Eigenvalue counting for multi-armed shrinking spirals and a radial-width bound for spiral waveguides

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Abstract

For the Dirichlet Laplacian on the region bounded by the two-armed Fermat spiral $r^2 = b^2\theta$, whose spectrum is discrete, Exner and Tater (2020) kept only the lowest transverse mode of the coils, estimated the number of eigenvalues below E by $b^4E^2/64$, found a significant excess in computed eigenvalues, and pointed to finer effects; Barseghyan and Exner cited this excess as an analogy for curvature effects. We prove that each of the two components has counting function $\zeta(3)\, b^4E^2/64 + O(E\log E)$. The excess is thus a leading-order effect: the n-th transverse mode contributes $b^4E^2/(64n^3)$, and curvature enters only the remainder. More generally, for the m-armed spirals $r = b\theta^{1/q}$, $q > 1$, the count is $c\,(b^2E)^{q/(q-1)}$ with an explicit constant proportional to $\zeta((q+1)/(q-1))$, up to a power-saving remainder. For a single arm the leading heat-trace term goes back to an unpublished 1984 preprint of van den Berg and Lewis; we extend it to several arms and to the counting function with remainders. The proof uses exact orthogonal coordinates, in which each component is a straight strip, and Dirichlet–Neumann bracketing. Second, for every single-armed spiral through the origin we prove that the Dirichlet Laplacian is at least π^2 over the squared local radial coil width. For the Archimedean spiral this proves that the discrete spectrum is empty, as Exner and Tater stated on numerical evidence; more generally, no spectrum lies below π^2/ℓ^2 whenever the radial coil width never exceeds its limit ℓ.

Keywords. Dirichlet Laplacian; spiral-shaped region; Fermat spiral; Archimedean spiral; eigenvalue counting function; non-classical Weyl asymptotics; Dirichlet–Neumann bracketing; Hardy inequality.
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1 Introduction

1.1 The questions

Exner and Tater [3] initiated the spectral analysis of the Dirichlet Laplacian $-\Delta_D$ on regions $\mathbb{R}^2\setminus\Gamma$, where Γ is a spiral. The decisive quantity is the width of the coils. For the simple spirals of [3], when it diverges the spectrum is $[0,\infty)$; when it has a finite positive limit the region is a waveguide with an essential spectrum starting at a positive threshold; when it tends to zero the spectrum is purely discrete [3, Propositions 5.1–5.3]. Section and proposition numbers of [3] refer to its arXiv version.

The basic example of the last class is the Fermat spiral $r^2 = b^2\theta$, taken with its two arms, which divides the plane into two congruent components [3, Remark 5.2 and Section 6]. In [3, Section 5.1] the authors ask how the eigenvalues are distributed; their count in [3, Figures 5 and 6] refers to one

component. They retain the lowest transverse Dirichlet eigenvalue π^2/d^2 of a coil of width d, which behaves like $c\,s^{2/3}$ in terms of the arclength s, treat s semiclassically, and obtain the estimate

$$N(E) \approx \frac{b^4 E^2}{64} \qquad (E \to \infty) \tag{1.1}$$

for the number $N(E)$ of eigenvalues below E. Their computed eigenvalues [3, Figures 5 and 6] show a significant excess over (1.1), which they attribute to the curvature and kinetic terms of their effective operator [3, (5.3)], and they point to finer effects in the asymptotics as a subject for later study. Barseghyan and Exner [1] proved Lieb–Thirring and Berezin-type bounds for shrinking spirals whose leading terms involve the coil width only, and they cited the excess of [3, Section 5.1] as an analogy for curvature effects at low energies [1, discussion after Corollary 3.2]. For a single arm through the origin with concave radial function, van den Berg and Lewis [9] had shown in an unpublished 1984 preprint that the heat trace is asymptotic to that of the unrolled horn, with every transverse mode kept and with no contribution of the curvature at leading order. The region of [3] has two arms, and [3, Figure 6] displays the counting function, not the heat trace. Comparing these works leaves the following gap.

Question 1.1. *What is the asymptotic behaviour, with a remainder, of the eigenvalue counting function of a component of the two-armed Fermat region, and which effect produces the excess over* (1.1)*?*

The second question concerns the Archimedean spiral $r = a\theta$. Exner and Tater proved that the essential spectrum of $\mathbb{R}^2 \setminus \Gamma$ is $[(2a)^{-2}, \infty)$ and that at most finitely many eigenvalues lie below it [3, Propositions 2.1–2.2 and Section 2]. The abstract of [3] states that the discrete spectrum is empty, and [3, Section 3.1] supports this with numerical evidence; the statement is repeated in [1, Remark 2.1] and in [2].

Question 1.2. *Is the discrete spectrum of the Dirichlet Laplacian on the Archimedean spiral region empty?*

1.2 Main results

We work with power spirals with any number of arms. Write $\mathrm{pol}(r,\varphi) = (r\cos\varphi, r\sin\varphi)$.

Definition 1.3. Let $q > 1$, $b > 0$ and $m \in \mathbb{N}$, and put $f(r) = (r/b)^q$ and $\Theta = 2\pi/m$. The *m-armed power spiral* is

$$\Gamma_{q,b,m} = \big\{\mathrm{pol}(r, f(r) + j\Theta) : \ r \geq 0,\ j = 0, \dots, m-1\big\},$$

the union of the m rotated copies of the arm $r = b\,\theta^{1/q}$, $\theta \geq 0$. Let $\Omega_{q,b,m} = \{\mathrm{pol}(r,\varphi) : \ r > 0,\ \varphi - f(r) \in (0,\Theta) + 2\pi\mathbb{Z}\}$, let H be the Dirichlet Laplacian on $\Omega_{q,b,m}$ and let $N(E)$ be the number of its eigenvalues below E, counted with multiplicity.

By Lemma 2.1, $\mathbb{R}^2 \setminus \Gamma_{q,b,m}$ is the disjoint union of m rotated copies of $\Omega_{q,b,m}$, so the counting function of the whole region is $mN(E)$. The Fermat region of [3] is $q = 2$, $m = 2$: its arms are $r = b\sqrt{\theta}$ and the rotation of this arm by π, which is the curve $r^2 = b^2\theta$ read with negative radii. Put

$$p = \frac{q}{q-1}, \qquad \kappa = \frac{q+1}{q-1}, \qquad c_{q,m} = \frac{\mathrm{B}(\kappa/2, 3/2)}{2\pi(q-1)}\, q^{1-\kappa}\Big(\frac{2}{m}\Big)^{\kappa}\zeta(\kappa), \tag{1.2}$$

where B is Euler's beta function and ζ Riemann's zeta function, and let $R_q(E) = E^{p/2}$ if $1 < q < 2$, $R_2(E) = E\log(2+E)$, and $R_q(E) = E$ if $q > 2$.

Theorem 1.4. *Let $q > 1$, $b > 0$ and $m \in \mathbb{N}$. Then H has compact resolvent and*

$$N(E) = c_{q,m}\,(b^2E)^p + O\big(R_q(b^2E)\big) \qquad (E \to \infty).$$

More precisely, $c_{q,m}(b^2E)^p = \sum_{n\geq 1} \mathcal{N}_n(E)$, where $\mathcal{N}_n(E)$ is the semiclassical count (4.3) *of the n-th transverse Dirichlet mode* $\sin(n\pi\psi/\Theta)$ *of the coil, and $\mathcal{N}_n(E) = \mathcal{N}_1(E)\,n^{-\kappa}$.*

Since $p > 1$ and $R_q(E) = o(E^p)$, the leading term dominates the area-type contribution $O(E)$ of any bounded part of the region; the law is non-classical, as the area of $\Omega_{q,b,m}$ is infinite. For one arm ($m = 1$) the leading term also follows, by Karamata's Tauberian theorem, from the heat-trace law of van den Berg and Lewis [9, Corollary, (14)]; see Remark 4.6. Theorem 1.4 adds the case of several arms and the explicit remainder.

