An explicit hyperbolicity threshold for xi Jensen polynomials

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Abstract

We give an explicit sufficient shift for hyperbolicity of the degree-d Jensen polynomials attached to Riemann's xi-function. Every integer $n \geq \max\{65536, \lceil 16\sqrt{d+1}\log(d+3)\rceil\}$ is sufficient. The estimate replaces the exponent loss in the qualitative $O_\varepsilon(d^{1/2+\varepsilon})$ threshold by a critical logarithmic factor and an effective onset. The proof quantitatively specializes the derivative-escape argument of Kim and Lee, using the classical zero strip, Gontcharoff interpolation, Cauchy's estimate and an elementary growth bound for xi. We also give a sharper scalar criterion and a portable exact checker for its rational enclosures.

1 The explicit bound

Write

$$\xi(s) = \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s), \qquad \xi(\tfrac{1}{2}+z) = \sum_{m\geq 0}\frac{\gamma(m)}{m!}z^{2m}.$$

Thus $\gamma(m) = m!\xi^{(2m)}(1/2)/(2m)! > 0$. We use

$$J^{d,n}(x) = \sum_{j=0}^{d}\binom{d}{j}\gamma(n+j)x^j, \qquad d \geq 1,\ n \geq 0. \tag{1}$$

A real polynomial is hyperbolic if all its zeros are real, counted with multiplicity. Logarithms throughout are natural.

Theorem 1.1. *For every integer $d \geq 1$, the polynomial $J^{d,n}$ is hyperbolic for every integer*

$$n \geq N(d) := \max\left\{65536, \left\lceil 16\sqrt{d+1}\log(d+3)\right\rceil\right\}. \tag{2}$$

Its zeros are negative.

The following sufficient test usually gives a smaller shift.

Theorem 1.2. *Let $d, n \geq 1$ be integers, and set*

$$H = 6n + 5d + 6, \qquad s = \tfrac{1}{2} + \sqrt{4H + \tfrac{1}{4}}.$$

If

$$2\log s + \frac{s}{2}\log\frac{s}{2} + \log 2 < n\log 3, \tag{3}$$

then $J^{d,n}$ is hyperbolic.

Griffin, Ono, Rolen and Zagier proved eventual hyperbolicity in every fixed degree by a Hermite limit [5]. An effective threshold of the form $n \geq ce^d$ was subsequently obtained in [6]. The same paper's finite-height argument gives hyperbolicity at every shift for $d \leq 9 \cdot 10^{24}$ [6, Corollary 1.3]. Kim and Lee proved the stronger qualitative bound $n \geq d^{c/2}$ for each fixed $c > 1$ and all sufficiently large degrees [1, Theorem 1.1]. The closest proof predecessor here is their published argument in Sections 2–3, which combines predecessor zeros, Gontcharoff's first interpolation estimate and Cauchy's formula. We keep that mechanism and replace its generic order comparison by an explicit growth estimate for the actual xi-function. The resulting quantitative conclusion is (2).

O'Sullivan's Hermite-linear-combination polynomials form a related, distinct family [7]. Hyperbolicity of (1) implies the corresponding Hermite conclusion. Zero localization and joint semicircle limits, such as those studied in [8], require information beyond a hyperbolicity threshold.

2 The classical interpolation tools

For an entire function f, let $M(R; f) = \max_{|z|\leq R} |f(z)|$. We use three classical facts in the forms recorded by Kim and Lee.

Lemma 2.1 (Jensen predecessor zeros)**.** *Let f be a nonconstant real entire function of order less than two. If $f'(z_1) = 0$ and $\operatorname{Im} z_1 > 0$, there is a zero $z_0 = \alpha_0 + i\beta_0$ of f, with $\beta_0 > 0$, such that*

$$|z_1 - \alpha_0| \leq \beta_0.$$

Consequently, if $f^{(n)}(z_n) = 0$ and $\operatorname{Im} z_n > 0$, one can choose zeros $z_j = \alpha_j + i\beta_j$ of $f^{(j)}$, $0 \leq j < n$, satisfying

$$(\alpha_{j+1} - \alpha_j)^2 + \beta_{j+1}^2 \leq \beta_j^2, \qquad \beta_j > 0. \tag{4}$$

This is Jensen's disk theorem; see [1, Theorem 3.1 and Corollary 3.2]. Differentiation preserves order, so it may be applied successively.

