-\lambda)x_2,\ \lambda y_1 + (1-\lambda)y_2) \in B$, so h is concave; the maximum exists because each vertical slice of B over $[-s_*, s_*]$ is a nonempty compact segment. For a concave function the difference quotients $(h(x) - h(s))/(x - s)$ are nonincreasing in $x \ne s$. At an interior point s they are therefore bounded above for $x > s$ by their values for $x < s$, so $d(s) = h'_+(s)$ exists, is finite, and satisfies $(h(x) - h(s))/(x - s) \le d(s)$ for $x > s$ and $\ge h'_-(s) \ge d(s)$ for $x < s$. Both inequalities give $h(x) \le h(s) + d(s)(x - s)$. If $s < t$, the same monotonicity gives $d(s) \ge (h(t) - h(s))/(t - s) \ge h'_-(t) \ge d(t)$. This proves (b).

For (a), $p(s) \in B$ and $(s, h(s) + \varepsilon) \notin B$ for $\varepsilon > 0$, so $p(s)$ lies on the boundary of B, where the norm equals 1. A concave function is continuous on the interior of its interval, and p is injective because its first coordinate is s.

For (c), let $(x,y) \in B$. Then $|x| \le s_*$, so (b) gives $y \le h(x) \le h(s) + d(s)(x-s)$, that is $y - d(s)x \le c(s)$. Applying this to the point $\rho\,(-d(s),1)/\sqrt{1+d(s)^2}$ of B gives $\rho\sqrt{1+d(s)^2} \le c(s)$, hence $c(s) \ge \rho > 0$. Thus $\varphi_s \le 1$ on B. Since B is the unit ball, homogeneity gives $\varphi_s(y) \le \|y\|$ for all y, and $\varphi_s(p(s)) = 1$ by the definition of $c(s)$.

For (d), the definitions give

$$1 - \varphi_s(p(t)) = \frac{h(s) + d(s)(t-s) - h(t)}{c(s)}.$$

The numerator is nonnegative by (b) at s. By (b) at t, $h(s) - h(t) \le d(t)(s-t)$, so the numerator is at most $(d(s) - d(t))(t-s)$. By the monotonicity in (b) this product equals $|d(s) - d(t)|\,|s-t|$, and $c(s) \ge \rho$ completes the proof. □

Put $V = d(-s_*/2) - d(s_*/2) \ge 0$ and fix $a \in (0, s_*/2]$ so small that

$$2aV/\rho \le \tfrac{1}{2}. \tag{5}$$

By the monotonicity in Lemma 3.1(b), $|d(s) - d(t)| \le V$ for $s,t \in [-a,a]$, and then Lemma 3.1(d) gives, for $s,t \in [-a,a]$,

$$\varphi_s(p(t)) \;\ge\; 1 - \frac{2aV}{\rho} \;\ge\; \frac{1}{2}. \tag{6}$$

The arc $\Gamma = \{p(s) :\ |s| \le a\}$ is nondegenerate by Lemma 3.1(a).

Lemma 3.2. *Let $t_0, t_1, \dots, t_q \in [-a,a]$ be monotone (nondecreasing or nonincreasing) with $|t_i - t_{i-1}| \le \mu$ for all i. Then*

$$\sum_{i=1}^{q} (1 - \varphi_{t_i}(p(t_{i-1}))) \;\le\; \frac{\mu V}{\rho}.$$

Proof. By Lemma 3.1(d) each term is at most $\mu\,|d(t_i) - d(t_{i-1})|/\rho$. Since d is monotone and the t_i are monotone, the numbers $d(t_i)$ are monotone, so $\sum_i |d(t_i) - d(t_{i-1})| = |d(t_q) - d(t_0)| \le V$. □

Proof of Theorem 1.3. Since $\dim X \ge 2$, X contains a two-dimensional subspace Y; apply the construction above in Y and let Γ be the arc just defined. The norm of X restricted to Y is the norm of Y, so it suffices to build the sequence in Y.

