Energy-Filtered Dynamical Localization in Constructive Dirichlet Towers

Scott Schubr
Luminary Oddities

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Abstract

For every dimension $d \geq 2$ and every measurable spatial weight $w \geq 1$ essentially bounded on bounded sets, we construct a connected tower of enlarging cubes whose ordinary Dirichlet Laplacian H has a complete orthonormal eigenbasis and essential spectrum $[0, \infty)$. Given any finite nondecreasing $r(E) \geq 1$ tending to infinity, the tower can be chosen so that weighted local-source evolution below energy E is bounded uniformly in time by $C_K \min\{r(E), \sqrt{E+1}\}$. More generally, any prescribed Borel spectral filter $0 < h \leq 1$ uniformly vanishing at high energy admits a uniform weighted local-source bound. Local Neumann approximation targets and a summable spectral-projection budget supply the uniform source estimate. With a rational initial room and a supplied computable sequence of local weight upper bounds and a supplied computable filter-tail modulus, the geometry and its finite spectral certificates can be produced by a procedure terminating at every stage. We prove this using crack-aware conforming primal and flux approximation, certified whole-resolvent compression and strict whole-cluster spectral tests. An explicit logarithmic aperture bound applies in the plane; in higher dimensions a heat-semigroup argument gives a linear sufficient bound. We also prove that the corresponding unfiltered all-energy bound fails for uniformly diverging weights. Thus an arbitrarily slow divergent energy factor can be prescribed, whereas a uniformly bounded factor is impossible. The geometry depends on the prescribed rate.

Keywords: Dirichlet Laplacian; dynamical localization; quasi-conical domain; spectral approximation; constructive geometry.

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1 Introduction

The Dirichlet Laplacian on an open set containing arbitrarily large balls has spectrum $[0, \infty)$, but its spectral type can depend strongly on the connecting geometry. Krejčiřík and Lotoreichik [12] constructed quasi-conical domains from enlarging cubes joined through small windows, with an embedded eigenvalue and no absolutely continuous spectral part. A subsequent adaptive construction [14] gives a complete eigenfunction basis in this geometry. Here we obtain uniform weighted bounds for the evolution of bounded-source inputs and a terminating construction of the geometry from effective finite data.

Eigenfunction decay and dynamical localization are distinct properties; their relationship and the role of eigenfunction constants are developed in [10, 11]. Deterministic dynamical localization is also known for other operators, including limit-periodic lattice operators [7]; strong dynamical localization for random Dirichlet quantum waveguides near a spectral edge is established in [8, Section 4.3]. Our setting is the ordinary Dirichlet Laplacian on one connected Euclidean domain, with the entire half-line as its essential spectral set. The spatial weight and energy rate are prescribed before the domain is chosen. All bounds below are operator norms, with constants depending on the bounded source region. No simplicity, finite multiplicity or translation-uniform constant is assumed.

For any prescribed real Borel $h : [0, \infty) \to (0, 1]$ with $\sup_{\lambda > T} h(\lambda) \to 0$, the main estimate is

$$\sup_{\|g\|_\infty \leq 1} \|w g(H) h(H) \mathbf{1}_{D_k}\| \leq A_k + 1 + 2^{1-k},$$

where D_k is the first k rooms and A_k is a predetermined local upper bound for w. The construction uses finite roomwise Neumann modes as approximation targets. This controls the complete class of locally sourced inputs after spectral smoothing. Finite reducing blocks then give a uniformly summable weighted series. With $h(\lambda) = \max\{r(\lambda)^{-1}, (1+\lambda)^{-1/2}\}$, the same estimate yields energy-filtered evolution uniformly for all times and all finite energy ceilings on the same domain, with the prescribed energy factor $\min\{r(E), \sqrt{E+1}\}$. A logarithmic choice yields every positive fractional smoothing order on one domain. The all-energy obstruction shows that bounded dependence on the energy ceiling is impossible for uniformly diverging spatial weights. These statements concern growth with the energy ceiling, not growth of energy in time.

To make the geometric selection effective, we prove a finite-domain certificate procedure in the exact Dirichlet screen geometry. A roomwise Neumann compression accounts explicitly for the omitted continuum modes. Conforming scalar and $H(\mathrm{div})$ flux residuals certify the retained columns. The required density at crack tips is proved by an orthogonal-complement argument and capacity cutoffs on the finite boundary skeleton. Standard first-order least-squares methods [6, 4] then provide a terminating finite certificate search. Compact spectral computability and rough-domain algorithms have substantial prior theory [5, 13]; our input is explicit rational screen incidence, and every spectral test includes the continuum error and the zero complement.

The aperture estimates use variational comparison and capacitary cutoffs, within the established domain-perturbation framework [9, 2]. Sharp singular-perturbation asymptotics are available in related settings [1, 3]. The estimates here are sufficient uniform bounds with explicit constants. They combine with rational spectral contours to choose each positive window and retain whole multiplicities. In the constructive statement the weight is specified through computable local upper bounds, together with a computable uniform tail modulus for the filter. Merely knowing that an arbitrary measurable weight is locally bounded or that a filter vanishes at infinity supplies no such algorithmic data.

Section 2 proves the weighted dynamical theorem and its prescribed-rate consequences and high-energy limitation. Section 3 gives quantitative aperture and spectral-projection bounds. Section 4 proves finite certificate production and the terminating geometric recursion.

2 Prescribed-weight dynamics

We first strengthen the complete eigenbasis construction of [14]. The cube and window geometry and its finite perturbation input originate in [12, 9]. Pure-point spectral type, even with exponentially decaying individual eigenfunctions, does not by itself give uniform bounds for evolving localized data [10, 11]. Deterministic constructions with uniform dynamical localization are also known for limit-periodic lattice operators [7]. Here the operator is the ordinary Dirichlet Laplacian on a connected quasi-conical Euclidean domain. We prescribe a spatial weight and control an entire local source space, with a spectral smoothing factor or a finite energy cutoff.

Write $Q_a(x) = x + (-a, a)^d$ and $e_1 = (1, 0, \dots, 0)$. For an open set D, let H_D be the self-adjoint operator of the form

$$h_D(u, v) = \int_D \nabla u \cdot \overline{\nabla v}, \qquad \operatorname{dom} h_D = H_0^1(D).$$

All form functions are extended by zero when comparing domains. We use operator norms on the physical L^2 spaces. Multiplication by a measurable weight w is understood as its closed maximal operator; a bounded weighted composition includes the assertion that its range lies in $\operatorname{dom} w$.

Theorem 2.1. *Fix $d \geq 2$, $a_1 > 0$, and a measurable function $w : \mathbb{R}^d \to [1, \infty)$ which is essentially bounded on bounded sets. Let $h : [0, \infty) \to (0, 1]$ be Borel and satisfy*

$$\lim_{T \to \infty} \sup_{\lambda > T} h(\lambda) = 0.$$

There exist half-widths a_n, centres $c_n e_1$, and apertures δ_n satisfying

$$\begin{aligned} c_1 = a_1, \qquad c_{n+1} = c_n + a_n + a_{n+1}, \\ a_n + 1 < a_{n+1} < a_n + 2, \qquad 0 < \delta_n < \min\{a_n, 2^{-n}\}, \end{aligned} \tag{1}$$

such that, on putting

$$\begin{aligned} \omega_n &= Q_{a_n}(c_n e_1), \\ \mathcal{W}_n &= Q_{\delta_n}((c_n + a_n)e_1), \\ D_n &= \bigcup_{m \le n} \omega_m \cup \bigcup_{m < n} \mathcal{W}_m, \qquad \Omega = \bigcup_n D_n, \end{aligned} \tag{2}$$

the Dirichlet Laplacian $H = H_\Omega$ has a complete orthonormal eigenbasis and

$$\sigma(H) = \sigma_{\mathrm{ess}}(H) = [0, \infty).$$

Its eigenvalues are positive, and their set is countable and dense in $[0, \infty)$. Both continuous spectral subspaces vanish.