Lemma 2.2 (Gontcharoff's first estimate)**.** *Suppose f is entire and $f^{(j)}(z_j) = 0$ for $0 \leq j < n$. If M is the maximum of $|f^{(n)}|$ on the convex hull of $\{z, z_0, \ldots, z_{n-1}\}$, then*

$$|f(z)| \leq \frac{M}{n!} \left(|z - z_0| + \sum_{j=0}^{n-2} |z_{j+1} - z_j| \right)^n.$$

Proof. Successive integration gives

$$f(z) = \int_{z_0}^{z} \int_{z_1}^{v_1} \cdots \int_{z_{n-1}}^{v_{n-1}} f^{(n)}(v_n)\, dv_n \cdots dv_1.$$

Parameterize the polygonal path from z_{n-1} through $z_{n-2}, \ldots, z_0$ to z by arc length on $[0, L]$. The ordered integration region is contained in $0 \leq t_n \leq \cdots \leq t_1 \leq L$, whose volume is $L^n/n!$. All path points lie in the indicated convex hull. This proves the estimate, as in [1, Theorem 3.3] and [2, p. 662], following Gontcharoff [3]. □

Lemma 2.3 (Finite-degree sector criterion)**.** *Let f be a real entire function of order less than two, and suppose all its zeros satisfy $|\operatorname{Im} z| \leq \delta |z|$, where $\delta > 0$. If $d \leq \delta^{-2}$, then*

$$\sum_{j=0}^{d} \binom{d}{j} f^{(j)}(0) x^j$$

is hyperbolic.

This is the consequence of Obreschkoff's theorem used in [1, Theorem 2.6 and Corollary 2.7]; see [4]. The entire-function version follows by locally uniform polynomial approximation with zeros in the same double sector, allowing real zeros, and closure of the hyperbolic polynomials of bounded degree.

3 Quantitative escape in a parabola

The next statement isolates the effective part of the argument.

Lemma 3.1. *Let f be a transcendental real entire function of order less than one, with $f(0) \neq 0$, and suppose its zeros lie in*

$$\mathcal{P} = \{\alpha + i\beta : \alpha \geq \beta^2\}.$$

For $n \geq 1$ and $X \geq 0$, put $H = 6n + 5X + 1$. If $f^{(n)}$ has a nonreal zero with real part at most X, then

$$|f(0)| \leq 3^{-n} M(4H; f). \tag{5}$$

Proof. Conjugation allows us to take the zero in the upper half-plane. Choose the predecessor chain of Lemma 2.1. Since (4) gives decreasing β_j, summation and Cauchy–Schwarz show that

$$|\alpha_n - \alpha_0| \leq \sqrt{n \sum_{j=0}^{n-1} (\alpha_{j+1} - \alpha_j)^2} \leq \sqrt{n}\, \beta_0.$$

The initial zero is in $\mathcal{P}$, so $\alpha_0 \leq X + \sqrt{n\alpha_0}$. Hence

$$\alpha_0 \leq A := (\sqrt{n} + \sqrt{X})^2 \leq 2(n + X), \qquad \beta_0 \leq \sqrt{A}.$$

Also, (4) implies

$$|z_{j+1} - z_j|^2 \leq 2\beta_j(\beta_j - \beta_{j+1}).$$

Thus the total chain length is at most

$$\sum_{j=0}^{n-1} |z_{j+1} - z_j| \leq \sqrt{2n}\, \beta_0 \leq \sqrt{2nA} \leq n + A/2.$$

