Estimating and testing effective conductivity from spatial samples

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Luminary Oddities

November 23, 2026

Abstract

We study the effective tensor of a positive periodic scalar conductor when the observations are independent locations sampled with density proportional to its local conductivity and the mean conductivity is known. For every nonzero symmetric tensor contrast we determine the exact minimax estimation rate over Hölder classes. Uniform root-sample risk starts at $s = d/4$, and bounded-scale centered Gaussian calibration fails uniformly at equality. For a fixed direction we identify exactly the densities with zero influence variance. At this composite null, the minimax separation for detecting conductivity loss is $n^{-4s/(4s+d)}$ for every $s > 0$, including above the root-sample estimation threshold. A weighted Fourier quadratic statistic achieves that separation without estimating a corrector. For unknown smoothness, a single test attains the sharp iterated-logarithm adaptive separation.

1 Cell problem and statistical results

Effective conductivity describes the macroscopic response of a heterogeneous medium. We consider direct samples of spatial locations whose density is proportional to the microscopic conductivity. This observation model allows us to ask how accurately an effective tensor contrast can be recovered and how small a physical conductivity loss can be detected.

For positive semidefinite contrasts, the null of zero loss has a geometric characterization and admits exact conditional calibration. Detecting loss at this null is possible at a finer scale than estimating the tensor contrast in smooth classes. The proof uses a weighted Fourier statistic, with no arithmetic assumption on the direction being tested.

On the unit-volume flat torus $\mathbb{T}^d$, let the local scalar conductivity f be normalized by its known mean. The data are n independent locations with density f. The class $\mathcal{F}_s$ consists of densities with $m \le f \le M$, with $0 < m < 1 < M$, and $\|f\|_{C^s} \le B$ for $B > \|1\|_{C^s}$. At integer $s > 0$ use bounded classical derivatives through order s. At noninteger s use the usual Hölder norm. If the known mean is other than one, the physical effective tensor is the tensor below multiplied by that mean.

For $e \in \mathbb{R}^d$ define the periodic effective tensor K_f by

$$e^\top K_f e = \min_{\phi \in H^1(\mathbb{T}^d)} \int_{\mathbb{T}^d} f|e + \nabla\phi|^2. \tag{1}$$

The minimizer $\chi_{f,e}$ is unique after its spatial mean is fixed at zero. Its equation is $\operatorname{div}(f(e + \nabla\chi_{f,e})) = 0$, the scalar symmetric instance of the periodic corrector equation [2, §2.1]. We study this fixed-cell observation model. Sampling multiple random realizations of a medium is a different experiment [5]. We use the same inverse-operator idea as the diffusion metric studied in [4, (2.1)], with the corrector source set by the prescribed macroscopic direction.

Theorem 1. *For any fixed nonzero real symmetric d by d matrix C, write $T_C(f) = \operatorname{tr}(CK_f)$. Its minimax root mean squared error satisfies*

$$\inf_{\widehat{T}} \sup_{f \in \mathcal{F}_s} \left(\mathbb{E}_f |\widehat{T} - T_C(f)|^2\right)^{1/2} \asymp n^{-\min(1/2,\, 4s/(4s+d))}.$$

If $s > d/4$, an estimator is uniformly asymptotically linear with influence function

$$\psi_{f,C}(x) = \sum_{i,j=1}^{d} C_{ij}\, q_{f,i}(x) \cdot q_{f,j}(x) - T_C(f), \qquad q_{f,i} = e_i + \nabla \chi_{f,e_i}.$$

At an interior density this is the canonical gradient. At densities with nonzero influence variance the estimator is asymptotically normal with the canonical variance. If $s = d/4$, no estimator has a uniform centered standard Gaussian limit after division by a positive uniformly bounded scale, even on a fixed L^∞ neighborhood of 1 inside the class. The scale can be random with a uniform deterministic upper bound.

For a fixed nonzero direction e, let $D_e(f) = |e|^2 - e^\top K_f e$. It is nonnegative by testing (1) at the constant potential.

