Fekete convergence from balanced integer predecessors in shrinking corridors

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October 25, 2026

Abstract

Let w be positive, measurable and nonincreasing, with divergent square integral. Consider real sequences satisfying exact subadditivity only for integer pairs $n \le m \le n + nw(\log n)$. We replace eventual legality of adjacent pairs by a weaker condition: every sufficiently large output has a legal decomposition whose two summands differ by $O(N^\alpha)$ for some $\alpha < 1$, or by $O(N/(\log N)^{3/2+\varepsilon})$ for some $\varepsilon > 0$. Under either condition, a_n/n converges in $\mathbb{R} \cup \{-\infty\}$, without sign, monotonicity or growth assumptions. The proof regularizes the sequence by growing local maxima, proves exact integer corridor inequalities for that majorant, and separately transfers its limit back by propagation of low values along dyadic orbits. An explicit staircase width has arbitrarily long windows of illegal adjacent pairs, satisfies the logarithmic condition, and fails every fixed power condition. Both this convergence example and an oscillating isolated-output example can have divergent integrals of every finite positive power of w. Thus analytic width budgets and integer connectivity play different roles.

1 Integer connectivity beyond adjacent pairs

Fekete's lemma asserts convergence of normalized real subadditive sequences. Restricted-pair versions need much less than all pairs. The classical arguments of de Bruijn and Erdős [1], and the fixed-relative-width theorem of Füredi and Ruzsa [2], show the importance of comparable summands and propagation of low slopes. The latter paper asks about sublinear widths in its Section 7. For shrinking corridors, the author's preceding paper [3] proves a square-integral criterion, with eventual adjacent-pair legality in the arbitrary-real class. The present paper weakens that discrete hypothesis. It concerns scalar sequences, not a realization by permutation classes or another counting structure.

All logarithms are natural. Fix $T \in \mathbb{R}$ and a measurable nonincreasing function $w : [T, \infty) \to (0, 1]$. A pair of positive integers (p, q) is *legal* if

$$p \le q \le p + pw(\log p). \tag{1}$$

Only indices large enough to lie in the domain will be used. A real sequence $a = (a_n)_{n \ge 1}$ is eventually corridor-subadditive if there is n_0 such that

$$a_{p+q} \le a_p + a_q \quad \text{for every legal pair with } p \ge n_0. \tag{2}$$

The endpoint in (1) is literal: there is no ceiling or replacement of a width less than one by one.

Theorem 1. *Assume*

$$\int_T^\infty w(s)^2 \, ds = \infty. \tag{3}$$

Suppose that one of the following conditions holds for every sufficiently large integer N:

1. *There is a legal split* $N = p + q$ *with*

$$q - p \le CN^{\alpha}, \qquad 0 < \alpha < 1, \tag{4}$$

for fixed finite $C > 0$ *and fixed* α.

2. *There is a legal split* $N = p + q$ *with*

$$q - p \le \frac{CN}{(\log N)^{\beta+1}}, \qquad \beta > \tfrac{1}{2}, \tag{5}$$

for fixed finite $C > 0$ *and fixed* β.

Then every real sequence satisfying (2) *has*

$$\lim_{n\to\infty} \frac{a_n}{n} \in \mathbb{R} \cup \{-\infty\}.$$

No sign, monotonicity or growth bound on a *is required.*

The two constants C and the onset of the splits may depend on w. The theorem gives no parameter-uniform effective onset or convergence rate. If adjacent pairs are eventually legal, either predecessor condition follows, but the converse fails by Proposition 6. The logarithmic conclusion is not just a restatement of the power cases: that proposition satisfies (5) with $\beta = 1$ and fails (4) for every $\alpha < 1$.

Relation to earlier mechanisms

The new step is a growing local-maximum regularization that acquires adjacent inequalities from balanced legal predecessors. Concave allocation of a perturbed output then preserves a smaller original corridor. Convergence of this majorant alone is insufficient; a separate low-value transfer is essential. Dyadic propagation and the underlying adjacent-corridor theorem are inherited mechanisms, explicitly credited below. Classical recursive minimization also studies balanced splits [4, 5]. Wallace's restricted-recursion work concerns particular extremal recurrences in sequential search; his author summary [6] displays the recurrence with output $n+2p+1$ and child-index sum $2n+2p$.

2 The local-maximum construction

We use the following previously established theorem with its stated integer hypothesis.

Theorem 2 (Adjacent-corridor theorem, [3])**.** *Let* $u : [S, \infty) \to [0, 1]$ *be measurable and nonincreasing, with* $\int_S^\infty u^2 = \infty$ *and* $nu(\log n) \ge 1$ *eventually. Every eventually corridor-subadditive real sequence for* u *has a normalized limit in* $\mathbb{R} \cup \{-\infty\}$.