Define the predetermined envelopes

$$B_n = 2\sqrt{d}\, n(a_1 + n), \qquad A_n = \operatorname*{ess\,sup}_{|x| \le B_n} w(x), \qquad C_k = A_k + 1 + 2^{1-k}. \tag{3}$$

For every $k \ge 1$,

$$\sup_{\substack{g \text{ bounded Borel on } [0,\infty) \\ \|g\|_\infty \le 1}} \left\| w\, g(H) h(H) \mathbf{1}_{D_k} \right\| \le C_k. \tag{4}$$

Every bounded measurable source set K is contained in some D_k up to a null subset of Ω, so the same bound holds with the rightmost indicator replaced by $\mathbf{1}_{K \cap \Omega}$.

The domain is selected after the spatial weight and spectral filter have been fixed. That one domain works for every source set and Borel function in the statement. The filter need not be continuous or monotone. The spatial indicator acts first and the filter second; the filtered state need not remain compactly supported. No spatial indicator is commuted through $h(H)$.

In particular, taking $h(\lambda) = (1 + \lambda)^{-1/2}$ gives

$$\sup_{\|g\|_\infty \le 1} \|w g(H)(H + 1)^{-1/2} \mathbf{1}_{D_k}\| \le C_k$$

and the finite-energy estimate

$$\|w g(H) \mathbf{1}_{[0,E]}(H) \mathbf{1}_{D_k}\| \le C_k \sqrt{E + 1} \qquad (E \ge 0). \tag{5}$$

Indeed, apply (4) to the bounded multiplier $g(\lambda)\mathbf{1}_{[0,E]}(\lambda)\sqrt{(1 + \lambda)/(1 + E)}$. All dynamical estimates below are uniform in time on taking $g(\lambda) = e^{-it\lambda}$.

The theorem concerns spectral type, rather than a compact resolvent or an eigenvalue at every point of the half-line. Its source constants need not be translation uniform.

2.1 Finite attachment and whole spectral clusters

For a bounded open set D, zero extension embeds $H_0^1(D)$ in H_0^1 of a surrounding box. Rellich compactness gives a compact resolvent, and the box Poincaré inequality gives a positive Dirichlet spectral bottom. When $0 < \delta_n < a_n < a_{n+1}$, both off-interface halves of $\mathcal{W}_n$ are already contained in adjacent rooms. Windows therefore change the disjoint room union only on null interface sets. The common Hilbert spaces are

$$\mathcal{H} = L^2(\Omega) = \bigoplus_{m \ge 1} L^2(\omega_m), \qquad \mathcal{H}_n = L^2(D_n) = \bigoplus_{m \le n} L^2(\omega_m). \tag{6}$$

The space $\mathcal{H}$ may be defined after the induction: every finite comparison uses only rooms already chosen, and later zero extension does not change its norm. Write R_n for the zero extension of $(H_{D_n} + 1)^{-1}$ to $\mathcal{H}$.

Lemma 2.2. *Fix D_n and a next cube ω_{n+1} of half-width $a > a_n + 1$. At their interface centre $p = (c_n + a_n)e_1$, put*

$$U_\delta = D_n \cup \omega_{n+1} \cup Q_\delta(p), \qquad U_0 = D_n \cup \omega_{n+1}.$$

As $\delta \downarrow 0$, on their common almost-everywhere L^2 space,

$$(H_{U_\delta} + 1)^{-1} \longrightarrow T_0 := (H_{D_n} + 1)^{-1} \oplus (H_{\omega_{n+1}} + 1)^{-1}$$

in operator norm.

Proof. All domains lie in a fixed bounded box. The zero-extended form spaces $V_\delta = H_0^1(U_\delta)$ decrease as the aperture closes, and their intersection is $V_0 = H_0^1(U_0)$. We verify the exceptional point in this assertion. A point has zero H^1 capacity in every $d \geq 2$: cutoffs vanishing near p and equal to one away from a shrinking neighbourhood have gradient energy $O(r^{d-2})$ for $d > 2$, and logarithmic cutoffs between radii r and $\sqrt{r}$ have energy $O(1/|\log r|)$ for $d = 2$.

If $u \in \bigcap_{\delta>0} V_\delta$, truncate its real and imaginary parts to bounded H^1 functions. Multiply a fixed truncation by these point cutoffs. The products converge in H^1: the cutoff gradient term is bounded by the truncation's supremum norm, and the remaining errors have shrinking support and integrable energy. For each fixed cutoff, choose δ so small that the entire added window is in its zero region. Multiplying smooth compact approximations in U_δ then gives functions supported inside U_0. Thus the product belongs to V_0. Closedness, followed by removal of the truncation, proves $u \in V_0$. The reverse inclusion is immediate. Earlier windows are fixed and separated from p.

For fixed f, the variational solutions are bounded in the ambient box's H_0^1 norm. Every weak limit lies in V_0 and, on testing against V_0, solves the limiting equation. The energy identity gives strong H^1, hence L^2, convergence. The resolvents are collectively compact by the same ambient bound. This upgrades strong to norm convergence for these self-adjoint operators: if $f_m \rightharpoonup f$ are unit inputs, every strong cluster limit of $T_m f_m$ equals $T_0 f$, because $\langle T_m f_m, h\rangle = \langle f_m, T_m h\rangle$ and $T_m h \to T_0 h$. Compactness also gives $T_0 f_m \to T_0 f$. A norm-failing sequence is therefore impossible. The limiting form on the disjoint open union is the direct sum stated above. □

Lemma 2.3. *Suppose $P_1 \leq \cdots \leq P_s$ are finite-rank spectral projections of R_n, selecting whole positive eigenspaces. A next half-width $a \in (a_n + 1, a_n + 2)$ can be chosen so that every selected eigenvalue is nonresonant with the new cube resolvent. For every $\eta > 0$, a sufficiently small positive aperture then gives nested finite-rank spectral projections P'_j of R_{n+1} with*

$$\operatorname{rank} P'_j = \operatorname{rank} P_j, \qquad \|P'_j - P_j\| \leq \eta.$$

Any additional positive upper bound on the aperture may be imposed.

Proof. The cube Dirichlet eigenvalues are $\pi^2|\nu|^2/(4a^2)$, $\nu \in \mathbb{N}^d$. For a selected resolvent value $\mu \in (0, 1)$, resonance excludes only

$$a = \frac{\pi}{2}\sqrt{\frac{|\nu|^2}{\mu^{-1} - 1}}.$$

Finitely many countable forbidden sets cannot fill the specified interval. After choosing a, each selected value is isolated in the disconnected compact resolvent T_0, with exactly its old eigenspace. Choose disjoint circles around the distinct selected values, excluding zero. The old projections are nested sums over these circles. Lemma 2.2 and the resolvent identity give norm convergence of the Riesz projections. Their ranks agree once the distance is below one, since projections at norm distance less than one have injective restrictions in both directions. Taking the same nested circle subsets preserves nestedness. A multiple eigenvalue may split inside its circle; its whole cluster is retained. There are only finitely many constraints. □

2.2 The weighted projection induction

The bounded-interval choice gives $a_n \le a_1 + 2(n-1)$ and

$$c_n + a_n = 2\sum_{m\le n} a_m \le 2na_1 + 2n(n-1).$$

Every transverse coordinate has absolute value at most a_n. Thus $D_n \subset \{|x| < B_n\}$ for the predetermined radii in (3); A_n is finite and nondecreasing.

Put

$$\varepsilon_n = \frac{2^{-n}}{1+A_{n+1}}, \qquad \sup_{\lambda>Z_n} h(\lambda) \le \varepsilon_n/4, \qquad L_n \ge \max\{n, 16Z_n\varepsilon_n^{-2}\}, \tag{7}$$

where $Z_n \ge n$ and L_n may be chosen rational. The tail condition on h guarantees these finite choices before the next room and aperture are selected. On each room choose its orthonormal Neumann eigenbasis, with energies $\pi^2|\nu|^2/(4a_m^2)$, $\nu \in \{0,1,\ldots\}^d$. Zero extensions of these modes are L^2 approximation targets. Let F_n retain all modes of energy at most L_n in the first n rooms, or any finite larger selection. Write $M_n = \operatorname{rank} F_n$ and set

$$\rho_n = \frac{\varepsilon_n}{2M_n}, \qquad \eta_n = 2^{-(n+2)} \min\{1, A_{n+1}^{-1}, \rho_1, \ldots, \rho_n\}. \tag{8}$$

This tolerance is convenient for the rational construction below; $\sqrt{M_n}\rho_n \le \varepsilon_n/2$.