Corollary 1.5 (The Fermat spiral)**.** *For each component Ω of the Fermat spiral region $\mathbb{R}^2 \setminus \{r^2 = b^2\theta\}$,*

$$N(E) = \zeta(3)\,\frac{b^4E^2}{64} \;+\; O(E\log E), \quad \textit{equivalently} \quad N(E) - \frac{b^4E^2}{64} = \big(\zeta(3)-1\big)\frac{b^4E^2}{64} \;+\; O(E\log E),$$

and the whole region has $\zeta(3)\,b^4E^2/32 + O(E\log E)$ eigenvalues below E. The n-th transverse mode contributes $b^4E^2/(64n^3)$; the estimate (1.1) *is the term $n = 1$.*

This answers Question 1.1. The ratio of $N(E)$ to the one-mode estimate (1.1) tends to $\zeta(3) = 1.2020569\ldots$, so the excess observed in [3, Figure 6] is the contribution of the transverse modes $n \geq 2$ and persists as $E \to \infty$. Every effect beyond the transverse-mode sum, including the curvature and kinetic terms of [3, (5.3)], is contained in the remainder $O(E\log E)$. Section 6 compares the law with the computed spectrum.

Corollary 1.6 (Heat trace)**.** *Under the hypotheses of Theorem 1.4, as $t \downarrow 0$,*

$$\operatorname{Tr} e^{-tH} = \Gamma(p+1)\,c_{q,m}\Big(\frac{b^2}{t}\Big)^p + O\big(\widetilde{R}_q(t)\big),$$

with $\widetilde{R}_q(t) = t^{-p/2}$ for $1 < q < 2$, $t^{-1}\log(1/t)$ for $q = 2$ and t^{-1} for $q > 2$. For a Fermat component, $\operatorname{Tr} e^{-tH} = \zeta(3)\,b^4/(32t^2) + O(t^{-1}\log(1/t))$.

For $m = 1$ the leading term of Corollary 1.6 is the constant of [9, Corollary, (14)] for the spiral $r = c\varphi^\alpha$ with $\alpha = 1/q$ and $c = b$; the two constants agree (Remark 4.6). The remainders and the cases $m \geq 2$ are added here.

The second result holds for general single-armed spirals. Let $r : [0,\infty) \to [0,\infty)$ be continuous and strictly increasing with $r(0) = 0$ and $r(\theta) \to \infty$, and let $\Gamma = \{\operatorname{pol}(r(\theta),\theta) : \theta \geq 0\}$. Every $x \in \mathbb{R}^2 \setminus \Gamma$ has a unique *unwrapped angle* $\vartheta(x) > 0$ with $x = \operatorname{pol}(|x|, \vartheta(x))$ and $r(\vartheta(x) - 2\pi) < |x| < r(\vartheta(x))$, where $r(\theta) := 0$ for $\theta < 0$ (Lemma 5.1). The *radial width* of the coil at the angle $\theta > 0$ is

$$\ell(\theta) = r(\theta) - r(\theta - 2\pi), \tag{1.3}$$

so $\ell(\theta) = r(\theta)$ for $0 < \theta < 2\pi$. In the notation of [3, (5.1)], $\ell = 2\pi a$ with the polar width function a.

Theorem 1.7. *For every $u \in H^1_0(\mathbb{R}^2 \setminus \Gamma)$,*

$$\int_{\mathbb{R}^2\setminus\Gamma} |\nabla u|^2\,\mathrm{d}x \;\geq\; \int_{\mathbb{R}^2\setminus\Gamma} \frac{\pi^2}{\ell(\vartheta(x))^2}\,|u(x)|^2\,\mathrm{d}x. \tag{1.4}$$

Consequently $-\Delta_D \geq \pi^2/L^2$ with $L = \sup_{\theta>0}\ell(\theta)$.

The hypothesis that the arm reaches the origin is essential: when the Archimedean arm is removed near the origin, leaving a central cavity, bound states below the threshold appear once the cavity is large enough [3, Proposition 3.1].

Corollary 1.8. (a) *For the Archimedean spiral $r = a\theta$, $a > 0$, the Dirichlet Laplacian on $\mathbb{R}^2 \setminus \Gamma$ satisfies $-\Delta_D \geq (2a)^{-2}$. Hence its spectrum is $[(2a)^{-2}, \infty)$ and its discrete spectrum is empty.*

(b) *More generally, if $\ell_\infty = \lim_{\theta\to\infty} \ell(\theta)$ exists in $(0, \infty)$ and $\ell(\theta) \leq \ell_\infty$ for all $\theta > 0$, then $-\Delta_D \geq \pi^2/\ell_\infty^2$. Whenever the essential spectrum equals $[\pi^2/\ell_\infty^2, \infty)$, as [3, Proposition 5.3] shows for the asymptotically Archimedean spirals considered there, the discrete spectrum is empty.*

Part (a) answers Question 1.2. Part (b) complements [3, Proposition 5.4], which produces infinitely many bound states when the spiral approaches its linear asymptote from below so that $\ell(\theta) > \ell_\infty$ for large θ; for instance $r(\theta) = a\theta - c(1 - e^{-\theta})$ with $0 < c < a$ has $\ell < 2\pi a = \ell_\infty$ everywhere, so its spectrum lies in $[(2a)^{-2}, \infty)$.

1.3 Relation to earlier work and route of the proofs

Non-classical eigenvalue asymptotics for regions of infinite area with discrete spectrum were found by Simon [6]. For straight horn-shaped regions $\{(x, y) : x > 0,\ -f_1(x) < y < f_2(x)\}$ van den Berg proved heat-trace and counting asymptotics in which every transverse mode is kept [7, 8], with zeta-function coefficients for power-law horns [8, Corollary 3]; see also [10]. A spiral component is a curved horn whose coils wind around each other. For a single arm $r = r(\varphi)$ through the origin with r concave, van den Berg and Lewis [9, Theorem, (10)–(13)] proved, under two growth conditions that include every power $r = c\varphi^\alpha$ with $0 < \alpha < 1$, that the heat trace is asymptotic to that of the unrolled horn $\{0 < y < 2\pi\dot{r}(\varphi)\}$ with every transverse mode kept, and they observed that the curvature is not felt as $t \downarrow 0$ [9, (8)–(9)]. For $r = c\varphi^\alpha$ they computed the constant, which involves $\zeta((1 + \alpha)/(1 - \alpha))$ [9, Corollary, (14)]. For $m = 1$, Corollary 1.6 recovers this constant with a remainder, and Karamata's theorem turns it into the leading term of Theorem 1.4. Barseghyan and Exner compared a shrinking spiral, in Fermi coordinates, with its unrolled horn up to factors $(1 - w)^{\pm 2}$ and used the horn law of [8] for the sharpness of their bounds [1, Section 4.1]; made in both directions with $w \to 0$, this comparison is a second route to leading terms without remainder. What the present paper adds is the case of several arms, which contains the Fermat region of [3]; the counting function with explicit remainders and the heat trace with remainders; the identification of the excess in [3, Figure 6]; the exact orthogonal coordinates of Lemma 2.2; and Theorem 1.7. The mechanism behind Theorem 1.4, quantized transverse modes with semiclassical motion along the coil, also governs the non-classical laws for twisted tubes with diverging twisting in [5]; the proofs below are self-contained.