Theorem 2. *The following are equivalent for any $f \in \mathcal{F}_s$:*

$$D_e(f) = 0, \qquad \partial_e f = 0 \text{ in distributions}, \qquad \operatorname{Var}_f(\psi_{f,ee^\top}) = 0.$$

For testing $D_e(f) = 0$ against $D_e(f) \geq \delta$, the minimax separation order over $\mathcal{F}_s$ is $r_n = n^{-4s/(4s+d)}$ for all $s > 0$. Precisely, given $0 < \eta < 1/2$, there is a constant A_η and a sequence of tests for $\delta = A_\eta r_n$ whose worst type I and type II errors are each at most η for all sufficiently large n. For a sufficiently small $a_\eta > 0$, every test at $\delta = a_\eta r_n$ has sum of worst errors bounded below by $1 - \eta$ for all sufficiently large n.

The testing result is stated in the physical loss D_e. The null admits nonconstant conductivities if the torus has nonzero integer frequencies orthogonal to e. No arithmetic condition on the direction is needed for the stated separation.

2 Resolvent identities and upper estimation bound

Put $A_f = -\operatorname{div}(f\nabla)$ on the spatial mean-zero space, and for a pilot density p write $a = f - p$. Let $r_i = \operatorname{div}(a q_{p,i})$. In weak form the corrector difference is $\chi_{f,e_i} - \chi_{p,e_i} = A_f^{-1} r_i$. Polarizing the completion of the square in (1) gives

$$T_C(f) - T_C(p) = \int a \psi_{p,C} - \sum_{i,j} C_{ij} \langle r_i, A_f^{-1} r_j \rangle. \tag{2}$$

For example, in one direction evaluate the f-energy of the p-corrector and subtract the energy of its difference from the f-corrector. This gives exactly (2) with $C = ee^\top$, and finite-dimensional polarization gives the general identity.

The $q_{p,i}$ are uniformly bounded by divergence-form Schauder applied to the smooth positive Hölder coefficient and the C^α vector datum pe_i, for fixed $0 < \alpha < \min(s, 1)$. The divergence-data resolvent gradients are bounded on each of L^2, L^3, L^4 uniformly in p and f. For the two exponents above two, use the interior $W^{1,q}$ estimates [1, §3] on finitely many periodic charts, then absorb

the solution's L^q norm by interpolation with its L^2 energy bound. Uniform convergence of the bounds for smooth approximants gives the estimates for the Hölder coefficients. In particular, for $w_i = A_p^{-1} r_i$, $\|\nabla w_i\|_q \le C\|a\|_q$ for these exponents. The quadratic term of (2) is

$$Q_p(a) = -\sum_{i,j} C_{ij} \langle r_i, A_p^{-1} r_j \rangle.$$

It is represented by a uniformly bounded self-adjoint L^2 operator H_p on mean-zero functions, extended by zero on constants. The identity $A_f^{-1} - A_p^{-1} = -A_f^{-1} A_a A_p^{-1}$ with $A_a = -\operatorname{div}(a\nabla)$ and the L^3 bounds give the remainder $O(\|a\|_3^3)$. With one additional resolvent step the cubic term and remainder are

$$C_p(a,a,a) = \sum_{i,j} C_{ij} \int a\, \nabla w_i \cdot \nabla w_j, \qquad |R_4| \le C\|a\|_4^4.$$

Indeed, the final term of the inverse expansion is $A_p^{-1} A_a A_f^{-1} A_a A_p^{-1}$. For each pair its energy pairing is bounded by $C\|a\nabla w_i\|_2 \|a \nabla w_j\|_2 \le C\|a\|_4^4$. The polarized cubic is a finite sum of integrals of three linear images, each map bounded in L^2 and L^4. In addition, $\|\psi_{p,C} - \psi_{f,C}\|_2 \le C\|a\|_2$ by the energy bound for corrector differences and the uniform supremum bounds on $q_{p,i}, q_{f,i}$.

For clarity we give the common estimation argument, using classical quadratic influence corrections [3] and retaining the mixed resolution terms. A pilot based on an independent fixed fraction of the data can be chosen with

$$\mathbb{E}_f \|a\|_3^6 \lesssim n^{-6s/(2s+d)}, \qquad \mathbb{E}_f \|a\|_4^8 \lesssim n^{-8s/(2s+d)},$$

using the L^3 or L^4 version as needed. Smooth the empirical law at scale $h = n^{-1/(2s+d)}$ with a moment-canceling periodic kernel. Its pointwise bias is $O(h^s)$ and the pointwise moments of order $2q$ of the centered estimator are $O((nh^d)^{-q} + (nh^d)^{-(2q-1)})$ for $q = 3, 4$. Integration and Jensen give the claimed norms. Choose a measurable n^{-1}-approximate projection of the pilot into $\mathcal{F}_s$ in the chosen L^q norm using a fixed countable dense subset.