Directions. For $j \ge 1$ let G_j be the grid $\{-a + 2ai/j^2 :\ 0 \le i \le j^2\}$, listed increasingly for odd j and decreasingly for even j. Concatenate $G_1, G_2, \dots$, deleting the first point of each G_j with $j \ge 2$ (it repeats the last point of G_{j-1}), and call the resulting sequence $s_1, s_2, \dots$; thus $s_1 = -a$. Put $e_k = p(s_k)$ and $\varphi_k = \varphi_{s_k}$ for $k \ge 1$, and $e_0 = e_1$. By Lemma 3.1(c), φ_k is norming at e_k. Put $u_k = 1 - \varphi_k(e_{k-1})$ for $k \ge 1$, so $u_1 = 0$. Each step (s_{k-1}, s_k) belongs to a single grid G_j, which is monotone with mesh $2a/j^2$, so Lemma 3.2 gives

$$\sum_{k \ge 1} u_k \;\le\; \sum_{j \ge 1} \frac{2a}{j^2} \cdot \frac{V}{\rho} \;<\; \infty,$$

and (6) gives $0 \le u_k \le \frac{1}{2}$.

Norm ratios. For $k \ge 1$ let $A_k = \prod_{j>k} (1-u_j)^{-1}$. Since $-\ln(1-u) \le 2u$ for $0 \le u \le \frac{1}{2}$, the product converges, $A_k \ge 1$, $(A_k)_{k \ge 1}$ is nonincreasing, and $\ln A_k \le 2\sum_{j>k} u_j \to 0$. Put $A_0 = 2A_1$. Then (3) holds: for $k \ge 1$, $A_k \varphi_{k+1}(e_k) = A_k(1 - u_{k+1}) = A_{k+1}$, and for $k = 0$, $A_0 \varphi_1(e_0) = A_0 \ge A_1$. Condition (4) holds by (6): $A_0 \varphi_k(e_0) = 2A_1 \varphi_{s_k}(p(-a)) \ge A_1 \ge A_k$.

The sequence. Choose blocks for P by Lemma 2.1 and put $v_n = nA_{\beta(n)}e_{\beta(n)}$. Lemma 2.2 shows that (v_n) is P-subadditive and that its norms are subadditive for all pairs, and $\|v_n\|/n = A_{\beta(n)}$ is nonincreasing and tends to 1. This is (a).

For (b), every e_k lies in Γ, which is compact, and $v_n/n = A_{\beta(n)}e_{\beta(n)}$ with $A_k \to 1$ and every block nonempty, so the limit points of (v_n/n) are those of (e_k) and lie in Γ. Conversely, let $|s| \le a$. Consecutive points of G_j are $2a/j^2$ apart, so the points of G_j that remain in the sequence include one within $4a/j^2$ of s. Different grids contribute different indices, so for every j_0 infinitely many k satisfy $|s_k - s| \le 4a/j_0^2$; by continuity of p, $p(s)$ is a limit point of (e_k). Hence the set of limit points of (v_n/n) is Γ, which contains the two distinct points $p(\pm a)$, so v_n/n does not converge. □

Proof of Corollary 1.4. Every partner set of P_f is contained in $[n, f(n)]$ and is therefore finite, so Theorem 1.3 applies to P_f in X. In dimension one, [4, Proposition 3.1] gives convergence under (1) for $\frac{1}{2}n \le m \le 2n$, which by symmetry of (1) is the corridor P_f with $f(n) = 2n$; in dimension zero every sequence is constant. □

Remark 3.3. Two computational checks accompany these proofs; neither is used in them. An exact rational computation treats the Euclidean plane with rotations $\tan(\eta_j/2) = 1/(j+1)$. It verifies the nonnegativity of the three coefficients of the quadratic form $\|v_n + v_m\|^2 - \|v_{n+m}\|^2$ for all block triples with $l \in \{k, k+1\}$ up to block 40, and it tests all pairs $n \le m \le f(n)$ with $n + m \le 1000$ for $f(n) = 2n$, $5n$, n^2 and $n^2 + 7n$; for each of these corridors it also exhibits a pair from non-adjacent blocks at which (1) fails. A 40-digit computation evaluates a variant of the construction of Theorem 1.3, with sweeps of mesh $2a/(4j^2)$, $a = 0.05$, block 0 equal to $\{1, \dots, 4\}$, and the lower bound (6) at $t = -a$ checked directly. It treats the ℓ^p planes for $p = 3/2, 3, 6$ and the planes normed by $|y| + |y_1|$ and $|y| + \frac{1}{2}\max(|y_1|, |y_2|)$, which have corners (the first one on the swept arc), with the exceptional set $E = \{1, 2, 3\}$ and the corridors $f(n) = 2n$ and n^2, and it tests all pairs of P with $n + m \le 1600$. It exhibits pairs from non-adjacent blocks at which (1) fails in every plane for $f(n) = 2n$, and in four of the five planes for $f(n) = n^2$ within the tested range $n + m \le 1600$.