At stage n maintain nested whole-eigenspace spectral projections $P_{1,n} \le \cdots \le P_{n,n}$ of R_n. Choose the new projection $Q_n = P_{n,n}$ as a positive cutoff

$$Q_n = \mathbf{1}_{[\tau_n,\infty)}(R_n), \qquad 0 < \tau_n \notin \sigma(R_n),$$

containing $P_{n-1,n}$ when $n > 1$, and satisfying

$$\|(I-Q_n)b\| \le \rho_n/2 \quad \text{for every orthonormal target } b \text{ in } F_n. \tag{9}$$

These finitely many constraints are possible because positive cutoffs of the compact, injective resolvent tend strongly to the identity on $\mathcal{H}_n$. In particular Q_n has positive finite rank. Apply Lemma 2.3 to choose the next room and a positive aperture satisfying (1), with

$$\|P_{j,n+1} - P_{j,n}\| \le \eta_n \qquad (1 \le j \le n). \tag{10}$$

Then choose Q_{n+1} at the following stage. Every constraint uses finite data already fixed; the radius envelope for the next room is known before choosing its width.

Two simultaneous estimates are

$$\sum_{l\ge n} \eta_l \le \rho_n 2^{-(n+1)} \le \rho_n/4, \qquad \sum_{l\ge j} A_{l+1}\eta_l \le 2^{-(j+1)} \le 1. \tag{11}$$

The first uses the occurrence of ρ_n in every later minimum; the second uses the weight cap. No future value of ρ_l is needed when choosing η_n.

2.3 The physical operator and completeness

Lemma 2.4. *The zero-extended finite resolvents satisfy $R_n \to R = (H_\Omega + 1)^{-1}$ strongly on $\mathcal{H}$.*

Proof. The spaces $V_n = H_0^1(D_n)$ are increasing closed subspaces of $V = H_0^1(\Omega)$ for the energy inner product

$$a(u,v) = \int_\Omega (\nabla u \cdot \overline{\nabla v} + u\overline{v}).$$

Their union is dense: the compact support of a smooth compactly supported function in Ω lies in one D_N, by a finite subcover of the increasing open exhaustion. The variational equations make $R_n f$ the energy-orthogonal projection of Rf onto V_n. These projections converge in energy norm, hence in L^2. No operator-norm convergence on the infinite tower is used. □

For fixed j, (10) gives a norm limit P_j of $P_{j,n}$. It is an orthogonal projection, and has the same finite rank as Q_j: their distance is at most $\sum_{n\geq j}\eta_n \leq 2^{-(j+1)} < 1$. Nestedness passes to the limit. Since $R_nP_{j,n} = P_{j,n}R_n$, Lemma 2.4, norm convergence of the projections and $\|R_n\| \leq 1$ imply

$$RP_j = P_jR. \tag{12}$$

For example, the difference of $R_nP_{j,n}f$ and RP_jf is bounded by $\|P_{j,n}-P_j\|\|f\|+\|(R_n-R)P_jf\|$. Thus P_j reduces H and commutes with $S = h(H)$ and every bounded Borel function of H.

For each target in F_n, (9) and (11) give

$$\|(I - P_n)b\| \leq \rho_n. \tag{13}$$

Every fixed room Neumann mode belongs to all sufficiently late target lists, since $L_n \to \infty$; also $\rho_n \to 0$. The room modes together are an orthonormal basis of $\mathcal{H}$. Consequently the nested P_n tend strongly to the identity. Put $P_0 = 0$ and $E_j = P_j - P_{j-1}$. These are orthogonal finite-rank reducing projections with strong sum I. Diagonalize R on each range. Its eigenvalues there are positive because the physical resolvent is injective. If $Ru = \mu u$, $\mu > 0$, then $u = \mu^{-1}Ru \in \operatorname{dom} H$ and $Hu = (\mu^{-1} - 1)u$. Their finite-block eigenbases therefore form a complete orthonormal eigenbasis for H. The limiting projections need only be reducing projections; coincident eigenvalues in different blocks cause no difficulty.

To identify the spectral set, fix $\lambda \geq 0$ and $\xi \in \mathbb{R}^d$ with $|\xi|^2 = \lambda$. For normalized $\chi \in C_c^\infty((-1,1)^d)$ define

$$u_n(x) = a_n^{-d/2}\chi\left(\frac{x - c_ne_1}{a_n}\right)e^{i\xi\cdot x}.$$

These unit vectors have disjoint supports inside the rooms, tend weakly to zero, and satisfy

$$\|(H-\lambda)u_n\|_2 \leq \frac{2|\xi|}{a_n}\|\nabla\chi\|_2 + \frac{1}{a_n^2}\|\Delta\chi\|_2 \longrightarrow 0.$$

Weyl's criterion and nonnegativity give $\sigma(H) = \sigma_{\mathrm{ess}}(H) = [0,\infty)$. If $Hu = 0$, the energy identity makes its zero extension have zero weak gradient on $\mathbb{R}^d$, so it is a square-integrable constant and hence zero. Completeness then gives the remaining spectral claims in Theorem 2.1.

2.4 From local form compactness to uniform source bounds

Multiplication by w is closed: if $f_m \to f$ and $wf_m \to h$ in L^2, simultaneous almost-everywhere subsequences give $h = wf$. The increment $P_{j,n+1} - P_{j,n}$ is supported in D_{n+1}, so

$$\|w(P_{j,n+1} - P_{j,n})\| \leq A_{n+1}\eta_n.$$

Together with $\|wQ_j\| \leq A_j$, the weighted telescope converges in operator norm by (11). Closedness identifies its limit as wP_j, and therefore

$$\|wP_j\| \leq A_j + 1, \qquad \|wE_j\| \leq A_j + 1. \tag{14}$$

This gives weighted ranges, but the following additional estimate is needed to sum the source contributions of infinitely many blocks.

Put $S = h(H)$, a self-adjoint contraction. Let $k \leq n$, and let F retain in the first k rooms the Neumann modes of energy at most L_n. Then $A = \mathbf{1}_{D_k} - F$ is an orthogonal projection. Restrictions of $u \in H_0^1(\Omega)$ to individual rooms belong to their Neumann form domains, including across open windows, and their expansions give

$$\|Au\|_2 \leq L_n^{-1/2}\|\nabla u\|_2.$$

Let $J_n = \mathbf{1}_{[0,Z_n]}(H)$. Spectral calculus gives $SJ_nf \in H_0^1(\Omega)$ and $\|\nabla SJ_nf\|_2 \leq \sqrt{Z_n}\|f\|_2$, since $h \leq 1$. The complementary part satisfies $\|S(I - J_n)\| \leq \sup_{\lambda>Z_n} h(\lambda)$. Consequently

$$\|(\mathbf{1}_{D_k} - F)S\| \leq \sqrt{Z_n/L_n} + \sup_{\lambda>Z_n} h(\lambda) \leq \varepsilon_n/2. \tag{15}$$

This is why the local comparison uses Neumann rather than Dirichlet room modes. The estimate is independent of future geometry; it does not require Sf itself to have finite Dirichlet energy.

Using $SP_n = P_nS$ and (13), the finite row bound is

$$\|FS(I-P_n)\|^2 \le \sum_{b\in\text{basis of } \operatorname{ran} F} \|(I-P_n)Sb\|^2 \le M_n\rho_n^2 \le \varepsilon_n^2/4.$$

Combining this with (15) and taking adjoints yields

$$\|(I-P_n)S\mathbf{1}_{D_k}\| \le \varepsilon_n \qquad (k \le n). \tag{16}$$

Only S and P_n were commuted; no spatial indicator was commuted.