Since $|z_0| \leq A + \sqrt{A} \leq 3A/2 + 1/2$, every node and the length

$$|z_0| + \sum_{j=0}^{n-2} |z_{j+1} - z_j|$$

are bounded by $2A + n + 1/2 \leq 5n + 4X + 1/2 < H$. Lemma 2.2 at $z = 0$ therefore gives

$$|f(0)| \leq M(H; f^{(n)}) H^n / n!.$$

A Cauchy circle of radius $3H$ centered at any point of $|z| \leq H$ lies in $|z| \leq 4H$. Consequently $M(H; f^{(n)})/n! \leq M(4H; f)/(3H)^n$, proving (5). □

4 The actual xi input

Set

$$\Xi(z) = \xi(1/2 + iz), \quad F(w) = \Xi(\sqrt{w}), \quad \Theta(w) = F(-w), \quad G(z) = F(z - 1/4).$$

The quotient notation defines entire functions because Ξ is even. The functions F and G have order $1/2$, $\Theta(w) = \sum_{m\ge 0} \gamma(m) w^m/m!$, and $G(0) = \xi(0) = 1/2$.

Lemma 4.1. *Every zero $\alpha + i\beta$ of G, and of each derivative of G, satisfies $\alpha \ge \beta^2$. For $R \ge 0$,*

$$M(R; G) \le \xi\left(\tfrac{1}{2} + \sqrt{R + \tfrac{1}{4}}\right). \tag{6}$$

For real $s \ge 2$,

$$\xi(s) \le s^2 (s/2)^{s/2}. \tag{7}$$

Proof. The classical xi zero strip says that a zero $t + iu$ of Ξ satisfies $|u| < 1/2$; the definitions and zero-strip use are also recorded in [1, Sections 1–2]. Its square, followed by translation by $1/4$, gives a zero of G with

$$\alpha = t^2 - u^2 + 1/4 \ge t^2 \ge 4t^2u^2 = \beta^2.$$

These are all its zeros. A real entire function of order less than one is a locally uniform limit of its finite genus-zero canonical products. The set $\mathcal{P}$ is closed and convex, so Gauss–Lucas and Hurwitz give the same containment for every derivative.

Since $G(z) = \Theta(1/4 - z)$ and Θ has positive coefficients,

$$|G(z)| \le \Theta(R + 1/4) = \xi\left(\tfrac{1}{2} + \sqrt{R + \tfrac{1}{4}}\right) \quad (|z| \le R),$$

which proves (6). For $s \ge 2$ the integral comparison for the zeta series gives $\zeta(s) \le 1 + (s-1)^{-1} \le 2$. Moreover, $\pi^{-s/2} \le 1$ and $\Gamma(x) \le x^x$ for $x \ge 1$. For the last inequality, log-convexity between adjacent positive integers bounds $\Gamma(x)$, for $m \le x \le m + 1$, by $m! \le m^m \le x^x$. Substitution in the definition of ξ proves (7). □

Proof of Theorem 1.2. Apply Lemma 3.1 to G, with $X = d + 1$. If $G^{(n)}$ had a nonreal zero of real part at most X, Lemma 4.1 would give

$$\tfrac{1}{2} \le 3^{-n} M(4H; G) \le 3^{-n} s^2 (s/2)^{s/2},$$

contrary to (3). Thus every nonreal zero $w = a + ib$ of $F^{(n)}$ has $\alpha = a + 1/4 > d + 1$ and $b^2 \le \alpha$. In particular,

$$\frac{|b|}{|w|} \le \frac{\sqrt{\alpha}}{\alpha - 1/4} < \frac{1}{\sqrt{d}},$$

because $(\alpha - 1/4)^2 - d\alpha > 0$ for $\alpha \ge d + 1$. Real zeros satisfy the required double-sector condition as well. Lemma 2.3, with $\delta = d^{-1/2}$, proves hyperbolicity of the degree-d Jensen polynomial of $F^{(n)}$. Finally,

$$F^{(n+j)}(0) = (-1)^{n+j} \gamma(n + j),$$

so its Jensen polynomial is $(-1)^n J^{d,n}(-x)$. Reflection proves the assertion. □