Let S be a real smooth Fourier cutoff of rank K, using a multiplier in $[0,1]$ equal to one near zero. It is self-adjoint and L^q-bounded at fixed exponents, and $\|(I-S)a\|_q \le CK^{-s/d}$ for $q = 2, 4$ by moment cancellation. Its represented basis functions have analysis operator of norm at most one and pointwise squared sum $O(K)$. Approximate H_p by $H_pS + SH_p - SH_pS$. Its bias in the quadratic form is $O(K^{-2s/d})$, since the omitted operator is $(I-S)H_p(I-S)$. Its rank is at most $2K$. Estimate this quadratic term by the corresponding order-two off-diagonal U-statistic centered against p in each slot. Conditional on p, the standard deviation is bounded by $C(\|a\|_2/\sqrt{n} + \sqrt{K}/n)$ by the first and second Hoeffding projections. The linear term uses the empirical mean of $\psi_{p,C}$ and has conditional variance $O(n^{-1})$.

For the cubic term, retain the terms with at most one $I - S$ argument. In expanded form these are the three terms with two S arguments and one identity argument, minus twice the all-S term. Each omitted term has at least two $I - S$ inputs. Place two of these in L^4 and the third input in L^2. The conditional bias is therefore $O(K^{-2s/d}\|a\|_2)$. Evaluate the resulting kernel on the product of three independent empirical signed measures centered against p, each of size proportional to n. For a kernel with j live slots and the others filled by a, its squared L^2 norm is $O(K^{j-1})$. To verify this, use the L^2 adjoint of the corresponding linear map in one live slot, choosing the unprojected slot if live, and use Bessel if that slot was projected. Each remaining live slot supplies a square function, whose L^4 norm is $O(\sqrt{K})$ by the Rademacher fourth moment and the uniform L^4 bound on the map. Each fixed slot has bounded L^4 norm. Hölder's inequality gives the claimed kernel bound. The conditional variance is thus $O(\sum_{j=1}^{3} K^{j-1}/n^j)$ by expanding each signed empirical measure

about its conditional mean $a\,dx$. Independent centered groups make the terms for different live sets orthogonal.

For $s \geq d/4$, the quadratic correction with $K \asymp n$ gives root-n risk, since $K^{-2s/d}$ and the L^3 pilot remainder are both $O(n^{-1/2})$ in root mean square. When $s > d/4$ use $n^{d/(4s)} \ll K \ll n$ instead, and cross-fit two folds. The root mean square bias and quadratic fluctuation are then $o(n^{-1/2})$. The influence stability just proved allows the centered empirical influence to be replaced by $\psi_{f,C}$, so weighting the two folds by their validation sizes gives the full-sample empirical mean and the asserted uniform expansion. At interior densities the tangent directions from smooth mean-zero additive perturbations are dense in $L_0^2(f)$, and (2) gives the path derivative. This identifies the canonical gradient. For any local smooth path the same expansion plus the triangular-array central limit theorem gives regularity at densities with positive variance.

For $s < d/4$ use both corrections, with ranks of orders $n^{2/(1+4x)}$ and n^{κ} respectively, where $x = s/d$ and

$$\kappa = \frac{2}{1+4x} - \frac{1}{2(1+2x)}.$$

Write $v = x/(1+2x)$ and $\gamma = 4x/(1+4x)$. The quadratic bias and standard deviation are $O(n^{-\gamma})$. The cubic bias is $O(n^{-2x\kappa - v}) = O(n^{-\gamma})$ in root mean square, while the fourth-order pilot remainder is $O(n^{-4v}) = o(n^{-\gamma})$. If $\kappa \leq 1$, the cubic variance is $O(n^{-1})$. Otherwise its dominant standard deviation is $O(n^{\kappa - 3/2}) = O(n^{-\gamma})$, since $3/2 - \kappa - \gamma = 1/2 - (1+4x)^{-1} + [2(1+2x)]^{-1} > 0$. This proves the upper estimation rate.