Fix k and g with $\|g\|_\infty \le 1$. The first k blocks combine as P_k, with

$$\|wP_kg(H)S\mathbf{1}_{D_k}\| \le A_k + 1.$$

For $j > k$, reducingness, (14), and (16) give

$$\|wE_jg(H)S\mathbf{1}_{D_k}\| \le (A_j+1)\varepsilon_{j-1} = 2^{-(j-1)}. \tag{17}$$

The weighted block series thus converges in norm, uniformly over these Borel functions, with bound $A_k + 1 + 2^{1-k}$. Its unweighted partial sums are $P_Ng(H)S\mathbf{1}_{D_k}$ and converge strongly to $g(H)S\mathbf{1}_{D_k}$. Closedness of multiplication by w identifies the weighted limit for every input and proves (4). Finally, room left faces escape to positive infinity, and windows add only null interfaces to the measurable room union. Every bounded K is therefore contained in some D_k up to null sets; right composition by $\mathbf{1}_{K\cap\Omega}$ proves its source bound. This completes the proof of Theorem 2.1.

Corollary 2.5 (Prescribed energy growth)**.** *Fix d, a_1, w as in Theorem 2.1, and let $r : [0,\infty) \to [1,\infty)$ be finite, nondecreasing and tend to infinity. The tower can be selected so that, on the same domain,*

$$\sup_{\|g\|_\infty\le 1} \|wg(H)\mathbf{1}_{[0,E]}(H)\mathbf{1}_{D_k}\| \le C_k \min\{r(E), \sqrt{E+1}\} \qquad (k \ge 1,\ E \ge 0), \tag{18}$$

and the half-resolvent source estimate

$$\sup_{\|g\|_\infty\le 1} \|wg(H)(H+1)^{-1/2}\mathbf{1}_{D_k}\| \le C_k$$

also holds. The spectral and geometric conclusions of Theorem 2.1 are retained.

Proof. Apply the theorem with

$$h_r(\lambda) = \max\{r(\lambda)^{-1}, (1+\lambda)^{-1/2}\}.$$

This filter is positive, nonincreasing, at most one, and tends to zero. On $[0,E]$, $h_r(\lambda) \ge h_r(E)$, so $g(\lambda)\mathbf{1}_{[0,E]}(\lambda)h_r(E)/h_r(\lambda)$ is a bounded Borel multiplier of norm at most one. Equation (4) gives (18). The quotient $(1+\lambda)^{-1/2}/h_r(\lambda)$ is also at most one, which gives the half-resolvent bound with its same constant. □

The rate describes the dependence on the energy ceiling; the evolution preserves energy. It can diverge as slowly as prescribed, but the domain is selected after the rate.

Corollary 2.6 (Every positive smoothing order)**.** *The domain in Corollary 2.5 for $r(\lambda) = 1 + \log(1+\lambda)$ has the logarithmic energy bound* (18) *and, simultaneously for every $\alpha > 0$,*

$$\sup_{\|g\|_\infty\le 1} \|wg(H)(H+1)^{-\alpha}\mathbf{1}_{D_k}\| \le C_k(1+1/\alpha).$$

Proof. For $t = 1 + \lambda \geq 1$,

$$\frac{(1+\lambda)^{-\alpha}}{h_r(\lambda)} \leq (1+\log t)t^{-\alpha} \leq 1 + \frac{1}{e\alpha} \leq 1 + \frac{1}{\alpha}.$$

Apply (4) to the resulting bounded quotient. □

Corollary 2.7. *If $w(x) \geq \omega(|x|)$ almost everywhere for a nondecreasing $\omega \geq 1$, the domain of Theorem 2.1 satisfies*

$$\sup_t \left\| \mathbf{1}_{\{|x|>r\}} e^{-itH} h(H) \mathbf{1}_{D_k} \right\| \leq \frac{C_k}{\omega(r)}.$$

For the domain of Corollary 2.5, the corresponding finite-energy bound is $C_k \min\{r_0(E), \sqrt{E+1}\}/\omega(r)$, where r_0 denotes its prescribed energy rate. Every vector in each finite reducing block also belongs to weighted L^2. In particular an eigenbasis with that property can be chosen.

Proof. The first assertion follows by multiplication on $\{|x| > r\}$ in (4); the finite-energy assertion follows from (18), and the block assertion from (14). □

2.5 An all-energy obstruction

The spectral smoothing or finite energy restriction has a precise role in a bound uniform over the whole source space.

Proposition 2.8. *On every positive-window straight-axis tower of the form (2) with $c_N \to \infty$, let K contain a neighbourhood of an interior axis point in the first room. If a measurable weight $v \geq 1$ satisfies*

$$\operatorname*{ess\,inf}_{|x|\geq r} v(x) \longrightarrow \infty,$$

then no finite bound is possible for

$$\sup_{t\in\mathbb{R}} \|ve^{-itH}\mathbf{1}_{K\cap\Omega}\|.$$

Proof. Fix an interior starting point $x_0 e_1$ and remote destinations $c_N e_1$. Every finite axis segment between them is a compact subset of Ω, since its finitely many traversed windows are positive. Choose a normalized smooth bump f_N, supported in a sufficiently small starting ball inside K, whose translates along the segment all remain compactly supported in Ω. Its radius may be taken at most one. For $q > 0$ put

$$z_q(t,x) = e^{iqx_1 - iq^2 t} f_N(x - 2qte_1), \qquad T_q = \frac{c_N - x_0}{2q}.$$

On $0 \leq t \leq T_q$ these functions are in $C_c^\infty(\Omega)$. Direct differentiation gives

$$(i\partial_t - H) z_q = e^{iqx_1 - iq^2 t} \Delta f_N(x - 2qte_1).$$

Unitarity and Duhamel's formula imply

$$\|e^{-iT_q H} z_q(0) - z_q(T_q)\|_2 \leq T_q \|\Delta f_N\|_2 \longrightarrow 0.$$

Project first onto the destination ball A_N, where the final bump has unit norm. For sufficiently large q, the actual evolved state has L^2 norm at least $1/2$ there. Its weighted norm, if defined, is at least $\frac{1}{2}\operatorname{ess\,inf}_{A_N} v$, which tends to infinity as the destination balls escape. If multiplication by v fails to be defined for an evolved input, the proposed bounded composition already fails. Equivalently the same lower bounds hold first for the bounded truncations $\min\{v, M\}$. □

The initial bumps and frequencies may depend on the destination; the source neighbourhood remains fixed. The proposition obstructs a uniform all-input, all-energy estimate.

Proposition 2.9 (Limits of prescribed rates)**.** *On a fixed tower with source set K and weight v as in Proposition 2.8, no finite bound can hold for*

$$\sup_{E\geq 0,\, t\in\mathbb{R}} \|ve^{-itH}\mathbf{1}_{[0,E]}(H)\mathbf{1}_{K\cap\Omega}\|.$$

Moreover, no such fixed tower satisfies the estimate

$$\|ve^{-itH}\mathbf{1}_{[0,E]}(H)\mathbf{1}_{K\cap\Omega}\| \leq C_r r(E) \qquad (E \geq 0,\ t \in \mathbb{R})$$

for every finite nondecreasing divergent rate r, even allowing the finite constant C_r to depend on r.

Proof. If the first supremum were bounded by C, replace v by the bounded truncation $v_M = \min\{v, M\}$ and fix an input and time. The strong limit $\mathbf{1}_{[0,E]}(H) \to I$ gives $\|v_M e^{-itH}\mathbf{1}_{K\cap\Omega} f\| \leq C\|f\|$. Monotone convergence as $M \to \infty$ proves the forbidden unfiltered bound of Proposition 2.8.