The proof of Theorem 1.4 rests on an exact coordinate system (Lemma 2.2). Each component is the image of a straight strip $\mathbb{R} \times (0, \Theta)$ in which the coordinate lines $\psi = \text{const}$ are the rotated arms and the lines $\sigma = \text{const}$ are their orthogonal trajectories. The metric is diagonal and the Dirichlet form becomes $\int (w^{-1}|v_\sigma|^2 + w|v_\psi|^2)$ with an explicit weight w. Section 3 localizes away from the centre at a fixed cost and brackets the rest by unit cells $(\varsigma, \varsigma+1) \times (0, \Theta)$ with Neumann or Dirichlet faces, where the coefficients are frozen at their extreme values. Section 4 sums the explicit cell counts over the transverse modes n: every mode yields a unimodal profile whose Riemann sums differ from its integral by at most twice its maximum, and the integrals add up to $c_{q,m}(b^2E)^p$. The relative variation of the coefficients over a cell is $O(1/\varsigma)$, and its sum over cells and modes gives the remainder; for $q = 2$ the sum over modes is harmonic, which is the origin of $\log E$.

Theorem 1.7 is proved in Section 5 in unwrapped polar coordinates. At fixed radius the angular interval of the region has length exactly 2π, which gives the angular energy $|u|^2/(4r^2)$; at fixed

unwrapped angle the radial interval has length $\ell(\theta)$, and the substitution $g = r^{1/2}u$ turns the radial energy into $|g'|^2 - |g|^2/(4r^2)$. The two terms $|g|^2/(4r^2)$ cancel, and the Dirichlet condition at both ends of every radial interval, including the origin, which lies on the spiral, gives (1.4). The bound of [3, Section 2] treats the end $r = 0$ of the inner radial intervals as free, which yields finiteness of the discrete spectrum; the cancellation above uses that the origin belongs to Γ.

2 Spiral components and their orthogonal coordinates

Throughout this section $q > 1$, $b > 0$, $m \in \mathbb{N}$ and $\Theta = 2\pi/m$ are fixed, $f(r) = (r/b)^q$, and $\Omega = \Omega_{q,b,m}$.

Lemma 2.1. *A point* $\mathrm{pol}(r,\varphi)$ *with* $r > 0$ *lies on* $\Gamma_{q,b,m}$ *if and only if* $\varphi - f(r) \in \Theta\mathbb{Z} + 2\pi\mathbb{Z} = \Theta\mathbb{Z}$, *and the origin lies on* $\Gamma_{q,b,m}$. *Consequently* $\mathbb{R}^2 \setminus \Gamma_{q,b,m}$ *is the disjoint union of the open connected sets* $\rho_{j\Theta}(\Omega)$, $j = 0, \dots, m-1$, *where* ρ_α *is the rotation by* α, *and its Dirichlet counting function is* $mN(E)$.

Proof. The first two statements are the definition of $\Gamma_{q,b,m}$ (the origin is the point $r = 0$). The map $P(r,\psi) = \mathrm{pol}(r, \psi + f(r))$ is a local diffeomorphism of $(0,\infty)\times\mathbb{R}$ onto $\mathbb{R}^2 \setminus \{0\}$, and $P(r,\psi) \in \Gamma_{q,b,m}$ exactly when $\psi \in \Theta\mathbb{Z}$. Hence $\mathbb{R}^2 \setminus \Gamma_{q,b,m}$ is the union of the sets $P((0,\infty)\times(j\Theta,(j+1)\Theta)) = \rho_{j\Theta}(\Omega)$, which are open, connected as continuous images of connected sets, and pairwise disjoint because the intervals $(j\Theta,(j+1)\Theta)$, $0 \le j < m$, are disjoint modulo 2π. The Dirichlet Laplacian of a disjoint union of open sets is the direct sum of the Dirichlet Laplacians of the pieces, and rotations preserve the spectrum. □

Put

$$F(r) = \Big(\frac{r}{b}\Big)^q - \frac{1}{q^2}\Big(\frac{b}{r}\Big)^q, \qquad F'(r) = f'(r) + \frac{1}{r^2 f'(r)} > 0. \tag{2.1}$$

The function F is an increasing bijection of $(0,\infty)$ onto $\mathbb{R}$. Its inverse is $\mathcal{R}(\tau) = b\,\xi(\tau)^{1/q}$ with

$$\xi(\tau) = \tfrac{1}{2}\big(\tau + \sqrt{\tau^2 + 4/q^2}\big), \qquad \tau < \xi(\tau) < \tau + \frac{1}{q^2\tau} \quad (\tau > 0), \tag{2.2}$$

because $\xi = (r/b)^q$ satisfies the quadratic equation $\xi - 1/(q^2\xi) = \tau$ when $F(r) = \tau$, and $\sqrt{\tau^2 + c} < \tau + c/(2\tau)$.

Lemma 2.2 (Orthogonal coordinates)**.** *The map*

$$\Phi : \mathbb{R}\times(0,\Theta) \to \Omega, \qquad \Phi(\sigma,\psi) = \mathrm{pol}(\rho,\ \psi + f(\rho)), \quad \rho = \mathcal{R}(\sigma - \psi),$$

is a C^∞ *diffeomorphism. With* $w = \rho f'(\rho) = q(\rho/b)^q$, *the Euclidean metric pulls back to*

$$\Phi^*(\,\mathrm{d}x^2) = h_\sigma^2\,\mathrm{d}\sigma^2 + h_\psi^2\,\mathrm{d}\psi^2, \qquad h_\sigma = \frac{\rho w}{\sqrt{1+w^2}}, \quad h_\psi = \frac{\rho}{\sqrt{1+w^2}}.$$

Consequently, for $u \in C_c^\infty(\Omega)$ *and* $v = u\circ\Phi \in C_c^\infty(\mathbb{R}\times(0,\Theta))$,

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x = \iint \Big(\frac{|v_\sigma|^2}{w} + w|v_\psi|^2\Big)\,\mathrm{d}\sigma\,\mathrm{d}\psi, \qquad \int_\Omega |u|^2\,\mathrm{d}x = \iint \omega\,|v|^2\,\mathrm{d}\sigma\,\mathrm{d}\psi, \qquad \omega = \frac{\rho^2 w}{1+w^2}, \tag{2.3}$$

where w *and* ω *are evaluated at* $\rho = \mathcal{R}(\sigma - \psi)$.

Proof. The map $(\sigma,\psi)\mapsto(\mathcal{R}(\sigma-\psi),\psi)$ is a diffeomorphism of $\mathbb{R}\times(0,\Theta)$ onto $(0,\infty)\times(0,\Theta)$, and P of Lemma 2.1 maps $(0,\infty)\times(0,\Theta)$ bijectively onto Ω, since $\Theta\le 2\pi$ and the interval is open; P is a local diffeomorphism, so Φ is a bijective local diffeomorphism. In the polar coordinates (ρ,φ) of the image, $\varphi=\psi+f(\rho)$ and $F(\rho)=\sigma-\psi$, so

$$\mathrm{d}\rho=\frac{\mathrm{d}\sigma-\mathrm{d}\psi}{F'(\rho)},\qquad \mathrm{d}\varphi=\mathrm{d}\psi+f'(\rho)\,\mathrm{d}\rho.$$

Write $k=1/F'(\rho)=\rho^2 f'/(1+w^2)$. Then $\mathrm{d}x^2=\mathrm{d}\rho^2+\rho^2\,\mathrm{d}\varphi^2=k^2(\,\mathrm{d}\sigma-\,\mathrm{d}\psi)^2+\rho^2(f'k\,\mathrm{d}\sigma+(1-f'k)\,\mathrm{d}\psi)^2$ and $1-f'k=1/(1+w^2)$. The coefficient of $\mathrm{d}\sigma\,\mathrm{d}\psi$ is $-2k^2+2\rho^2 f'k(1-f'k)=-2k^2+2\rho^4 f'^2/(1+w^2)^2=0$. The coefficient of $\mathrm{d}\sigma^2$ is $k^2(1+\rho^2 f'^2)=\rho^2 w^2/(1+w^2)$, and that of $\mathrm{d}\psi^2$ is $k^2+\rho^2/(1+w^2)^2=\rho^2/(1+w^2)$. The metric is nondegenerate, so Φ is a diffeomorphism. In orthogonal coordinates $|\nabla u|^2=h_\sigma^{-2}|v_\sigma|^2+h_\psi^{-2}|v_\psi|^2$ and $\mathrm{d}x=h_\sigma h_\psi\,\mathrm{d}\sigma\,\mathrm{d}\psi$; since $h_\psi/h_\sigma=1/w$ and $h_\sigma h_\psi=\omega$, (2.3) follows. □

The verification script `coords_check.py` repeats this computation symbolically. The lines $\psi=$ const are the rotated arms; the lines $\sigma=$ const are their orthogonal trajectories. For the Fermat spiral ($q=2$) these are the curves $\varphi-b^2/(4\rho^2)=$ const, and $w=2\rho^2/b^2$.