5 The closed onset and exact scalar checks

Proof of Theorem 1.1. Let $X = d+1$ and $Y = \max(n, X)$. Then $H = 6n + 5X + 1 \le 12Y$ and $s \le 8\sqrt{Y}$. Since $s/2 \ge 1$,

$$\begin{aligned} 2\log s + (s/2)\log(s/2) + \log 2 &\le 2\log(8\sqrt{Y}) + 4\sqrt{Y}\log(4\sqrt{Y}) + \log 2 \\ &\le 15\sqrt{Y}\log(Y+2). \end{aligned}$$

For the last bound, use $8\sqrt{Y} \le (Y+2)^2$, $4\sqrt{Y} \le (Y+2)^2$ and $\log 2 \le \sqrt{Y}\log(Y+2)$ for $Y \ge 1$; these give the stronger coefficient 13. The first elementary inequality follows on putting $t = \sqrt{Y} \ge 1$: $t^4 + 4t^2 + 4 - 8t$ is positive at 1 and increasing.

If $n \le X$, hypothesis (2) gives $15\sqrt{Y}\log(Y+2) < n$. If $n \ge X$, we have $n \ge 65536$ and

$$\log(n+2) \le \sqrt{n}/16.$$

Indeed, $\sqrt{t}/16 - \log(t+2)$ is increasing for $t \ge 1024$, since $t + 2 \ge 32\sqrt{t}$, and is positive at 65536 because $\log 65538 < 16$. This last strict inequality follows, for example, from $e > 5/2$ and $(5/2)^{16} > 65538$. Thus $15\sqrt{Y}\log(Y+2) \le 15n/16 < n$ also in this case. Since $\log 3 > 1$, condition (3) holds in both cases. Theorem 1.2 now applies. The positive coefficients and nonzero constant term of (1) make every real zero negative. □

For integer-only computation one may use the sufficient onset

$$N_{\mathrm{int}}(d) = \max\{65536, 16\lceil\sqrt{d+1}\rceil\lceil\log_2(d+3)\rceil\}.$$

Here $\lceil\log_2 m\rceil$ is the bit length of $m-1$, and the square-root ceiling is obtained from the integer square root. Because $\log 2 < 1$, this onset dominates (2).

The supplied `verify.py` computes this integer onset and certifies selected instances of (3) with rational logarithm enclosures. After reduction to $1 \le x \le 2$, it uses

$$\log x = 2\sum_{j\ge 0} \frac{z^{2j+1}}{2j+1}, \qquad z = \frac{x-1}{x+1},$$

with the omitted positive tail bounded geometrically. The analytic arguments above establish the degree and shift quantifiers; the supplementary calculations verify the scalar implementation.

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References

[1] Y.-O. Kim and J. Lee, *A note on the zeros of Jensen polynomials*, J. Korean Math. Soc. **59** (2022),775–787. doi:10.4134/JKMS.j210622.

[2] Y.-O. Kim, *Critical zeros and nonreal zeros of successive derivatives of real entire functions*, J. Korean Math. Soc. **33** (1996),657–667.

[3] W. Gontcharoff, *Recherches sur les dérivées successives des fonctions analytiques*, Ann. Sci. École Norm. Sup. (3) **47** (1930),1–78.

[4] N. Obreschkoff, *Sur une généralisation du théorème de Poulain et Hermite pour les zéros réels des polynômes réels*, Acta Math. Acad. Sci. Hungar. **12** (1961),175–184.

[5] M. Griffin, K. Ono, L. Rolen and D. Zagier, *Jensen polynomials for the Riemann zeta function and other sequences*, Proc. Natl. Acad. Sci. USA **116** (2019),11103–11110.

[6] M. J. Griffin, K. Ono, L. Rolen, J. Thorner, Z. Tripp and I. Wagner, *Jensen polynomials for the Riemann xi-function*, Adv. Math. **397** (2022),108186.

[7] C. O'Sullivan, *Zeros of Jensen polynomials and asymptotics for the Riemann xi function*, Res. Math. Sci. **8** (2021),46.

[8] J. Holland, *A new hyperbolicity wedge and a joint semicircle limit for Jensen polynomials of Riemann's xi-function*, arXiv:2608.08682v1 (2026).