For the second assertion, suppose every rate admitted such a constant. The particular rate $1+E$ makes

$$A(E) = \sup_{0\leq F\leq E,\, t\in\mathbb{R}} \|ve^{-itH}\mathbf{1}_{[0,F]}(H)\mathbf{1}_{K\cap\Omega}\|$$

finite for every E. It is nondecreasing and, by the first assertion, tends to infinity. The rate $r(E) = \sqrt{\max\{1, A(E)\}}$ is therefore finite, nondecreasing and divergent. Its presumed estimate gives $A(E) \leq C_r r(E)$, contradicting $A(E) \to \infty$. The qualitative corollary concerns every specified escape rate. □

For completeness the analogous filter quantifier is also necessary. No fixed tower in Proposition 2.8 has finite time-uniform weighted source bounds for every $f \in C_0([0,\infty))$, with a bound allowed to depend on f. Indeed, the bounded linear maps

$$T_{M,t} : C_0([0,\infty)) \longrightarrow \mathcal{B}(\mathcal{H}), \qquad T_{M,t} f = v_M e^{-itH} f(H)\mathbf{1}_{K\cap\Omega}$$

would be pointwise bounded. The uniform boundedness principle would give $\sup_{M,t} \|T_{M,t}\| < \infty$. Continuous compactly supported filters of norm at most one tending pointwise to one then recover the unfiltered bound by strong spectral convergence and monotone convergence in M, again a contradiction. Thus Corollary 2.6 concerns a common dominated collection of filters, not all vanishing filters on one domain.

3 Quantitative aperture bounds

The small-window limit used above has an explicit modulus. The two-dimensional estimate requires only the location of the added set; it does not require boundary regularity at the window endpoints. Throughout this section, $D_0 \subset D_\delta \subset \mathbb{R}^d$ are open, $|D_\delta \setminus D_0| = 0$, and

$$D_\delta \setminus D_0 \subset B_\delta(p).$$

The Dirichlet resolvents $T_0 = (H_{D_0} + 1)^{-1}$ and $T_\delta = (H_{D_\delta} + 1)^{-1}$ act on their common L^2 space. For a cube window in dimension d, the added interface lies instead in $B_{d\delta}(p)$; replacing the radius by $d\delta$ is sufficient.

Lemma 3.1 (Energy comparison)**.** *Let $a(u,v) = \langle\nabla u, \nabla v\rangle + \langle u, v\rangle$. Then $0 \leq T_\delta - T_0 \leq I$. If $u = T_\delta f$, then*

$$\langle (T_\delta - T_0) f, f\rangle = \operatorname{dist}_a(u, H^1_0(D_0))^2. \tag{19}$$

If a bounded Lipschitz function ψ equals one on a neighbourhood of $\overline{B_\delta(p)}$, then

$$\operatorname{dist}_a(u, H^1_0(D_0))^2 \leq \|\psi u\|_a^2 = \operatorname{Re}\langle f, \psi^2 u\rangle + \int |u|^2 |\nabla\psi|^2. \tag{20}$$

Here ψ is real valued.

Proof. The variational equations make $T_0 f$ the energy-orthogonal projection of u onto the closed subspace $H_0^1(D_0)$ of $H_0^1(D_\delta)$. Expanding the squared error gives (19), hence positivity. Both resolvents are positive contractions, so their positive difference is at most I. To see $(1-\psi)u \in H_0^1(D_0)$, approximate u by compactly supported smooth functions in D_δ and multiply them by $1-\psi$. The resulting supports have positive separation from the entire added set and are compact in D_0; smooth approximation there gives the assertion. Thus ψu bounds the projection error. Testing the weak equation with $\psi^2 u$ and expanding $|\nabla(\psi u)|^2$ proves the last identity. □

Theorem 3.2 (Planar modulus)**.** *For $d = 2$ and every $\delta < \rho < R$,*

$$\|T_\delta - T_0\| \le \min\left\{1, \frac{R}{2} + \frac{1}{2\log(R/\rho)}\right\}. \tag{21}$$

Consequently, if $0 < \delta < 1/2$,

$$\|T_\delta - T_0\| \le \min\left\{1, \sqrt{\delta/2} + \frac{1}{\log(1/(2\delta))}\right\}. \tag{22}$$

For every $\varepsilon > 0$, the dyadic rule

$$0 < \delta \le 2^{-N}, \qquad N = 1 + \lceil 3/\varepsilon \rceil \quad \Longrightarrow \quad \|T_\delta - T_0\| < \varepsilon \tag{23}$$

is sufficient.

Proof. The Dirichlet heat semigroup on any open set is dominated by the free heat semigroup after zero extension. Integrating in time, $|T_\delta f| \le G * |\widetilde{f}|$, where $\widehat{G}(\xi) = (1+|\xi|^2)^{-1}$ with the Fourier convention $\widehat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)\,dx$. In dimension two,

$$\|G\|_2^2 = (2\pi)^{-2} \int_{\mathbb{R}^2} (1+|\xi|^2)^{-2}\, d\xi = \frac{1}{4\pi}, \qquad \|T_\delta\|_{2\to\infty} \le \frac{1}{2\sqrt{\pi}}.$$

Use the radial logarithmic cutoff equal to one on $B_\rho(p)$, zero outside $B_R(p)$, and $\log(R/|x-p|)/\log(R/\rho)$ in between. Its gradient energy is $2\pi/\log(R/\rho)$. For $\|f\|_2 = 1$, the first term in (20) is bounded by $\|T_\delta f\|_\infty |B_R|^{1/2} \le R/2$. The second is at most $1/(2\log(R/\rho))$. Taking the supremum in (19) proves (21). Set $\rho = 2\delta$ and $R = \sqrt{2\delta}$ to obtain (22). Writing $L = \log(1/(2\delta))$, its nontrivial bound is $\frac{1}{2}e^{-L/2} + 1/L < 2/L$. Under (23), $L \ge (N-1)\log 2 > 2/\varepsilon$, since $\log 2 > 2/3$. This proves the strict tolerance. □

Theorem 3.3 (A linear modulus in higher dimensions)**.** *If $d \ge 3$, $D_\delta \setminus D_0 \subset B_\delta(p)$, and $0 < \delta < \rho \le 1$, then*

$$\|T_\delta - T_0\| < 5\rho^{3/2} + 25\rho \le 30\rho. \tag{24}$$

For a cube-window half-width s, it is therefore sufficient to take

$$0 < s \le \min\{1/d, \varepsilon/(60d)\} \tag{25}$$

to make the resolvent difference less than ε.

Proof. Put $q = 2d/(d-3)$, interpreting $q = \infty$ for $d = 3$, and $p = 2d/3$. Interpolating the heat bounds $\|e^{-tH_D}\|_{2\to 2} \le 1$ and $\|e^{-tH_D}\|_{2\to\infty} \le (8\pi t)^{-d/4}$ with parameter $3/d$ gives

$$\|e^{-tH_D}\|_{2\to q} \le (8\pi t)^{-3/4}, \qquad \|(H_D+1)^{-1}\|_{2\to q} \le C := \frac{\Gamma(1/4)}{(8\pi)^{3/4}} < 5/8.$$

The last elementary bound follows by splitting the defining gamma integral at one: its two pieces are at most 4 and less than 1. Also $\pi > 2$ gives $(8\pi)^{3/4} > 8$. Choose $\psi = 1$ on $B_\rho(p)$, $\psi = 0$ outside $B_{2\rho}(p)$, and $|\nabla\psi| \le 1/\rho$. Since $1/2 = 1/q + 1/p$, Hölder's inequality in (20) bounds its two terms by

$$C(2^d V_d)^{3/(2d)} \rho^{3/2}, \qquad C^2 (2^d V_d)^{3/d} \rho,$$

respectively, where $V_d = |B_1(0)|$. The inequalities $V_d \le 2^d$ and $C < 5/8$ give the constants 5 and 25. For the cube window take $\rho = ds$, strictly larger than the interface radius $\sqrt{d-1}\,s$. The rule (25) gives $\rho \le 1$ and $30\rho \le \varepsilon/2$, proving the claim. In this application the radius hypothesis uses the actual interface radius, not s. □

Lemma 3.4 (Whole-cluster continuation)**.** *Let T and T' be bounded self-adjoint operators with $\|T'-T\| \le e < g$, and let Γ be a finite system of positively oriented, disjoint contours of total length ℓ, at distance at least g from $\sigma(T)$. Then*

$$\|P_{T'}(\Gamma) - P_T(\Gamma)\| \le \frac{\ell e}{2\pi g(g-e)}. \tag{26}$$

If this bound is below one and the selections are finite rank, their ranks are equal. For s disjoint circles of radius r with clearance at least r, the bound is $se/(r-e)$.