Lemma 2.3 (Scaling)**.** *$\Omega_{q,b,m}=b\,\Omega_{q,1,m}$, so the counting functions satisfy $N_{q,b,m}(E)=N_{q,1,m}(b^2E)$.*

Proof. $\mathrm{pol}(r,\varphi)\in\Omega_{q,b,m}$ if and only if $\mathrm{pol}(r/b,\varphi)\in\Omega_{q,1,m}$, and dilation by b multiplies Dirichlet eigenvalues by b^{-2}. □

By Lemmas 2.1 and 2.2, $H_0^1(\Omega)$ is identified, via $u\mapsto u\circ\Phi$, with the closure $\mathcal{D}$ of $C_c^\infty(\mathbb{R}\times(0,\Theta))$ in the norm of the form $\mathfrak{q}[v]=\iint(w^{-1}|v_\sigma|^2+w|v_\psi|^2)$ on $L^2(\mathbb{R}\times(0,\Theta),\omega\,\mathrm{d}\sigma\,\mathrm{d}\psi)$, and H is the operator of $\mathfrak{q}$.

3 Counting tools

For a closed semibounded quadratic form $\mathfrak{a}$ in a Hilbert space with operator A, let $N(\mathfrak{a};E)$ be the dimension of the spectral subspace of A for $(-\infty,E)$. By the min-max principle [4, Section XIII.1], $N(\mathfrak{a};E)$ is the supremum of $\dim L$ over subspaces L of the form domain with $\mathfrak{a}[u]<E\|u\|^2$ for all $u\in L\setminus\{0\}$ (Glazman's lemma). Bracketing by Dirichlet and Neumann conditions is classical [4, Section XIII.15]; we use it in the following form.

Lemma 3.1. *Let $\mathfrak{a}$ and $\mathfrak{a}_k$, $k\in K$ countable, be closed semibounded forms in Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_k$, and let J_k map the form domain of $\mathfrak{a}$ linearly into that of $\mathfrak{a}_k$, with $\sum_k\|J_k u\|^2=\|u\|^2$. Let $E,E_k\in\mathbb{R}$ satisfy*

$$\mathfrak{a}[u]-E\|u\|^2\;\ge\;\sum_k\bigl(\mathfrak{a}_k[J_k u]-E_k\|J_k u\|^2\bigr)$$

for every u in the form domain of $\mathfrak{a}$. Then $N(\mathfrak{a};E)\le\sum_k N(\mathfrak{a}_k;E_k)$.

Proof. Assume $n:=\sum_k N(\mathfrak{a}_k;E_k)<\infty$, and let P_k be the spectral projection of $\mathfrak{a}_k$ for $(-\infty,E_k)$. If a subspace L as in Glazman's lemma had dimension $n+1$, the linear map $u\mapsto(P_kJ_ku)_k$ from L into a space of dimension n would vanish at some $u\neq0$. For this u every J_ku lies in the spectral subspace of $[E_k,\infty)$, so $\mathfrak{a}_k[J_ku]\ge E_k\|J_ku\|^2$ for all k, hence $\mathfrak{a}[u]\ge E\|u\|^2$, a contradiction. □

Fix $s_*\in\mathbb{R}$ and functions $\chi_1,\chi_2\in C^\infty(\mathbb{R})$ with $\chi_1^2+\chi_2^2=1$, $\chi_1=1$ on $(-\infty,s_*-\frac34]$ and $\chi_1=0$ on $[s_*-\frac14,\infty)$. Let $\Omega_{\mathrm{in}}=\Phi(\{\sigma<s_*\})$, $\Omega_{\mathrm{out}}=\Phi(\{\sigma>s_*-1\})$ and $T=\Phi(\{s_*-1\le\sigma\le s_*\})$.

Lemma 3.2 (Localization)**.** *There is $c_\chi < \infty$, depending on q, b, m, s_* and the χ_i, such that for all E*

$$N(E) \le N_D(E + c_\chi; \Omega_{\mathrm{in}}) + N(\mathfrak{q}_{\mathrm{out}} - c_\chi \mathbf{1}_T; E),$$

where $N_D(\cdot\,; \Omega_{\mathrm{in}})$ is the Dirichlet counting function of Ω_{in} and $\mathfrak{q}_{\mathrm{out}}$ is the Dirichlet form of Ω_{out}. Moreover Ω_{in} lies in the disc of radius $\mathcal{R}(s_*)$, and $N_D(E; \Omega_{\mathrm{in}}) \le \mathcal{R}(s_*)^2 E/\pi$ for $E \ge 0$.

Proof. Put $X_i = \chi_i \circ \sigma$, a smooth function on Ω with $|\nabla X_i| = |\chi_i'(\sigma)|/h_\sigma$. For $u \in C_c^\infty(\Omega)$, expanding $|\nabla(X_i u)|^2$ and using $\sum_i X_i^2 = 1$ and $\sum_i X_i \nabla X_i = 0$ gives

$$\int |\nabla u|^2 = \sum_{i=1}^{2} \int |\nabla(X_i u)|^2 - \int V |u|^2, \qquad V = \sum_i |\nabla X_i|^2,$$

and by density the identity holds on $H_0^1(\Omega)$. The function V vanishes outside T. On T the radius $\rho = \mathcal{R}(\sigma - \psi)$ lies in $[\mathcal{R}(s_* - 1 - \Theta), \mathcal{R}(s_*)]$, a compact subset of $(0, \infty)$, so h_σ is bounded below there and $V \le c_\chi$. Since $\int V|u|^2 \le c_\chi \int_T (X_1^2 + X_2^2)|u|^2 \le c_\chi \|X_1 u\|^2 + c_\chi \|\mathbf{1}_T X_2 u\|^2$,

$$\int |\nabla u|^2 - E\|u\|^2 \ge \Big(\int |\nabla(X_1 u)|^2 - (E + c_\chi)\|X_1 u\|^2 \Big) + \Big(\int |\nabla(X_2 u)|^2 - c_\chi \|\mathbf{1}_T X_2 u\|^2 - E\|X_2 u\|^2 \Big).$$

For $u \in C_c^\infty(\Omega)$ the support of $X_1 u$ is a compact subset of Ω_{in} and that of $X_2 u$ a compact subset of Ω_{out}, so by density $X_1 u \in H_0^1(\Omega_{\mathrm{in}})$ and $X_2 u \in H_0^1(\Omega_{\mathrm{out}})$ for $u \in H_0^1(\Omega)$. Hence Lemma 3.1 with $J_i u = X_i u$ gives the first claim. Every point of Ω_{in} has $\rho = \mathcal{R}(\sigma - \psi) < \mathcal{R}(s_*)$, so Ω_{in} lies in the disc B of radius $\mathcal{R}(s_*)$, which lies in a square of side $2\mathcal{R}(s_*)$. Domain monotonicity of Dirichlet eigenvalues and the explicit spectrum of the square give $N_D(E; \Omega_{\mathrm{in}}) \le \#\{(k,l) \in \mathbb{N}^2 : \pi^2(k^2 + l^2) < 4\mathcal{R}(s_*)^2 E\} \le \mathcal{R}(s_*)^2 E/\pi$. □