3 Oriented lower bounds for general tensor contrasts

At $p = 1$ the correctors vanish and the influence is identically zero. Write $f = 1 + a$ and use (2). The leading quadratic term for the deficit $\operatorname{tr} C - T_C(f)$ is

$$Q_C(a) = \sum_{k \in \mathbb{Z}^d \setminus \{0\}} \frac{k^\top C k}{|k|^2} \, |\widehat{a}(k)|^2. \tag{3}$$

Replacing A_f^{-1} by A_1^{-1} in the quadratic resolvent pairing makes an error at most $C_0 \|a\|_\infty \|a\|_2^2$ by the energy resolvent identity. For indefinite C, we choose bumps whose quotient $Q_C(a)/\|a\|_2^2$ stays away from zero with a fixed sign.

Fix a unit eigenvector v of C with nonzero eigenvalue λ. In a rotated interior cube use v as the first coordinate. In the unit template cube take a fixed smooth interior bump multiplied by $\sin(Lx_1)$, where L is a fixed large parameter. Call the result ζ. The L^2 norm of its transverse gradient is bounded independently of L. The norm of its first derivative is bounded below by a positive multiple of L for sufficiently large L, by integration of the oscillatory square of the bump. Given a small prescribed $\epsilon_0 > 0$, fix L such that

$$\|\nabla_\perp \zeta\|_2 \leq \epsilon_0 \|\partial_1 \zeta\|_2.$$

Use $m_h \asymp h^{-d}$ separated cells of side h in that cube. In cell j take the translated and scaled profile $V_j = \rho h^{s+1} \zeta((x - x_j)/h)$, where the coordinates in the argument are the rotated ones. Let $a_j = \partial_v V_j$ and use alternatives $f_\epsilon = 1 + a_\epsilon$ with $a_\epsilon = \sum_j \epsilon_j a_j$ and $\epsilon_j = \pm 1$. For sufficiently small fixed $\rho > 0$, all belong to $\mathcal{F}_s$ by the scaled supports. Each a_j has zero mass and $\|a_j\|_2^2 \asymp \rho^2 h^{2s+d}$, with constants allowed to depend on the fixed template. For $V = \sum_j \epsilon_j V_j$ we have $a_\epsilon = \partial_v V$ and $\|\nabla_\perp V\|_2 \leq \epsilon_0 \|a_\epsilon\|_2$ by disjointness.

Since $Cv = \lambda v$, its mixed longitudinal and transverse entries vanish in this frame. Put $k_\perp = k - (v\cdot k)v$. The difference between the symbol in (3) and λ has magnitude at most $(\|C\|+|\lambda|)|k_\perp|^2/|k|^2$. By Parseval and $a_\epsilon = \partial_v V$,

$$\sum_{k\neq 0} \frac{|k_\perp|^2}{|k|^2}|\widehat{a}_\epsilon(k)|^2 \le \|\nabla_\perp V\|_2^2 \le \epsilon_0^2\|a_\epsilon\|_2^2.$$

This computation uses the Fourier modes of the original torus. The rotation serves only to construct the compactly supported profiles in the local cube. Thus for ϵ_0 small enough, the leading quadratic deficit has sign λ for every sign vector and magnitude at least $|\lambda|\|a_\epsilon\|_2^2/2$. Since $\|a_\epsilon\|_\infty = O(\rho h^s)$, the true contrast has the opposite sign and a gap from $T_C(1)$ of order $\rho^2 h^{2s}$, uniformly in signs as $h \to 0$.

The uniform mixture Q of the n-sample laws of these alternatives satisfies, under the uniform product law P_0,

$$1 + \chi^2(\mathsf{Q}, \mathsf{P}_0) \le \prod_j \cosh(n\|a_j\|_2^2) \le \exp(C_1 n^2 \rho^4 h^{4s+d}). \tag{4}$$

The first inequality follows by integrating two likelihood ratios, obtaining $(1 + \sum_j \epsilon_j\epsilon_j'\|a_j\|_2^2)^n$, and averaging its exponential bound over the independent signs. For $h = tn^{-2/(4s+d)}$ and sufficiently small fixed t, the total variation is bounded away from one. The common oriented parameter gap yields the lower bound $cn^{-4s/(4s+d)}$ by thresholding an estimator halfway into the gap.

For a parametric lower bound choose a fixed sufficiently small cell scale and a corresponding smooth a with $T_C(1+a) \neq T_C(1)$, keeping all inequalities defining the class strict along $1 + \tau a$, $0 \le \tau \le 1$. Differentiability from (2) and the mean value theorem give nonzero derivative at some interior τ_*. The pair τ_* and $\tau_* + b/\sqrt{n}$ has parameter gap of order $n^{-1/2}$ with product chi-square bounded by $\exp(C_2 b^2) - 1$. This proves the parametric lower bound as well.