Proof. The Neumann series gives $\|(z-T')^{-1}\| \le (g-e)^{-1}$ on Γ. Integrating the resolvent identity gives (26). The spectral selections are orthogonal projections. Projections at distance below one restrict injectively to each other's ranges, so their finite ranks agree. All estimates apply to any subset of the same disjoint contours. □

3.1 An exact two-square certificate

Consider adjacent squares with half-widths 1 and $5/2$. The first square's eigenvalue $5\pi^2/4$ has multiplicity two, from indices $(1,2)$ and $(2,1)$. It cannot be an eigenvalue of the second square, whose energies are $\pi^2(m^2+n^2)/25$: resonance would require $m^2+n^2 = 125/4$. The possible sums below 32 are

$$2, 5, 8, 10, 13, 17, 18, 20, 25, 26, 29.$$

Put $x = \pi^2$, so $49/5 < x < 10$. Comparing the selected resolvent value $(1+5x/4)^{-1}$ with its nearest complementary values gives the following strict lower bounds:

complement	new 32	new 29	old 2	old 8	0
distance	49/31050	49/9450	49/540	7/270	2/27.

For example, use $|c-t|x/((1+cx)(1+tx)) > |c-t|(49/5)/((1+10c)(1+10t))$ for $c = 5/4$. Ordering the remaining levels shows these comparisons cover the entire complement. Every displayed bound exceeds $1/1000$. A circle of radius $r = 1/3000$ therefore has full spectral clearance at least r.

For $\varepsilon = 1/600000$, the simple rule (23) gives the sufficient opening $\delta = 2^{-1800001}$. A better choice in (21) is

$$R = \varepsilon^2, \qquad \rho = 2\delta, \qquad \delta = 2^{-450044}.$$

Indeed $2/3 < \log 2 < 7/10$ and $600000 < 2^{20}$ imply

$$\log(R/\rho) > 300000 + 2/3, \qquad \frac{\varepsilon^2}{2} + \frac{1}{2(300000+2/3)} < \varepsilon.$$

The projection drift is consequently strictly below $\varepsilon/(r-\varepsilon) = 1/199 < 1/100$, and the cluster has rank two. The eigenvalues inside it may split. These are conservative sufficient openings. The accompanying exact-arithmetic program verifies the finite mode and rational calculations, not the analytic operator arguments.

4 Finite certificates and a terminating construction

Fix an integer $d \ge 2$. A *rational finite tower* has finitely many cubes with rational centres and half-widths, and positive rational window half-widths. Its input records the screen faces and the open windows, as well as their coordinates. This is stronger information than membership queries for an arbitrary open set; compare [13].

Theorem 4.1 (Effective construction)**.** *Let $d \geq 2$ be an integer, let $a_1 > 0$ be rational, and let a total algorithm produce a nondecreasing sequence of rational numbers $U_n \geq 1$. Let $h : [0,\infty) \to (0,1]$ be real Borel, and suppose a total algorithm returns nonnegative rational $T(m)$ such that*

$$\sup_{\lambda > T(m)} h(\lambda) \leq 2^{-m} \qquad (m \geq 1). \tag{27}$$

There is a deterministic procedure which, for each requested N, terminates with the first N rational room half-widths and centres in $\mathbb{R}^d$, the first $N-1$ positive dyadic apertures, and finite spectral certificates for the completed attachments. The geometry satisfies

$$a_n + 1 < a_{n+1} < a_n + 2, \qquad c_1 = a_1, \qquad c_{n+1} = c_n + a_n + a_{n+1}, \qquad 0 < \delta_n < \min\{a_n, 2^{-n}\}.$$

Its union Ω is connected and effectively open. For every measurable $w \geq 1$ satisfying

$$\operatorname*{ess\,sup}_{|x| \leq B_n} w(x) \leq U_n, \qquad B_n = 2\sqrt{d}\, n(a_1 + n), \tag{28}$$

the ordinary Dirichlet Laplacian H on this same domain has a complete orthonormal eigenbasis, spectral and essential spectral sets $[0,\infty)$, and

$$\sup_{\substack{g \text{ bounded Borel} \\ \|g\|_\infty \leq 1}} \left\| w g(H) h(H) \mathbf{1}_{D_k} \right\| \leq U_k + 1 + 2^{1-k} \qquad (k \geq 1). \tag{29}$$

In particular, let $r : [0,\infty) \to [1,\infty)$ be finite, nondecreasing and divergent, and suppose a total algorithm supplies rational $T_r(m) \geq 0$ with $r(\lambda) \geq 2^m$ for $\lambda > T_r(m)$. Taking

$$h(\lambda) = \max\{r(\lambda)^{-1}, (1+\lambda)^{-1/2}\}, \qquad T(m) = \max\{T_r(m), 2^{2m}\},$$

the same procedure constructs a domain satisfying

$$\sup_{\|g\|_\infty \leq 1} \|w g(H) \mathbf{1}_{[0,E]}(H) \mathbf{1}_{D_k}\| \leq (U_k + 1 + 2^{1-k}) \min\{r(E), \sqrt{E+1}\}.$$

This domain also retains the half-resolvent source bound with the same constant.

The inputs are an effective spatial upper envelope and a promised uniform filter-tail modulus. No evaluation of h at spectral points is needed. Correctness of an arbitrary input program's promises is a hypothesis, not a property decided by the algorithm. For a computable nondecreasing divergent r, a divergence modulus can be found by searching integer arguments and certified lower enclosures until $r(N) > 2^m$. Monotonicity makes $T_r(m) = N$ valid, and divergence gives termination. Thus the logarithmic example of Corollary 2.6 is included. A finite run returns a finite prefix; effective openness does not decide membership at arbitrary boundary points. We prove the theorem by constructing all finite certificates and then implementing the induction of Theorem 2.1.

4.1 Approximation spaces on the actual cracked domain

Fix a rational finite tower D. Draw rational coordinate hyperplanes through all room and window faces, subdividing the rooms into boxes. Triangulate each box by ordering its normalized coordinates, and refine all coordinate intervals dyadically, using the same ordered triangulation in each child box. This family is nested: the coarse order of two coordinates is determined first by their dyadic child bits and, when the bits agree, by their fractional coordinates. Every child simplex therefore lies in a coarse simplex. The induced triangulations agree on common faces. There are finitely many initial box shapes; all descendants are translated uniform scalings of their ordered simplices. Thus the family has a fixed shape-regularity constant and maximal diameter tending to zero. Faces are paired precisely when they are genuine interior faces of D, including windows. The two coincident sides of a Dirichlet screen remain unpaired boundary faces. Write $\mathcal{T}_l$ for these rational crack-aware meshes.

Let V_l be the conforming continuous piecewise-affine space with zero values on every boundary and screen face. Let Q_l consist of fields of the form $q(x) = a + bx$ on each simplex, with $a \in \mathbb{R}^d$ and $b \in \mathbb{R}$, whose normal components match on genuine interior faces. No normal boundary condition is imposed. Thus

$$V_l \subset H_0^1(D), \qquad Q_l \subset H(\mathrm{div}; D).$$

The flux inclusion follows by elementwise integration by parts; normal terms cancel at paired faces, whereas compactly supported tests do not cross screen faces. All defining linear constraints have rational coefficients. One can use unnormalized rational face normals, or integral fluxes, without introducing square roots of face Gram determinants.