Cells. For $\varsigma \in \mathbb{R}$ let $\widetilde{Q}_\varsigma = (\varsigma, \varsigma + 1) \times (0, \Theta)$ and $Q_\varsigma = \Phi(\widetilde{Q}_\varsigma)$. Let $w_\pm$ and $\omega_\pm$ be the supremum and infimum of w and ω over $\widetilde{Q}_\varsigma$. On $\widetilde{Q}_\varsigma$ the coefficients of $\mathfrak{q}$ and the weight ω are bounded above and below, so the closure of $\{v \in C^\infty(\overline{\widetilde{Q}_\varsigma}) :\ v = 0 \text{ near } \psi \in \{0, \Theta\}\}$ in the form norm is $D_\varsigma^N = \{v \in H^1(\widetilde{Q}_\varsigma) :\ v = 0 \text{ on } \psi \in \{0, \Theta\}\}$. Let $\mathfrak{q}_\varsigma^N$ and $\mathfrak{q}_\varsigma^D$ be the form $\mathfrak{q}$ restricted to $\widetilde{Q}_\varsigma$ on D_ς^N and on $H_0^1(\widetilde{Q}_\varsigma)$, in $L^2(\widetilde{Q}_\varsigma, \omega \,\mathrm{d}\sigma\,\mathrm{d}\psi)$, and write $\nu_n = n\pi/\Theta = nm/2$.

Lemma 3.3 (Frozen cells)**.** *For $E \ge 0$,*

$$N(\mathfrak{q}_\varsigma^N; E) \le \sum_{n \ge 1} \Big[\frac{1}{\pi} \sqrt{w_+ \omega_+} \Big(E - \nu_n^2 \frac{w_-}{\omega_+} \Big)_+^{1/2} + \mathbf{1}\Big\{ \nu_n^2 \frac{w_-}{\omega_+} < E \Big\} \Big], \tag{3.1}$$

$$N(\mathfrak{q}_\varsigma^D; E) \ge \sum_{n \ge 1} \Big[\frac{1}{\pi} \sqrt{w_- \omega_-} \Big(E - \nu_n^2 \frac{w_+}{\omega_-} \Big)_+^{1/2} - \mathbf{1}\Big\{ \nu_n^2 \frac{w_+}{\omega_-} < E \Big\} \Big]. \tag{3.2}$$

Proof. For $v \in D_\varsigma^N$ we have $\mathfrak{q}[v] - E\|v\|_\omega^2 \ge \mathfrak{m}[v] - E\omega_+ \|v\|^2_{L^2(\widetilde{Q}_\varsigma)}$ with $\mathfrak{m}[v] = w_+^{-1}\|v_\sigma\|^2 + w_-\|v_\psi\|^2$ on D_ς^N in the unweighted space $L^2(\widetilde{Q}_\varsigma)$. The two forms have the same domain, so Glazman's lemma gives $N(\mathfrak{q}_\varsigma^N; E) \le N(\mathfrak{m}; E\omega_+)$. The functions $\cos(k\pi(\sigma - \varsigma)) \sin(\nu_n \psi)$, $k \ge 0$, $n \ge 1$, form an orthogonal eigenbasis of $\mathfrak{m}$ with eigenvalues $k^2\pi^2/w_+ + w_-\nu_n^2$. For fixed n the number of $k \ge 0$ with $k^2\pi^2 < w_+(E\omega_+ - w_-\nu_n^2)$ is at most $1 + \pi^{-1}\sqrt{w_+(E\omega_+ - w_-\nu_n^2)}$ when the bracket is positive and 0 otherwise; this is (3.1). For (3.2), $\mathfrak{q}[v] - E\|v\|_\omega^2 \le w_-^{-1}\|v_\sigma\|^2 + w_+\|v_\psi\|^2 - E\omega_-\|v\|^2$ on $H_0^1(\widetilde{Q}_\varsigma)$, whose eigenbasis is $\sin(k\pi(\sigma - \varsigma)) \sin(\nu_n \psi)$, $k, n \ge 1$; the number of $k \ge 1$ below a level $X > 0$ is at least $X - 1$. □

4 Proof of the counting law

By Lemma 2.3 we may take $b = 1$. Then $f(r) = r^q$, $w = q\,\xi$ and, by (2.3),

$$\omega = \frac{\xi^{2/q}\, q\xi}{1 + q^2\xi^2} = \frac{\xi^e}{q}\Big(1 + \frac{1}{q^2\xi^2}\Big)^{-1}, \qquad e = \frac{2}{q} - 1 \in (-1, 1), \tag{4.1}$$

where $\xi = \xi(\sigma - \psi)$. Put $\gamma = 2 - 2/q \in (0, 2)$, so that $1 + e = 2/q$ and $1 - e = \gamma$. Constants C depend only on q and m and may change from line to line.

Lemma 4.1 (Coefficients on a cell)**.** *Let $\varsigma \ge s_0 := 16(\Theta + 2)/\min\{\gamma, 1\}$ and $\varepsilon = (\Theta + 2)/\varsigma$. On $\widetilde{Q}_\varsigma$,*

$$\sqrt{w_+\omega_+} \le \varsigma^{1/q}(1 + 2\varepsilon), \quad \sqrt{w_-\omega_-} \ge \varsigma^{1/q}(1 - 3\varepsilon), \quad \frac{w_-}{\omega_+} \ge q^2\varsigma^\gamma(1 - 3\varepsilon), \quad \frac{w_+}{\omega_-} \le q^2\varsigma^\gamma(1 + 4\varepsilon).$$

Proof. On $\widetilde{Q}_\varsigma$ we have $\tau = \sigma - \psi \in (\varsigma - \Theta, \varsigma + 1)$ and $\tau > 16$, so by (2.2) $\xi \in (\varsigma - \Theta, \varsigma + 17/16) \subset [\varsigma(1 - \varepsilon), \varsigma(1 + \varepsilon)]$, and $\varepsilon \le 1/16$. Hence $w = q\xi \in [q\varsigma(1 - \varepsilon), q\varsigma(1 + \varepsilon)]$. Since $|e| < 1$ and $\varepsilon \le 1/2$, $\xi^e \in [\varsigma^e(1 - \varepsilon), \varsigma^e(1 + 2\varepsilon)]$: for $e \ge 0$ use $(1 \pm \varepsilon)^e$ between $1 - \varepsilon$ and $1 + \varepsilon$; for $e < 0$ use $(1 - \varepsilon)^e \le (1 - \varepsilon)^{-1} \le 1 + 2\varepsilon$ and $(1 + \varepsilon)^e \ge (1 + \varepsilon)^{-1} \ge 1 - \varepsilon$. Also $q^{-2}\xi^{-2} \le \xi^{-2} \le 4/\varsigma^2 \le (\Theta + 2)/\varsigma = \varepsilon$, because $\xi \ge \varsigma(1 - \varepsilon) \ge \varsigma/2$ and $\varsigma \ge 32$, so the last factor in (4.1) lies in $[1 - \varepsilon, 1]$. Therefore $\omega \in [q^{-1}\varsigma^e(1 - 2\varepsilon),\ q^{-1}\varsigma^e(1 + 2\varepsilon)]$, and

$$w_+\omega_+ \le \varsigma^{2/q}(1 + \varepsilon)(1 + 2\varepsilon) \le \varsigma^{2/q}(1 + 4\varepsilon), \qquad w_-\omega_- \ge \varsigma^{2/q}(1 - \varepsilon)(1 - 2\varepsilon) \ge \varsigma^{2/q}(1 - 3\varepsilon),$$

$$\frac{w_-}{\omega_+} \ge q^2\varsigma^\gamma\frac{1 - \varepsilon}{1 + 2\varepsilon} \ge q^2\varsigma^\gamma(1 - 3\varepsilon), \qquad \frac{w_+}{\omega_-} \le q^2\varsigma^\gamma\frac{1 + \varepsilon}{1 - 2\varepsilon} \le q^2\varsigma^\gamma(1 + 4\varepsilon),$$

using $\varepsilon \le 1/8$. Taking square roots gives the first two bounds. □

For $n \ge 1$ and $E > 0$ define the *mode profile*

$$g_n(s) = \frac{1}{\pi}\, s_+^{1/q}\big(E - \nu_n^2 q^2 s_+^\gamma\big)_+^{1/2} \qquad (s \in \mathbb{R}), \qquad z_n = \Big(\frac{\sqrt{E}}{q\nu_n}\Big)^{2/\gamma}, \tag{4.2}$$

so that $g_n > 0$ exactly on $(0, z_n)$. Note $z_n^{1/q} = (\sqrt{E}/(q\nu_n))^{1/(q-1)}$ and $z_n^{1+1/q} = (\sqrt{E}/(q\nu_n))^\kappa$.