The cited proof uses a prefix envelope and dyadic record indices to remove monotonicity; it is stronger than a convergence theorem for nonnegative monotone sequences. We will prove its adjacent-pair hypothesis for a new majorant, rather than assume it for a.

In case (4) set

$$h(x) = Hx^{\alpha}, \qquad 2^{\alpha-1}(H + C) < H. \tag{6}$$

Such a finite $H > 0$ exists. In case (5) put, on a late tail,

$$h(x) = \frac{Hx}{(\log x)^{\beta}}, \qquad H\beta \log 2 > 2C. \tag{7}$$

In both cases h is eventually increasing and concave, $h \to \infty$, $h = o(x)$, and $h' \to 0$. Define

$$k_n = \lfloor h(n) \rfloor, \qquad b_n = \max_{|j-n| \le k_n} a_j \tag{8}$$

for all sufficiently large n, with integer j in the maximum. Early values of b may be chosen arbitrarily. Each maximum is finite.

Lemma 3. *Eventually*

$$b_{2n} \le 2b_n, \qquad b_{2n+1} \le b_n + b_{n+1}. \tag{9}$$

Proof. For every j in the b_{2n} or b_{2n+1} window, choose a legal predecessor split promised by the appropriate hypothesis. Both children tend to infinity, since their difference is $o(j)$. Their distance from the relevant half-center is bounded by half the output-window radius, half the predecessor gap, and an absolute rounding constant.

In the power case this bound is

$$2^{\alpha-1}(H + C + o(1))n^\alpha + O(1) < Hn^\alpha - 2$$

eventually, by (6). In the logarithmic case it is at most

$$\frac{h(2n)}{2} + (C + o(1))\frac{n}{(\log n)^{\beta+1}} + O(1).$$

Writing $\ell = \log n$, direct expansion gives

$$h(n) - \frac{h(2n)}{2} = (H\beta \log 2 + o(1))\frac{n}{\ell^{\beta+1}}. \tag{10}$$

The choice in (7) leaves an unbounded spare margin. Also $h(n+1) - h(n) = o(1)$, so floors and the center shift by one are paid. Thus both children lie in $[n - k_n, n + k_n]$ for an even center. For an odd center, both lie in the intersection

$$[n - k_n, n + k_n] \cap [n + 1 - k_{n+1}, n + 1 + k_{n+1}].$$

Apply the actual inequality (2) to each selected split. In the odd case its right side is at most $2\min(b_n, b_{n+1}) \le b_n + b_{n+1}$, even if these values are negative. Taking maxima proves (9). □

Lemma 4. *Put $v(s) = w(s + \log 2)/8$. Then b is eventually corridor-subadditive for the literal width v.*

Proof. The diagonal case follows from Lemma 3. For an off-diagonal pair write $m = n + d$, where

$$1 \le d \le \frac{nw(\log(2n))}{8}, \qquad 1 \le \frac{m}{n} \le \frac{9}{8}. \tag{11}$$

Let $j = n + m + e$ be any integer in the b_{n+m} window. Set

$$\theta = \frac{h(n)}{h(n) + h(m)}, \qquad p = n + \lfloor \theta e \rfloor, \qquad q = j - p. \tag{12}$$

We check the window inclusions and original legality separately.

The spare allocation margin $h(n)+h(m)-h(n+m)$ tends to infinity uniformly for $1 \le m/n \le 9/8$. For $h(x) = Hx^\alpha$, it equals

$$Hn^\alpha[1 + r^\alpha - (1 + r)^\alpha], \qquad r = m/n,$$

and the continuous bracket is positive on that compact interval. For (7), uniformly on the same interval,

$$h(n)+h(m)-h(n+m)=\frac{H\beta n}{\ell^{\beta+1}}[(1+r)\log(1+r)-r\log r]+O\!\left(\frac{n}{\ell^{\beta+2}}\right). \tag{13}$$

The bracket is at least $2\log 2$. This expansion follows by Taylor's formula for $(1+z)^{-\beta}$; $\log r$ and $\log(1+r)$ are bounded uniformly. Both θ and $1-\theta$ stay bounded away from zero. Consequently

$$\theta h(n+m)<h(n)-2, \qquad (1-\theta)h(n+m)<h(m)-2$$

eventually. The floor error in (12) is at most one. Thus $|p-n|\le k_n$ and $|q-m|\le k_m$.