We shall use the following elementary boundary observation. If $v \in H^1(D)$ has zero one-sided traces on every open boundary face, then $v \in H_0^1(D)$. Indeed, first truncate v to a bounded H^1 function. The boundary skeleton is contained in finitely many affine subspaces of codimension two. Around each use a logarithmic cutoff depending on distance to that subspace, zero up to radius r^2 and one beyond radius r. In a bounded ambient box, integration in the two normal directions and bounded tangential volume give gradient energy $O(1/|\log r|)$. The exceptional support has measure tending to zero. Products of these finitely many cutoffs remove all edges, corners and crack tips and have the same limiting properties. Multiplication converges in $H^1(D)$: the cutoff-gradient term is bounded by the truncation's supremum norm, and the other terms have shrinking support and integrable energy. Away from the removed skeleton a finite partition of unity reduces to interior balls and straight one-sided half-balls. Zero face traces permit zero extension and approximation inside each half-ball. This applies separately to both sides of a screen. Closedness, followed by removal of the cutoffs and truncations, proves the observation and the stated scalar conformity.

Lemma 4.2 (Density on the crack-aware meshes)**.** *The union of V_l is dense in $H_0^1(D)$ in its energy norm, and the union of Q_l is dense in $H(\mathrm{div}; D)$ in its graph norm. Rational coefficients are dense in both unions.*

Proof. For the scalar assertion, approximate a compactly supported smooth function by its nodal interpolants. On sufficiently fine meshes these vanish near the boundary and converge in H^1. Density of $C_c^\infty(D)$ gives the assertion.

For the flux assertion, let $q \in H(\mathrm{div}; D)$ be graph-orthogonal to every Q_l, and put $f = \mathrm{div}\, q$. Smooth compactly supported vector fields are graph-norm limits of their Raviart–Thomas interpolants on these meshes. This last claim requires only smooth fields: the interpolant reproduces constants, and its divergence is the elementwise mean of the divergence. Taylor approximation and the fixed shape regularity give convergence. Consequently

$$\int_D q \cdot \overline{r} + f\, \overline{\mathrm{div}\, r} = 0 \qquad (r \in C_c^\infty(D; \mathbb{R}^d)),$$

with complexification if necessary. Hence $q = \nabla f$ in distributions and $f \in H^1(D)$.

For a boundary face e of a simplex T, the local field

$$r_e(x) = \frac{x - v_e}{d|T|},$$

where v_e is the opposite vertex, has integral outward flux one on e and zero normal component on the other faces. Its zero extension to the other simplices belongs to Q_l. Orthogonality and Green's identity on T imply

$$0 = \int_T \nabla f \cdot r_e + f\, \mathrm{div}\, r_e = \frac{1}{|e|} \int_e \mathrm{Tr}_T\, f.$$

The same holds on every refined subface. Their step functions are dense in $L^2(e)$, so the one-sided trace vanishes on every open boundary face. Screen sides are tested independently. The boundary observation above gives $f \in H_0^1(D)$, including along crack edges and tips. Since $q = \nabla f$ and $\mathrm{div}\, q = f$,

$$\|f\|_2^2 = -\int_D q \cdot \overline{\nabla f} = -\|q\|_2^2.$$

Thus the graph-orthogonal complement is zero. Nestedness makes the union a linear space, proving density. Rational bases of the finite constraint spaces give the rational-coefficient assertion. □

The absence of a normal boundary condition in Q_l is essential. The exact gradient of a Dirichlet solution generally has nonzero normal boundary flux.

4.2 A total physical resolvent certificate

Proposition 4.3. *Given D and a rational $\beta > 0$, a terminating algorithm returns a rational symmetric matrix A and a synthesis isometry B for finitely many explicitly specified roomwise Neumann modes such that*

$$\left\|(H_D + 1)^{-1} - BAB^*\right\| < \beta. \tag{30}$$

The certificate includes the omitted continuum modes and the embedded operator's infinite zero complement.

Proof. Write $R = (H_D + 1)^{-1}$. Let $F = BB^*$ retain all roomwise Neumann modes of energy at most L, allowing extra modes. Restriction of $v \in H_0^1(D)$ to each room lies in its Neumann form domain, so

$$\|(I - F)v\|_2 \le L^{-1/2}\|\nabla v\|_2.$$

Spectral calculus gives $\|\nabla R\| \le 1/2$. Therefore

$$\|(I - F)R\| \le \frac{1}{2\sqrt{L}}, \qquad \|R - FRF\| \le \frac{1}{\sqrt{L}}. \tag{31}$$

The second estimate uses self-adjointness and $R - FRF = (I - F)R + FR(I - F)$.

For an orthonormal mode b_j in $\operatorname{ran} F$, choose conforming $u_j \in V_l$ and $q_j \in Q_l$ and set

$$\eta_j^2 = \|b_j - u_j + \operatorname{div} q_j\|_2^2 + \|q_j - \nabla u_j\|_2^2. \tag{32}$$

Integration by parts against $v \in H_0^1(D)$ gives

$$\langle b_j, v\rangle - a(u_j, v) = \langle b_j - u_j + \operatorname{div} q_j, v\rangle + \langle q_j - \nabla u_j, \nabla v\rangle,$$

where a is the energy inner product for $H_D + 1$. Hence $\|Rb_j - u_j\|_a \le \eta_j$. If $\mathcal{U}$ has columns u_j, Hermitian compression gives

$$A_0 = \tfrac{1}{2}(B^*\mathcal{U} + \mathcal{U}^*B), \qquad \|B^*RB - A_0\| \le \Big(\sum_j \eta_j^2\Big)^{1/2}.$$

Approximating A_0 by a rational symmetric A with norm error at most τ proves the sufficient certificate

$$\|R - BAB^*\| \le L^{-1/2} + \Big(\sum_j \eta_j^2\Big)^{1/2} + \tau. \tag{33}$$

We verify that a finite search can meet any prescribed error. The exact pair $u = Rb_j$, $q = \nabla u$ satisfies $\operatorname{div} q = u - b_j$ and has zero residual in (32). Lemma 4.2 makes rational trial/flux residuals arbitrarily small. All their continuum norms are effectively enclosable. On each rational simplex, b_j is a product of cosines of π times rational-affine functions, with computable normalization. Rational enclosures of π and square roots, together with the Taylor remainder

$$\left|\cos z - \sum_{m=0}^{s} \frac{(-1)^m z^{2m}}{(2m)!}\right| \le \frac{|z|^{2s+2}}{(2s+2)!} \qquad (z \in \mathbb{R}),$$

give rational polynomial approximants with certified uniform error tending to zero. Polynomial moments on a rational simplex are exact rational numbers, using $\int_{\{x_i \ge 0,\, \sum_i x_i \le 1\}} x^\alpha\, dx = \prod_i \alpha_i!/(|\alpha| + d)!$. The

polynomial approximation error bounds the residual norm error by the triangle inequality. Compression integrals are enclosed in the same way. Thus every strict residual or rounding margin is eventually certified.

Choose rational L with $L^{-1/2} < \beta/4$. If $a_{\max}$ is the largest room half-width, choose an integer K satisfying $9(K+1)^2 > 4a_{\max}^2 L$ and retain every room index $0 \le \nu_i \le K$ for $1 \le i \le d$. Since $\pi > 3$, every omitted mode has energy greater than L. With $M = \operatorname{rank} F$, request each residual below $\beta/(4M)$ and choose $\tau < \beta/4$. Then (33) is below $3\beta/4$. Fairly enumerate mesh levels, rational coefficients and integration precisions, accepting only these strict inequalities. Density provides a fixed finite witness with positive margin; the enclosures eventually certify it. The search therefore terminates. □

Remark 4.4. Coefficient enumeration can be replaced by finite first-order least squares [4, 6]. On $V_l \times Q_l$ minimize the squared residual (32). Its Gram matrix G_l is rational and positive definite: a zero homogeneous residual gives $q = \nabla u$, $\operatorname{div} q = u$, and the Dirichlet energy identity forces $u = q = 0$. Only the right-hand side contains computable cosine integrals. Let $K_G = \sum_{i,j} |(G_l)_{ij}|$, $K_I = \max_i \sum_j |(G_l^{-1})_{ij}|$, and $C = \max(1, K_G)\max(1, K_I)$. Approximating each right-hand-side entry within $2^{-l}/C$ and solving the rational system exactly adds at most 2^{-l} to the residual norm. Its certified upper bound can be enclosed within another 2^{-l}. The exact finite minima tend to zero by density, so this refinement procedure also terminates without a regularity rate.