Lemma 4.2 (Mode profiles)**.** (a) *g_n is unimodal, $\sup g_n \le \pi^{-1} z_n^{1/q}\sqrt{E}$, and*

$$\mathcal{N}_n(E) := \int_{\mathbb{R}} g_n(s)\,\mathrm{d}s = \frac{\mathrm{B}(\kappa/2, 3/2)}{2\pi(q - 1)}\, q^{1-\kappa}\, E^p\, \nu_n^{-\kappa}, \qquad \sum_{n\ge1} \mathcal{N}_n(E) = c_{q,m}E^p. \tag{4.3}$$

(b) *If $g : \mathbb{R} \to [0, \infty)$ is integrable, nondecreasing on $(-\infty, a_0]$ and nonincreasing on $[a_0, \infty)$, then $\big|\sum_{j\in\mathbb{Z}} g(t + j) - \int_{\mathbb{R}} g\big| \le 2\sup g$ for every $t \in \mathbb{R}$.*

Proof. (a) On $(0, z_n)$ put $G = \pi^2 g_n^2 = s^{2/q}(E - \nu_n^2 q^2 s^\gamma)$. Since $2/q + \gamma = 2$, $G'(s) = s^{2/q-1}\big(\frac{2}{q}E - 2\nu_n^2 q^2 s^\gamma\big)$, and the bracket is strictly decreasing, so G and g_n increase and then decrease. The bound on $\sup g_n$ is immediate. The substitution $s = z_n t$ and then $y = t^\gamma$ give

$$\int_{\mathbb{R}} g_n = \frac{\sqrt{E}}{\pi}\, z_n^{1+1/q}\int_0^1 t^{1/q}(1 - t^\gamma)^{1/2}\,\mathrm{d}t = \frac{\sqrt{E}}{\pi}\Big(\frac{\sqrt{E}}{q\nu_n}\Big)^\kappa\frac{1}{\gamma}\,\mathrm{B}\Big(\frac{1 + 1/q}{\gamma}, \frac{3}{2}\Big).$$

Since $(1 + 1/q)/\gamma = \kappa/2$, $1/\gamma = q/(2(q - 1))$ and $(1 + \kappa)/2 = p$, this is (4.3). Summing, $\sum_n \nu_n^{-\kappa} = (2/m)^\kappa\zeta(\kappa)$ with $\kappa > 1$.

(b) Let t_J be the largest lattice point $t+j$ in $(-\infty, a_0]$. For lattice points $t_i \le t_J$, monotonicity gives $\int_{t_i-1}^{t_i} g \le g(t_i)$, and $g(t_i) \le \int_{t_i}^{t_i+1} g$ when $t_i < t_J$. Hence the sum over the lattice points in $(-\infty, a_0]$ lies between $\int_{-\infty}^{t_J} g \ge \int_{-\infty}^{a_0} g - \sup g$ and $\int_{-\infty}^{t_J} g + g(t_J) \le \int_{-\infty}^{a_0} g + \sup g$. The same argument applies on (a_0, ∞), and the two errors add. □

Lemma 4.3 (Shifts)**.** *Let $c = 8(\Theta+2)/\min\{\gamma, 1\}$ and $c' = 3c + 2(\Theta+2)$. For $\varsigma \ge s_0$, with ε as in Lemma 4.1,*

$$(1-3\varepsilon)\varsigma^\gamma \ge (\varsigma - c)^\gamma, \qquad (1+4\varepsilon)\varsigma^\gamma \le (\varsigma + c)^\gamma,$$
$$\varsigma^{1/q}(1+2\varepsilon) \le (\varsigma - c)^{1/q}\Big(1 + \frac{c'}{\varsigma}\Big), \qquad \varsigma^{1/q}(1-3\varepsilon) \ge (\varsigma + c)^{1/q}\Big(1 - \frac{c'}{\varsigma}\Big).$$

Proof. Put $y = c/\varsigma \le 1/2$ and $\delta = \min\{\gamma, 1\}$. By concavity or convexity of $t \mapsto t^\gamma$, $(1-y)^\gamma \le 1 - \delta y$ and $(1+y)^\gamma \ge 1 + \delta y/2$ on $[0,1]$; since $\delta y = 8(\Theta+2)/\varsigma \ge 3\varepsilon$ and $\delta y/2 \ge 4\varepsilon$, the first two inequalities follow. Next $(\varsigma/(\varsigma - c))^{1/q} \le (1-y)^{-1} \le 1 + 2y$ and $(1+2y)(1+2\varepsilon) \le 1 + (2c + 2(\Theta+2) + 4c\varepsilon)/\varsigma \le 1 + c'/\varsigma$, because $4\varepsilon \le 1$. Finally $(\varsigma/(\varsigma + c))^{1/q} \ge 1 - y$ and $(1-y)(1-3\varepsilon) \ge 1 - c'/\varsigma$. (The last inequality is trivial when $c' > \varsigma$; below it is used only for $\varsigma \ge 2c'$, where $1 - c'/\varsigma \ge \frac{1}{2}$.) □

Proposition 4.4 (Upper bound)**.** *There are C and E_0, depending only on q and m, such that $N(E) \le c_{q,m} E^p + C\, R_q(E)$ for $E \ge E_0$.*

Proof. Choose $s_* = s_0 + 2c'$ in Lemma 3.2 (so that $s_* - 1 \ge s_0$, $s_* - c - 1 \ge 0$ and $s_* \ge 2c'$) and cover $\{\sigma > s_* - 1\}$ by the cells $\widetilde{Q}_{\varsigma_j}$, $\varsigma_j = s_* - 1 + j$, $j \ge 0$, up to the null set of cut lines. Restriction to the cells maps $H^1_0(\Omega_{\mathrm{out}})$ isometrically into $\bigoplus_j D^N_{\varsigma_j}$ and splits the form; T is the first cell. Lemmas 3.1 and 3.2 give

$$N(E) \le \mathcal{R}(s_*)^2 (E + c_\chi)/\pi + N(\mathfrak{q}^N_{\varsigma_0}; E + c_\chi) + \sum_{j \ge 1} N(\mathfrak{q}^N_{\varsigma_j}; E). \tag{4.4}$$

By Lemmas 3.3, 4.1 and 4.3, for every cell and every $E' \ge 0$,

$$N(\mathfrak{q}^N_{\varsigma_j}; E') \le \sum_{n \ge 1} \Big[\Big(1 + \frac{c'}{\varsigma_j}\Big) g_n^{(E')}(\varsigma_j - c) + \mathbf{1}\{\nu_n^2 q^2 (\varsigma_j - c)^\gamma < E'\}\Big], \tag{4.5}$$

where $g_n^{(E')}$ is the profile (4.2) at energy E'. The cell $j = 0$ at energy $E + c_\chi$ contributes at most $C(E+1)$: each term is at most $C\sqrt{E+1}$, and at most $C\sqrt{E+1}$ modes are active. For $j \ge 1$ we bound the three parts of (4.5) summed over j and n.