4 Indices with unbounded partner sets

We use the following geometric lemma of Kulikov and Shao, whose proof for general uniformly convex spaces they credit to F. Petrov.

Lemma 4.1 ([4, Lemma 2.1])**.** *Let X be uniformly convex and $\varepsilon > 0$. There is $\lambda = \lambda(\varepsilon) < 1$ such that all nonzero $u, v \in X$ with $\|u/\|u\| - v/\|v\|\| \ge \varepsilon$ and $\|v\| \le 2\|u\|$ satisfy $\|u + v\| \le \|u\| + \lambda\|v\|$.*

Proof of Theorem 1.6. Let (v_n) be P-subadditive and $a_n = \|v_n\|$. By the triangle inequality, $a_{n+m} \le a_n + a_m$ whenever $N_0 \le n \le m \le \kappa n$, so [3, Theorem 2] gives $a_n/n \to L = \inf_{k \ge N_0} a_k/k \in [0, \infty)$. If $L = 0$ then $v_n/n \to 0$. Otherwise we replace v_n by v_n/L and assume $L = 1$. Then $a_k \ge k$ for $k \ge N_0$, and for each $\delta \in (0, \frac{1}{3}]$ there is $N(\delta) \ge N_0$ with $a_k \le (1+\delta)k$ for $k \ge N(\delta)$. Put $w_k = v_k/a_k$ for $k \ge N_0$.

Step 1: comparable indices (this is [4, Proposition 2.4] with the ratio κ in place of 4). Let $\varepsilon > 0$, $\lambda = \lambda(\varepsilon)$, and fix $\delta_1 \in (0, \frac{1}{3}]$ with $1 - \lambda(1 + \delta_1) > \delta_1\kappa$. We claim $\|w_n - w_m\| < \varepsilon$ whenever $N(\delta_1) \le n \le m \le \kappa n$. Otherwise Lemma 4.1 applies to $u = v_m$, $v = v_n$, because $a_n \le (1+\delta_1)n \le 2m \le 2a_m$. With $(n, m) \in P$ by (i), this gives

$$n + m \le a_{n+m} \le \|v_n + v_m\| \le a_m + \lambda a_n \le (1 + \delta_1)(m + \lambda n),$$

so $n(1 - \lambda(1+\delta_1)) \le \delta_1 m \le \delta_1\kappa n$, contrary to the choice of δ_1. Write $N_1(\varepsilon) = N(\delta_1)$; it depends only on ε, κ and the sequence.

Step 2: long range from an index with a cofinite partner set. Let $\varepsilon > 0$ and $\lambda = \lambda(\varepsilon)$, and fix $\delta_2 \in (0, \frac{1}{3}]$ with $(\kappa-1)\bigl(1-\lambda(1+\delta_2)\bigr) \geq 2\delta_2$. Let $n \geq \max(N_1(\varepsilon), N(\delta_2))$ have a cofinite partner set, and let $m \geq M_n' = \max\bigl(M_n,\, n,\, 2n/(\kappa-1)\bigr)$. We claim $\|w_n - w_m\| < 2\varepsilon$. Suppose instead $\|w_n - w_m\| \geq 2\varepsilon$, and let $q = \lfloor(\kappa-1)m/n\rfloor$, so that $m+qn \leq \kappa m$ and $qn > (\kappa-1)m - n \geq (\kappa-1)m/2$. We show by induction on $r = 0, 1, \ldots, q$, as in [4, (2.2)], that

$$a_{m+rn} \leq a_m + r\lambda a_n. \tag{7}$$

The case $r = 0$ is trivial. Let $r < q$ and assume (7) for r. Since $m \leq m + rn \leq \kappa m$ and $m \geq N_1(\varepsilon)$, Step 1 gives $\|w_m - w_{m+rn}\| < \varepsilon$, so $\|w_n - w_{m+rn}\| > \varepsilon$. Moreover $a_n \leq (1+\delta_2)n \leq 2(m+rn) \leq 2a_{m+rn}$, and $(n, m+rn) \in P$ because $m + rn \geq m \geq M_n$. Lemma 4.1 with $u = v_{m+rn}$, $v = v_n$ gives