To check legality, exact subtraction gives

$$q-p=d+(1-2\theta)e+2\{\theta e\}.$$

The mean-value theorem, eventual concavity and the positive allocation margin imply

$$|(1-2\theta)e|\le\frac{[h(m)-h(n)]h(n+m)}{h(n)+h(m)}\le h'(n)d=o(d)$$

uniformly in (11). Therefore, for sufficiently large n,

$$q-p\ge d/2>0, \qquad q-p\le 2d+2\le 4d\le\frac{nw(\log(2n))}{2}.$$

Since $h=o(n)$, we have $n/2\le p\le 2n$ eventually. Nonincrease of w gives

$$pw(\log p)\ge\frac{n}{2}w(\log(2n))\ge q-p.$$

Thus (p,q) is an original legal pair with $p+q=j$. Its true subadditivity inequality bounds a_j by b_n+b_m. Take the maximum over the entire output window. No continuity of w has been used. □

3 Convergence and transfer of low values

The function

$$V(s)=\max\{v(s),e^{-s}\}$$

is positive, nonincreasing and at most one on a late tail. Its square integral diverges. An integer pair allowed by V is either v-allowed or diagonal/adjacent, since $ne^{-\log n}=1$. Lemmas 3 and 4 therefore give every V-corridor inequality for b. Theorem 2 yields

$$b_n/n\longrightarrow L\in\mathbb{R}\cup\{-\infty\}. \tag{14}$$

As $a_n\le b_n$, the case $L=-\infty$ already proves the desired conclusion. The finite case needs an additional argument.

Lemma 5 (Dyadic escape)**.** *For every fixed positive integer n, the values*

$$\frac{xw(\log x)}{h(x)}, \qquad x=2^k n,$$

are unbounded as $k\to\infty$.

Proof. If these ratios were bounded eventually, then on that logarithmic lattice $w(s)$ would be bounded by a constant multiple of $e^{-(1-\alpha)s}$ in the power case, or of $s^{-\beta}$ in the logarithmic case. Monotonicity fills the intervals of length $\log 2$ between successive lattice points, changing the constants only. Either bound makes w^2 integrable on a tail; in the second case this uses $2\beta > 1$. This contradicts (3). □

Completion of Theorem 1. Suppose L in (14) is finite, but a_n/n does not tend to L. As $a \le b$, there are $\delta > 0$ and arbitrarily large n with $a_n/n \le L - \delta$. Fix such an n in the subadditivity tail. Diagonal legality gives

$$a_x/x \le a_n/n \le L - \delta \quad (x = 2^k n).$$

By Lemma 5 there are arbitrarily large such x with

$$xw(\log x) \ge 4h(2x), \qquad h(2x) \le x/2. \tag{15}$$

Here $h(2x)/h(x)$ is bounded on a late tail for both choices of h. For every integer t with $|t| \le k_{2x}$ the pair consisting of x and $x + t$ is legal. For $t \ge 0$, use the first inequality in (15). For $t < 0$, the smaller child is $x + t \ge x/2$ and

$$(x + t)w(\log(x + t)) \ge (x/2)w(\log x) \ge 2h(2x) \ge |t|.$$

For any $\varepsilon > 0$, (14) gives $a_y \le b_y \le (L+\varepsilon)y$ for all sufficiently large y. Hence, uniformly throughout the output window,

$$a_{2x+t} \le a_x + a_{x+t} \le (L - \delta)x + (L + \varepsilon)(x + t).$$

Taking the maximum and dividing by $2x$ gives

$$\frac{b_{2x}}{2x} \le L - \frac{\delta}{2} + \frac{\varepsilon}{2} + o(1),$$

since $h(2x) = o(x)$. Choose $\varepsilon < \delta/2$ and let these dyadic x tend to infinity. This contradicts (14). Thus the limit of a_n/n is L as required. □

4 A width beyond adjacent and power predecessors

Proposition 6. *There is a positive nonincreasing measurable width $w \to 0$ satisfying* (3) *and the logarithmic predecessor condition with $\beta = 1$, such that adjacent pairs are illegal in infinitely many windows of lengths tending to infinity. For this width no power predecessor condition* (4)*, with fixed $\alpha < 1$ and finite C, holds eventually. The example can also be chosen so that $\int_0^\infty w(s)^p\, ds = \infty$ for every finite $p > 0$.*

Proof. Set $b_1 = 16$ and recursively define

$$A_j = 2^{b_j}, \quad L_j = \left\lfloor \frac{A_j}{16 b_j^2} \right\rfloor, \quad M_j = A_j + L_j, \quad b_{j+1} = M_j^3. \tag{16}$$

Let $w(s) = 1$ for $0 \le s < \log A_1$, and

$$w(s) = M_j^{-1} \quad (\log A_j \le s < \log A_{j+1}).$$

It has the asserted regularity. Its jth contribution to the square integral is

$$\frac{(M_j^3 - b_j)\log 2}{M_j^2} = \left(M_j - \frac{b_j}{M_j^2}\right)\log 2 \longrightarrow \infty.$$

At integers $A_j \le n \le A_j + L_j - 1$ we have $nw(\log n) < 1$; therefore $(n, n+1)$ is illegal. For $M_j \le n < A_{j+1}$ it is legal.