A mode n contributes only if $z_n > \varsigma_1 - c \ge 1$, that is $\nu_n < \sqrt{E}/q$; there are at most $n_E \le C\sqrt{E}$ such modes. By Lemma 4.2(b) applied to g_n (with $g_n = 0$ near $-\infty$),

$$\sum_{j \ge 1} g_n(\varsigma_j - c) \le \mathcal{N}_n(E) + 2 \sup g_n \le \mathcal{N}_n(E) + \tfrac{2}{\pi} z_n^{1/q} \sqrt{E}.$$

Next, $g_n(\varsigma - c) \le \pi^{-1} \varsigma^{1/q} \sqrt{E}$ and vanishes for $\varsigma \ge z_n + c$, so, comparing the sum of the decreasing function $\varsigma^{1/q-1}$ with an integral,

$$\sum_{j \ge 1} \frac{c'}{\varsigma_j} g_n(\varsigma_j - c) \le \frac{c'\sqrt{E}}{\pi} \int_0^{z_n + c} t^{1/q - 1}\, \mathrm{d}t \le C\sqrt{E}\,(z_n^{1/q} + 1).$$

For the indicators, the number of n with $\nu_n < \sqrt{E}/(q(\varsigma-c)^{\gamma/2})$ is at most $C\sqrt{E}(\varsigma-c)^{-\gamma/2}$, and it vanishes for $\varsigma - c \ge z_1$; as $\gamma/2 = 1 - 1/q$,

$$\sum_{j\ge 1}\sum_{n} \mathbf{1}\{\nu_n^2 q^2(\varsigma_j - c)^\gamma < E\} \le C\sqrt{E}\int_0^{z_1} t^{1/q-1}\,\mathrm{d}t \le C\sqrt{E}\, z_1^{1/q} = C\,E^{p/2},$$

because $\sqrt{E}\, z_1^{1/q} = \sqrt{E}\,(\sqrt{E}/(q\nu_1))^{1/(q-1)}$ and $\frac{1}{2} + \frac{1}{2(q-1)} = \frac{p}{2}$. Collecting, with $\sum_n \mathcal{N}_n(E) = c_{q,m}E^p$ from (4.3),

$$N(E) \le c_{q,m}E^p + C\sqrt{E}\sum_{n\le n_E}(z_n^{1/q}+1) + C\,E^{p/2} + C(E+1).$$

Here $\sqrt{E}\sum_{n\le n_E} z_n^{1/q} = CE^{p/2}\sum_{n\le n_E} n^{-1/(q-1)}$, which is $O(E^{p/2})$ for $q<2$, $O(E\log E)$ for $q=2$, and $O(E^{p/2}n_E^{1-1/(q-1)}) = O(E)$ for $q>2$; and $\sqrt{E}\,n_E \le CE$. Since $E^{p/2} \ge E$ when $q<2$, every error term is $O(R_q(E))$. □

Proposition 4.5 (Lower bound)**.** *There are C and E_0, depending only on q and m, such that $N(E) \ge c_{q,m}E^p - C\,R_q(E)$ for $E \ge E_0$.*

Proof. Let $\varsigma_j = s_* + j$, $j \ge 0$, with s_* as before. The sets Q_{ς_j} are disjoint open subsets of Ω, and extension by zero embeds $\bigoplus_j H_0^1(\widetilde{Q}_{\varsigma_j})$ into $\mathcal{D}$ with additive form and norm, so Glazman's lemma gives $N(E) \ge \sum_j N(\mathfrak{q}^D_{\varsigma_j}; E)$. By Lemmas 3.3, 4.1 and 4.3,

$$N(E) \ge \sum_{n\ge1}\sum_{j\ge0}\Big[\Big(1-\frac{c'}{\varsigma_j}\Big)g_n(\varsigma_j+c) - \mathbf{1}\{\nu_n^2q^2(\varsigma_j-c)^\gamma < E\}\Big];$$

for the indicator we used $w_+/\omega_- \ge w_-/\omega_+ \ge q^2\varsigma^\gamma(1-3\varepsilon) \ge q^2(\varsigma-c)^\gamma$. Apply Lemma 4.2(b) to $g_n\mathbf{1}_{[s_*+c,\infty)}$, which is again unimodal: $\sum_{j\ge0} g_n(\varsigma_j+c) \ge \int_{s_*+c}^\infty g_n - 2\sup g_n\,\mathbf{1}\{z_n > s_*+c\}$. The terms with c'/ς_j and the indicators are bounded exactly as in the proof of Proposition 4.4, and so is $\sum_n 2\sup g_n\mathbf{1}\{z_n > s_*+c\}$. It remains to compare $\sum_n\int_{s_*+c}^\infty g_n$ with $\sum_n \mathcal{N}_n(E)$. Put $S = s_*+c$. For the at most n_E modes with $z_n > S$, $\int_0^S g_n \le \pi^{-1}S^{1+1/q}\sqrt{E}$, which sums to $O(E)$. The modes with $z_n \le S$ satisfy $\nu_n \ge \nu_* := \sqrt{E}/(qS^{\gamma/2})$, and for them $\int_{\mathbb{R}} g_n \le \pi^{-1}\sqrt{E}\,z_n^{1+1/q} = \pi^{-1}\sqrt{E}(\sqrt{E}/(q\nu_n))^\kappa$; since $\kappa > 1$ and $\nu_* \ge 1$ for $E \ge E_0$,

$$\sum_{n:\,\nu_n\ge\nu_*} \sqrt{E}\Big(\frac{\sqrt{E}}{q\nu_n}\Big)^\kappa \le C\,E^{(1+\kappa)/2}\,\nu_*^{1-\kappa} = C\,E.$$

Hence $\sum_n\int_S^\infty g_n \ge c_{q,m}E^p - CE$, and the proposition follows. □

Proof of Theorem 1.4. Propositions 4.4 and 4.5 prove the asymptotics for $b=1$, and Lemma 2.3 gives general b. Since $N(E) < \infty$ for every E, the spectrum of H is discrete and the resolvent is compact. The identities for $\mathcal{N}_n$ are (4.3); for general b, $\mathcal{N}_n(E)$ denotes the profile integral at energy b^2E. □

Proof of Corollary 1.5. For $q=2$ and $m=2$: $p=2$, $\kappa=3$, $\nu_n = n$, $\mathrm{B}(3/2,3/2) = \pi/8$, so $\mathcal{N}_n(E) = b^4E^2/(64n^3)$ and $c_{2,2} = \zeta(3)/64$. The whole region has two components (Lemma 2.1). For $n=1$ the profile is $\pi^{-1}\sqrt{s}\,(b^2E - 4s)_+^{1/2}$ in the variable $s = (r/b)^2$, which is the arm angle θ. The substitution $s_{\mathrm{arc}} = \frac{2}{3}b\,\theta^{3/2}$, the leading behaviour of the arclength in [3, Section 5.1], turns its integral into the one-mode phase volume $\pi^{-1}\int(E - c\,s_{\mathrm{arc}}^{2/3})_+^{1/2}\,\mathrm{d}s_{\mathrm{arc}} = b^4E^2/64$ with $c = (2/b)^2(3/(2b))^{2/3}$, which yields (1.1). □

Proof of Corollary 1.6. Write $N(E) = c_{q,m}(b^2E)^p + \eta(E)$. By Theorem 1.4, $|\eta(E)| \le C(1+R_q(b^2E))$ for all $E \ge 0$, since N is bounded on bounded intervals. Integration by parts gives $\operatorname{Tr} e^{-tH} = \int_{[0,\infty)} e^{-tE}\,\mathrm{d}N(E) = t\int_0^\infty e^{-tE}N(E)\,\mathrm{d}E$, and $t\int_0^\infty e^{-tE}E^p\,\mathrm{d}E = \Gamma(p+1)t^{-p}$. The contribution of η is at most $Ct\int_0^\infty e^{-tE}(1+R_q(b^2E))\,\mathrm{d}E$, which is $O(t^{-p/2})$, $O(t^{-1}\log(1/t))$ or $O(t^{-1})$ in the three cases. For the Fermat spiral, $\Gamma(3)\zeta(3)/64 = \zeta(3)/32$. □