At $s = d/4$ put $h = tn^{-1/d}$. Multiply the contrast and estimator by a common sign if necessary so all alternative shifts are positive. The smallest shift on the root-n scale is at least $c\rho^2 t^{d/2}$. Equation (4) gives total variation at most $C_3\rho^2 t^d$. Under uniform centered Gaussian calibration the null event that the estimator lies below the null parameter has probability $1/2 + o(1)$. Under every alternative this probability is at most $\Phi(-c\rho^2 t^{d/2}/M_0) + o(1)$, where M_0 bounds the scale. This follows also for bounded positive random scales by event inclusion. For a sufficiently small fixed t the probability gap exceeds the total variation bound. This proves the last claim of Theorem 1 even on any fixed L^∞ neighborhood of 1.

4 No-loss directions and optimal tests

Fix $e \neq 0$. In (2) with $p = 1$ the exact deficit is

$$D_e(f) = \langle \partial_e f, A_f^{-1}\partial_e f\rangle.$$

Hence $D_e(f) = 0$ exactly when $\partial_e f = 0$ in distributions. Under this condition the corrector vanishes and the influence is identically zero. Conversely, zero influence variance means $|e + \nabla\chi_{f,e}|^2$ is constant since its continuous version agrees with a constant almost everywhere and $f \ge m$. At a maximum of the C^1 periodic corrector its gradient vanishes, so that constant is $|e|^2$. Integrating against unweighted volume gives $\int |\nabla\chi_{f,e}|^2 = 0$, since the mean gradient vanishes. The corrector therefore vanishes everywhere and the weak corrector equation yields $\partial_e f = 0$.

The order bounds $mA_1 \le A_f \le MA_1$ give

$$M^{-1}Q_e(f) \le D_e(f) \le m^{-1}Q_e(f), \qquad Q_e(f) = \sum_{k\neq 0} \frac{(e\cdot k)^2}{|k|^2}|\widehat{f}(k)|^2. \tag{5}$$

For an integer J let $Q_{e,J}$ be the truncated sum over $0 < |k|_\infty \le J$. The usual L^2 Fourier approximation bound gives $0 \le Q_e - Q_{e,J} \le CJ^{-2s}$ uniformly over the class. Estimate $Q_{e,J}$ with the order-two off-diagonal U-statistic $\widehat{Q}_{e,J}$ for the real kernel

$$B_{e,J}(x,y) = \sum_{0<|k|_\infty\le J} \frac{(e\cdot k)^2}{|k|^2} \exp\big(2\pi i k\cdot(x-y)\big).$$

Its mean is $Q_{e,J}$. Its squared kernel norm in unweighted volume is $O(J^d)$. Its first Hoeffding projection has squared norm at most $CQ_{e,J}$ in $L^2(f)$, by Parseval, the bounded weights, and $f \le M$. Thus

$$\mathrm{Var}_f(\widehat{Q}_{e,J}) \le C\big(Q_{e,J}/n + J^d/n^2\big). \tag{6}$$

At the null, $Q_{e,J} = 0$, and the statistic is degenerate for every J. With $J \asymp n^{2/(4s+d)}$ we have $J^{-2s} \asymp J^{d/2}/n \asymp r_n$. Reject when $\widehat{Q}_{e,J} > L_\eta r_n$. By (6), L_η can be chosen to make type I error at most η. Under $D_e(f) \ge A_\eta r_n$, (5) and the tail bound give $Q_{e,J} \ge (mA_\eta - C)r_n$. Choose A_η sufficiently large so this exceeds twice the rejection threshold. Dividing (6) by the squared gap and using $nr_n \to \infty$ makes the type II error at most η for all sufficiently large n.

For the converse take $C = ee^\top$ in the preceding lower-bound construction. Every alternative has positive deficit of order $\rho^2 h^{2s}$. Use $h = tn^{-2/(4s+d)}$ for sufficiently small fixed t that the mixture has total variation below η. The null class contains the uniform density, so every test has worst-error sum at least $1-\eta$ for a sufficiently small multiple of r_n. This proves Theorem 2.