$$a_{m+(r+1)n} \leq \|v_n + v_{m+rn}\| \leq a_{m+rn} + \lambda a_n \leq a_m + (r+1)\lambda a_n,$$

which is (7) for $r+1$. For $r = q$, (7) and the bounds on a_k give $m + qn \leq (1+\delta_2)(m + \lambda qn)$, that is $qn\bigl(1-\lambda(1+\delta_2)\bigr) \leq \delta_2 m$. With $qn > (\kappa-1)m/2$ this yields $(\kappa-1)\bigl(1-\lambda(1+\delta_2)\bigr) < 2\delta_2$, contrary to the choice of δ_2.

Step 3: the Cauchy property. Let $\varepsilon > 0$, and let δ_2 be the number chosen for ε in Step 2. By (ii) there is an index $n \geq \max(N_1(\varepsilon), N(\delta_2))$ with a cofinite partner set. For $m, m' \geq M_n'$, Step 2 gives $\|w_m - w_{m'}\| < 4\varepsilon$. So (w_m) is a Cauchy sequence; it converges to some w because X is complete, and $v_m/m = (a_m/m)\, w_m \to w$. □

Proof of Corollary 1.7. (1) If infinitely many partner sets are cofinite, Theorem 1.6 applies. Otherwise the infinite partner sets, which are exactly the cofinite ones, are finitely many, and Theorem 1.3 shows that Fekete's lemma fails for P in X.

(2) Let N_0 be such that $f(n) \geq \kappa n$ for $n \geq N_0$. Then P contains every pair with $N_0 \leq n \leq m \leq \kappa n$, which is property (i). The partner set of $n \in S$ is $\{m :\ m \geq n\}$, which is cofinite, and the partner set of $n \notin S$ is contained in $[n, f(n)]$, which is finite. Part (1) applies. □

Remark 4.2. The sequences of Theorem 1.3 have subadditive norms for all pairs, so the scalar lemmas of Fekete, de Bruijn–Erdős and Füredi–Ruzsa hold for $(\|v_n\|)$ with room to spare. When X is uniformly convex, the vector inequality (1) holds on P and fails for some pairs from non-adjacent blocks: by [4, Theorem 1.1], applied in the plane Y, every sequence in a uniformly convex Banach space with divergent v_n/n has such pairs. When the unit circle of Y contains a segment, the coordinates of Section 3 can be chosen so that the arc Γ lies inside it; then all the vectors e_k share one norming functional φ, every u_k vanishes, $\varphi(v_n) = nA_{\beta(n)}$, and (1) holds for all pairs, in line with [4, Corollary 1.2].

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References

[1] N. G. de Bruijn and P. Erdős. Some linear and some quadratic recursion formulas. II. *Indag. Math. (Proceedings)* 55 (= Indag. Math. 14) (1952), 152–163. https://doi.org/10.1016/S1385-7258(52)50021-0.

[2] M. Fekete. Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten. *Math. Z.* 17 (1923), 228–249. https://doi.org/10.1007/BF01504345.

[3] Z. Füredi and I. Z. Ruzsa. Nearly subadditive sequences. *Acta Math. Hungar.* 161 (2020), no. 2, 401–411. https://doi.org/10.1007/s10474-020-01071-0. Theorem numbers as in arXiv:1810.11723v1.

[4] A. Kulikov and F. Shao. Fekete's lemma in Banach spaces. Preprint, arXiv:2411.17380v1, 2024. https://arxiv.org/abs/2411.17380v1.

[5] S. Schubr. *Fekete convergence from balanced integer predecessors in shrinking corridors*, edition 2. Luminary Research Archive, 2026. https://doi.org/10.5281/zenodo.24472766.

[6] S. Schubr. *A square-integral threshold for subadditivity on shrinking corridors*, edition 6. Luminary Research Archive, 2026. https://doi.org/10.5281/zenodo.24407840.

[7] F. Shao and A. Kulikov. High dimensional Fekete's subadditive lemma: does $|\vec{x}_{n+m}| \leq |\vec{x}_n + \vec{x}_m|$ imply the convergence of $\{\vec{x}_n/n\}$? MathOverflow question 459773, 2023. https://mathoverflow.net/q/459773.