Every even output has a diagonal split. For an odd output $N = 2n + 1$ outside a forbidden window use $(n, n+1)$. Inside the jth forbidden window use

$$p = A_j - 1, \qquad q = 2n - A_j + 2, \qquad q - p = 2(n - A_j) + 3 \le 2L_j + 1. \tag{17}$$

At $p = A_j - 1$, the width is at least $1/b_j$: before the first drop it is one, and subsequently $b_j = M_{j-1}^3 \ge M_{j-1}$. For $b_j \ge 16$,

$$\frac{A_j - 1}{b_j} \ge \frac{A_j}{8b_j^2} + 1 \ge 2L_j + 1.$$

Thus (17) is actually legal. Moreover $A_j/(16b_j^2) \ge 1$, so

$$2L_j + 1 \le \frac{3A_j}{16b_j^2} < \frac{A_j}{4b_j^2}.$$

In this window $2A_j < N < 3A_j$ and $\log N \le b_j$ for $b_j \ge 16$. Consequently

$$q - p \le \frac{A_j}{4b_j^2} \le \frac{N}{(\log N)^2}.$$

The adjacent gap one also satisfies this bound eventually. This proves the logarithmic predecessor condition.

At the last forbidden output $N_j = 2(A_j + L_j - 1) + 1$, any smaller child $p \in [A_j, \lfloor N_j/2 \rfloor]$ has $pw(\log p) < 1$ and cannot produce a legal odd sum. Thus every legal split has $p \le A_j - 1$ and gap at least $2L_j + 1$. The split in (17) attains this bound. Since

$$L_j \sim \frac{A_j}{16b_j^2}, \qquad b_j = \log_2 A_j,$$

the minimum gap divided by N_j^α tends to infinity for each fixed $\alpha < 1$. No fixed power condition can apply.

For the last assertion replace only the last recurrence in (16) by $b_{j+1} = M_j^{j+3}$. All legality arguments used only $b_j \ge M_{j-1}$ and remain valid. For each fixed $p > 0$ the jth plateau contributes

$$\left(M_j^{j+3-p} - \frac{b_j}{M_j^p} \right) \log 2 \longrightarrow \infty$$

to $\int w^p$. Here eventually $j + 3 - p \ge 1$, and $b_j/M_j^p \to 0$. This proves simultaneous divergence for all finite positive powers. □

5 Connectivity and isolated-output obstructions

The preceding positive example can be compared with an isolated-output construction obtained in an unpublished strengthening of the predecessor manuscript [3]. We include its argument in full to make the distinction explicit; the new contribution here is the positive predecessor theorem and its strict-width example.

Lemma 7 (Isolated-output obstruction)**.** *Let $\mathcal{P}$ be any permitted set of positive-integer pairs. If infinitely many integers are not outputs $p+q$ of pairs in $\mathcal{P}$, then there is a nonnegative integer sequence, bounded above by n, satisfying every $\mathcal{P}$-subadditivity inequality, whose normalized slopes do not converge.*

Proof. Choose an infinite subset U of unrepresented outputs with infinite complement, and let $a_n = n\mathbf{1}_{n\in U}$. Every tested left side vanishes and every right side is nonnegative. Slopes are one on U and zero on its complement. □

For completeness the relevant all-power width has $c_1 = 4$,

$$B_j = 2^{c_j}, \quad c_{j+1} = c_j + 2^{(j+2)c_j+6}, \qquad u(s) = \frac{1}{8B_j} \quad (c_j \log 2 \le s < c_{j+1} \log 2),$$

with $u = 1$ before the first plateau. The integral of u^p on that plateau is

$$2^{(j+2-p)c_j+6-3p} \log 2,$$

which tends to infinity for every fixed $p > 0$. Yet $3B_j + 1$ is an unrepresented odd output. Indeed a legal split of this output would have $B_j + 1 \le n \le 3B_j/2$, by the global inequality $m \le 2n$. In that forced plateau its integer gap is at most $n/(8B_j) \le 3/16 < 1$, forcing equality of the children and an even output, a contradiction. Lemma 7 now applies.

Thus divergence of all finite-power integrals coexists with universal convergence for one width and failure for another. Theorem 1 uses quantitative integer predecessor access to distinguish them.

Reproducibility. The proof is analytic and does not depend on finite computation. The accompanying `verify.py` uses only the Python standard library to check exact integer window containments, literal parent allocations, and staircase endpoint controls. It emits a deterministic `verification.json`; these checks corroborate the finite algebra, not the universal asymptotic theorem.

Use of AI tools. AI tools assisted in developing and checking the arguments and preparing the manuscript.

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