5 Testing with unknown smoothness

Fix $0 < s_- < s_+ < \infty$ and let s be unknown in this interval. Write $\mathcal{F}_s$ with common positivity and Hölder bounds as above. For all sufficiently large n put $\ell_n = \log\log n$ and

$$r_n^{\mathrm{ad}}(s) = \left(\frac{\sqrt{\ell_n}}{n}\right)^{4s/(4s+d)}.$$

Theorem 3. *There is a single test with worst type I error at most η over the union of no-loss nulls for $s \in [s_-, s_+]$, and worst type II error at most η over $f \in \mathcal{F}_s$ and $D_e(f) \ge A_\eta r_n^{\mathrm{ad}}(s)$, simultaneously over the same interval. For a sufficiently small $a > 0$, the sum of the worst errors of any test over this null and alternatives at $D_e(f) \ge a r_n^{\mathrm{ad}}(s)$ is at least $1 - o(1)$.*

Multi-resolution testing and the adaptive iterated logarithm are classical in nonparametric testing [8, 6]. We need a threshold valid under the entire directional null and an adaptive lower bound in the physical loss, which can assign very small weight to directions nearly orthogonal to e.

For the upper bound use the same $B_{e,J}$ kernel on two independent halves, averaging over all pairs across the halves. The mean and the variance bound (6) remain valid by expansion around the true law in each half. Under every null the kernel is canonical. Write $R = J^d$. Its $L^2(f\otimes f)$ norm and maximum row $L^2(f)$ norm are at most $C\sqrt{R}$, and its absolute supremum is at most CR. Its operator norm on $L^2(f)$ is bounded, since under the isometry to unweighted L^2 it is a uniformly

bounded Fourier multiplier conjugated by multiplication by $\sqrt{f}$ on both sides. The canonical two-sample inequality of Giné, Latała and Zinn [7, Theorem 3.3], applied to the two equally sized halves (discard one observation if necessary), therefore gives a tail at most $C\exp(-t)$ at threshold

$$C\left(\frac{\sqrt{Rt}}{n}+\frac{t}{n}+\frac{\sqrt{R}\,t^{3/2}}{n^{3/2}}+\frac{Rt^2}{n^2}\right).$$

The four unnormalized bounds in that inequality are respectively $O(n\sqrt{R})$, $O(n)$, $O(\sqrt{nR})$, and $O(R)$.

Take the union over dyadic J between n^{c_0} and n^{c_1}, where

$$0<c_0<\frac{2}{4s_+ + d}, \qquad \frac{2}{4s_- + d}<c_1<\frac{2}{d}.$$

There are $O(\log n)$ resolutions. With t a sufficiently large constant times ℓ_n, the first threshold term dominates the other three uniformly in this range. A union bound therefore controls type I error at all nulls using thresholds of order $\sqrt{\ell_n R}/n$. For each s the grid contains a J within a constant factor of $(n^2/\ell_n)^{1/(4s+d)}$. Its approximation tail and rejection threshold are both $O(r_n^{\mathrm{ad}}(s))$ uniformly in s. The type II bound at this resolution follows from (5), (6) and the argument for Theorem 2. In particular, $nr_n^{\mathrm{ad}}(s)\to\infty$ uniformly on the indicated compact interval.

For the lower bound use the oriented templates for $C=ee^\top$, with the same fixed ρ and template at each resolution. Select $L_n\asymp \log n/\log\ell_n$ scales h_j in the range parametrized by

$$h_j^{4s_j+d}=\ell_n/n^2, \qquad s_j\in[s_-,s_+],$$

so that the ratio of successive scales is at least $\ell_n^{4/d}$. This is possible by spacing $\log(1/h_j)$ uniformly at increments $(4/d)\log\ell_n$. Write P_j for the within-scale sign mixture of product laws and Z_j for its density relative to the uniform product law. All alternatives on level j give $D_e(f)\ge c\rho^2 h_j^{2s_j}=c\rho^2 r_n^{\mathrm{ad}}(s_j)$ with uniform constants. By (4), $\mathbb{E}_0 Z_j^2\le\exp(C\rho^4\ell_n)$.