4.3 Strict finite tests and whole labelled clusters

Write $K = BAB^*$ and suppose $\|R - K\| < \beta$. Both operators are self-adjoint, so their spectral sets are within β of each other. The spectrum of K consists of $\sigma(A)$ and an infinite zero complement. Rational Hermitian matrices have computable algebraic eigenvalue enclosures and separated spectral projectors, including repeated roots; these are standard finite spectral computations, consistent with the compact-operator framework in [5].

For a rational rectangular contour system Γ of total length ℓ, a certified distance $\operatorname{dist}(\Gamma, \sigma(K)) \ge \gamma > \beta$ gives

$$\|P_R(\Gamma) - P_K(\Gamma)\| \le \frac{\ell\beta}{2\pi\gamma(\gamma - \beta)}. \tag{34}$$

One may replace 2π by 6 for rational upper bounds. Every fixed true gap is eventually certified as β and the matrix enclosures shrink. If the bound is below one, the two projections have the same rank. For a room-Neumann target b, compute $\|(I - P_K(\Gamma))b\|$ in the canonical room coordinates and add (34). This semidecides every strict target-approximation inequality. Zero is always included in the complementary spectral cover.

We now give the recursion. With the envelope of (28), put

$$\varepsilon_n = \frac{2^{-n}}{1 + U_{n+1}}, \quad \rho_n = \frac{\varepsilon_n}{2M_n}, \quad \eta_n = 2^{-(n+2)} \min\{1, U_{n+1}^{-1}, \rho_1, \dots, \rho_n\}. \tag{35}$$

Choose m_n with $2^{-m_n} \le \varepsilon_n/4$, a rational $Z_n \ge \max\{n, T(m_n)\}$, and a rational

$$L_n \ge \max\{n, 16 Z_n \varepsilon_n^{-2}\}.$$

The low-energy and high-energy filter split in Theorem 2.1 then bounds the local Neumann omission by $\sqrt{Z_n/L_n} + 2^{-m_n} \le \varepsilon_n/2$. Here F_n retains on each of the first n rooms all indices $0 \le \nu_i \le K_n$ for $1 \le i \le d$, where $9(K_n+1)^2 > 4a_n^2 L_n$, and $M_n = n(K_n+1)^d$. These lists contain every mode of energy at most L_n. The rational tolerance satisfies $\sqrt{M_n}\rho_n \le \varepsilon_n/2$, and

$$\sum_{l \ge n} \eta_l \le 2^{-(n+1)}\rho_n \le \rho_n/4, \qquad \sum_{l \ge j} U_{l+1}\eta_l \le 2^{-(j+1)}. \tag{36}$$

At stage n the older nested projections are specified by rational real interval systems, represented by rectangles of imaginary heights ± 1, with positive endpoints and certified spectral clearance. Let $h_0 > 0$

be the minimum left endpoint for the largest older selection; put $h_0 = 1$ at the first stage. Dovetail positive rational $t < h_0/2$ and certificate accuracies until the rectangle with real sides $t, 2$ has certified clearance and its projection Q_n satisfies

$$\|(I - Q_n)b\| < \rho_n/2 \qquad (b \text{ in the target basis of } F_n).$$

It contains all older selections. This search terminates because positive cutoffs of the compact injective finite resolvent tend strongly to the identity. Some small rational threshold avoids the spectrum and has a strict target margin. No uncertified threshold can block the dovetail.

For the next width enumerate rational $a \in (a_n + 1, a_n + 2)$. The finite old selected spectrum lies above t. A resonance with a new-cube mode requires

$$|\nu|^2 = \frac{4a^2(\mu^{-1} - 1)}{\pi^2} \le \frac{4(a_n + 2)^2(t^{-1} - 1)}{\pi^2}.$$

Thus only finitely many indices can resonate in this bounded width interval. Each selected value and index excludes at most one width. Only finitely many widths are bad, so a good rational width exists.

Dovetail widths, precision and finite disjoint rational intervals $I_l \subset (t, 2)$ subject to the following strict tests:

1. Enlarge every old matrix eigenvalue enclosure by its operator error β, and include the enlarged zero enclosure. None may contain or straddle t, and every enlarged enclosure above t must lie entirely in $\bigcup_l I_l$.

2. Each I_l is wholly inside or wholly outside every older interval system. Record these constant historical labels.

3. No new-room level lies in $\overline{I_l}$, and the interval endpoints have positive certified distance from the full disconnected spectrum.

The new-room check is finite: choose an integer $J \ge 2(a_n + 2)\sqrt{2/t}$. If any positive Dirichlet coordinate index exceeds J, its resolvent value is below $t/2$. The remaining levels have computable outward enclosures.

For a good width the selected old spectrum is a finite set disjoint from the new spectrum and the previous label boundaries. Small rational intervals around it satisfy all tests with positive margin. Whole equal-value eigenspaces have identical historical labels and need not be split. Thus sufficiently accurate calls to Proposition 4.3 certify one such finite record. The first test uses the *enlarged full spectral cover*; bare matrix eigenvalue approximations would not suffice. The labels reconstruct exactly every older projection and Q_n. Ranks are certified by (34). For the disconnected comparison, take the direct sum of certificates for its old-tower and single-cube components.

4.4 Opening the window and completing the proof

The common rectangles have rational total perimeter ℓ and a certified rational distance $g > 0$ from the disconnected spectrum, including zero. Put

$$\varepsilon^{\mathrm{ap}} = \min\{g/4, \eta_n g^2/\ell\}.$$

For $d = 2$, Theorem 3.2 supplies an aperture by taking $m \ge 1 + \lceil 3/\varepsilon^{\mathrm{ap}} \rceil$. For $d \ge 3$, Theorem 3.3 supplies one by increasing m until $2^{-m} \le \min\{1/d, \varepsilon^{\mathrm{ap}}/(60d)\}$. In either case increase m further until $\delta_n = 2^{-m} < \min\{a_n, 2^{-n}\}$. The actual resolvent error ε_0 is strictly below $\varepsilon^{\mathrm{ap}}$. The contour estimate of Lemma 3.4, applied to any subset of the common rectangles, gives

$$\|P_{\mathrm{new}} - P_{\mathrm{old}}\| \le \frac{\ell\varepsilon_0}{2\pi g(g - \varepsilon_0)} < \frac{2\ell\varepsilon^{\mathrm{ap}}}{9g^2} < \eta_n.$$

Complete ranks are preserved, and labelled subsets of the same rectangles remain nested. Their new clearance is at least $g - \varepsilon^{\mathrm{ap}} > 0$, providing the next finite state.

Every search above has a fixed finite witness with a positive margin. In a fair dovetail, each bounded list of candidates is tested at a bounded requested accuracy before both bounds increase. Each finite certificate call terminates by Proposition 4.3; bad candidates never require an infinite wait for an equality decision. Fixing an enumeration order makes the procedure deterministic. Induction therefore returns every finite prefix.

Finally (35)–(36) realize the projection and source budgets used in Theorem 2.1, with U_n replacing the local weight supremum. The physical resolvent limit and dense-target argument give finite-rank reducing projections P_j with dense union. Their orthogonal differences $E_j = P_j - P_{j-1}$ yield a complete eigenbasis. The weighted estimates are

$$\|wP_k\| \le U_k + 1, \qquad \|wE_j g(H)h(H)\mathbf{1}_{D_k}\| \le (U_j + 1)\varepsilon_{j-1} = 2^{-(j-1)} \quad (j > k).$$

Closedness of multiplication by w identifies the weighted series, and summation proves (29). The room Weyl sequences give the essential half-line. These arguments establish Theorem 4.1 for the effectively chosen sequence, without a gap assumption on its infinite-volume spectrum.

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