For two different scales h_j,h_k, let U_{jk} have entries $n\langle a_{j,b},a_{k,c}\rangle$ for the bumps at the respective levels. Denote their common within-level L^2 norms by g_j,g_k and the numbers of cells by m_j,m_k. Normalized bumps are orthonormal within each level by disjointness, so

$$\|U_{jk}\|_{\mathrm{op}}\le ng_jg_k, \qquad \|U_{jk}\|_{\mathrm{HS}}^2\le n^2g_j^2g_k^2\min(m_j,m_k).$$

The squared right side of the operator estimate tends to zero, since $n^2g_j^4m_j\lesssim\rho^4\ell_n$ and all cell counts grow polynomially. The Hilbert–Schmidt bound is at most

$$C\rho^4\ell_n\left(\frac{\min(m_j,m_k)}{\max(m_j,m_k)}\right)^{1/2}=O(\rho^4/\ell_n)$$

by scale spacing. Integrating two likelihood ratios and using $(1+x)^n\le\exp(nx)$ gives $\mathbb{E}_0 Z_jZ_k\le\mathbb{E}_{\epsilon,\epsilon'}\exp(\epsilon^\top U_{jk}\epsilon')$. This last expression is at most $\det(I-U_{jk}^\top U_{jk})^{-1/2}$ when the operator norm is less than one. To see this, take the expectation over one sign vector using $\cosh x\le e^{x^2/2}$. Represent the resulting exponential of a squared norm as a Gaussian Laplace transform and use the same sign inequality for the other vector. The determinant is then the Gaussian quadratic integral. With the stated operator bound, the determinant expression is at most $\exp(C\|U_{jk}\|_{\mathrm{HS}}^2)=1+o(1)$ uniformly in distinct levels.

The equally weighted mixture across all levels therefore has chi-square distance from the uniform product law at most $L_n^{-1}\exp(C\rho^4\ell_n)+o(1)=o(1)$ for sufficiently small fixed ρ. The uniform density is a common null. Total variation convergence gives the stated lower bound on worst errors. The weighted directional signal is bounded below at each scale, with no arithmetic assumption on the direction.

6 Exact calibration and positive semidefinite contrasts

For the directional tests, exact finite-sample calibration is available through the compact subgroup $H_e = \overline{\{te \bmod \mathbb{Z}^d : t \in \mathbb{R}\}}$. A continuous density satisfying $\partial_e f = 0$ is invariant under H_e, by its Fourier coefficients. Conditional on the observed quotient locations in $\mathbb{T}^d/H_e$, the samples under the null are independent and Haar-uniform on their respective cosets. This is the Haar disintegration, since the invariant density is constant on each coset. Thus the conditional law of the statistic under the null is obtained by adding an independent Haar element of H_e to each sample point. Use the single-resolution statistic from Theorem 2, or the maximum of the threshold-normalized split statistics in Theorem 3. The randomized upper-tail probability, using an independent uniform variable at an atom, gives an exact level-α test for any $0 < \alpha < 1$.

Both separation rates remain valid for this conditional calibration. Indeed, under any alternative the unconditional shifted sample is independent with density $\bar{f}$, the Haar average of f over H_e. It belongs to the no-loss null and inherits the density bounds. Choose a deterministic threshold b for the statistic so that its unconditional exceedance probability under every null is at most α_0. Markov's inequality bounds the probability that the conditional shifted exceedance probability at b is at least α by α_0/α. Whenever the observed statistic exceeds b and that shifted probability is less than α, even the atom-randomized tail probability at the observed statistic is less than α. Thus the conditional test's type II error is at most the deterministic test's type II error plus α_0/α. Arbitrarily small fixed α_0 is available at the same separation orders.

The no-loss results also hold for any fixed nonzero positive semidefinite contrast C. Put $D_C = \operatorname{tr} C - T_C$ and use the Fourier weights $k^\top C k/|k|^2$. If $C = \sum_\ell \lambda_\ell v_\ell v_\ell^\top$ in an orthonormal eigenbasis, the comparison (5) follows by summing the directional inequalities with nonnegative weights. The null is equivalent to $\partial_{v_\ell} f = 0$ for all $\lambda_\ell > 0$. For the converse from zero influence variance, the corresponding weighted sum of squared corrector fields must be constant. Its unweighted mean is

$$\operatorname{tr} C + \sum_\ell \lambda_\ell \int |\nabla \chi_{f,v_\ell}|^2,$$

because every correction has mean gradient zero. Its weighted mean is $T_C(f) \le \operatorname{tr} C$. Both means equal the constant, so every positive-weight corrector vanishes. The estimation and testing bounds above apply to the nonnegative bounded symbol $k^\top C k/|k|^2$ by the same variance and operator estimates. The oriented mixture lower bound uses any strictly positive eigenvalue. For exact calibration replace H_e by the closure of the image of range(C) in the torus, under which the null densities are invariant.

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