Remark 4.6 (Comparison with van den Berg and Lewis). Since $\kappa/2+3/2 = (2q-1)/(q-1) = p+1$, we have $\Gamma(p+1)\,\mathrm{B}(\kappa/2,3/2) = \Gamma(\kappa/2)\sqrt{\pi}/2$, and therefore

$$\Gamma(p+1)\,c_{q,1} = \frac{\Gamma(\kappa/2)\,q^{1-\kappa}\,2^{\kappa-2}\,\zeta(\kappa)}{\sqrt{\pi}\,(q-1)}.$$

With $\alpha = 1/q$ one has $1/(q-1) = \alpha/(1-\alpha)$, $q^{1-\kappa} = \alpha^{\kappa-1}$, $\kappa = (1+\alpha)/(1-\alpha)$ and $\kappa - 2 = (3\alpha-1)/(1-\alpha)$, so the right-hand side equals

$$\frac{1}{(1-\alpha)\sqrt{\pi}}\,2^{(3\alpha-1)/(1-\alpha)}\,\alpha^{(1+\alpha)/(1-\alpha)}\,\Gamma\Big(\frac{1+\alpha}{2-2\alpha}\Big)\,\zeta\Big(\frac{1+\alpha}{1-\alpha}\Big),$$

which is the limit of $\operatorname{Tr} e^{-tH}\,(t/c^2)^{1/(1-\alpha)}$ found in [9, Corollary, (14)] for the single-armed spiral $r = c\varphi^\alpha$, with $c = b$. The script `constants_check.py` confirms the identity numerically.

5 The radial-width bound

In this section $r : [0,\infty) \to [0,\infty)$ is continuous and strictly increasing with $r(0) = 0$ and $r(\theta) \to \infty$, $r(\theta) := 0$ for $\theta < 0$, and $\Gamma = \{\mathrm{pol}(r(\theta),\theta) : \theta \ge 0\}$; the origin lies on Γ. Let r^{-1} be the inverse function of r on $[0,\infty)$.

Lemma 5.1 (Unwrapped polar coordinates)**.** *Let $\Sigma = \{(\rho,\vartheta) :\ \vartheta > 0,\ r(\vartheta-2\pi) < \rho < r(\vartheta)\}$. The map $(\rho,\vartheta) \mapsto \mathrm{pol}(\rho,\vartheta)$ is a diffeomorphism of Σ onto $\mathbb{R}^2 \setminus \Gamma$. For fixed $\rho > 0$ the slice $\{\vartheta : (\rho,\vartheta) \in \Sigma\}$ is the interval $(r^{-1}(\rho), r^{-1}(\rho)+2\pi)$; for fixed $\vartheta > 0$ the slice $\{\rho : (\rho,\vartheta) \in \Sigma\}$ is the interval $(r(\vartheta-2\pi), r(\vartheta))$, of length $\ell(\vartheta)$. The endpoints of both slices correspond to points of Γ.*

Proof. For $\rho > 0$, the condition $r(\vartheta-2\pi) < \rho < r(\vartheta)$ is equivalent to $r^{-1}(\rho) < \vartheta < r^{-1}(\rho) + 2\pi$, by strict monotonicity (for $\vartheta < 2\pi$ the left inequality holds trivially, and so does $\vartheta - 2\pi < r^{-1}(\rho)$). A point $\mathrm{pol}(\rho,\varphi)$, $\rho > 0$, lies on Γ if and only if $\varphi \equiv r^{-1}(\rho) \pmod{2\pi}$; otherwise exactly one representative of φ modulo 2π lies in the open interval $(r^{-1}(\rho), r^{-1}(\rho)+2\pi)$. Hence the map is a bijection onto $\mathbb{R}^2 \setminus \Gamma$, and it is a local diffeomorphism because $\rho > 0$. The endpoint $\vartheta = r^{-1}(\rho)$ of an angular slice gives $\mathrm{pol}(\rho, r^{-1}(\rho)) \in \Gamma$, and $r^{-1}(\rho) + 2\pi$ gives the same point. The endpoint $r(\vartheta)$ of a radial slice gives $\mathrm{pol}(r(\vartheta),\vartheta) \in \Gamma$; the endpoint $r(\vartheta-2\pi)$ gives the point of Γ with parameter $\vartheta - 2\pi$ if $\vartheta \ge 2\pi$, and the origin, which lies on Γ, if $\vartheta < 2\pi$. □

Proof of Theorem 1.7. Let $u \in C_c^\infty(\mathbb{R}^2 \setminus \Gamma)$ and $v(\rho,\vartheta) = u(\mathrm{pol}(\rho,\vartheta))$. By Lemma 5.1, $v \in C_c^\infty(\Sigma)$, and every slice of v has compact support in the open slice interval; in particular v vanishes for small ρ, because the origin lies on Γ. In these coordinates

$$\int |\nabla u|^2\,\mathrm{d}x = \iint_\Sigma \Big(\rho|v_\rho|^2 + \frac{1}{\rho}|v_\vartheta|^2\Big)\,\mathrm{d}\rho\,\mathrm{d}\vartheta.$$

Angular part. For fixed ρ the function $v(\rho,\cdot)$ has compact support in an interval of length 2π, so the one-dimensional Dirichlet inequality gives $\int |v_\vartheta|^2\,\mathrm{d}\vartheta \ge \frac{1}{4}\int |v|^2\,\mathrm{d}\vartheta$.

Radial part. For fixed ϑ put $g(\rho) = \rho^{1/2} v(\rho, \vartheta)$, a smooth function with compact support in the slice $(r(\vartheta - 2\pi), r(\vartheta)) \subset (0, \infty)$. Then $\rho |v_\rho|^2 = |g' - g/(2\rho)|^2$, and integrating $\operatorname{Re}(g'\bar{g})/\rho = (|g|^2)'/(2\rho)$ by parts,

$$\int \rho |v_\rho|^2 \, \mathrm{d}\rho = \int \Big(|g'|^2 - \frac{|g|^2}{4\rho^2} \Big) \, \mathrm{d}\rho \;\geq\; \frac{\pi^2}{\ell(\vartheta)^2} \int |g|^2 \, \mathrm{d}\rho - \int \frac{|v|^2}{4\rho} \, \mathrm{d}\rho,$$

by the one-dimensional Dirichlet inequality on an interval of length $\ell(\vartheta)$.

Adding the two parts and using Fubini's theorem for the integrable function $|v|^2/(4\rho)$,

$$\int |\nabla u|^2 \, \mathrm{d}x \geq \iint_\Sigma \frac{\pi^2}{\ell(\vartheta)^2} |g|^2 \, \mathrm{d}\rho \, \mathrm{d}\vartheta - \iint_\Sigma \frac{|v|^2}{4\rho} + \iint_\Sigma \frac{|v|^2}{4\rho} = \iint_\Sigma \frac{\pi^2}{\ell(\vartheta)^2} |v|^2 \rho \, \mathrm{d}\rho \, \mathrm{d}\vartheta,$$

which is (1.4) for u. For general $u \in H^1_0(\mathbb{R}^2 \setminus \Gamma)$ take $u_k \in C^\infty_c$ with $u_k \to u$ in H^1 and almost everywhere; Fatou's lemma on the right-hand side of (1.4) gives the claim. The operator bound follows because $\pi^2/\ell^2 \geq \pi^2/L^2$. □

Proof of Corollary 1.8. (a) For $r = a\theta$, $\ell(\theta) = a\theta < 2\pi a$ for $\theta < 2\pi$ and $\ell(\theta) = 2\pi a$ for $\theta \geq 2\pi$, so $L = 2\pi a$ and $-\Delta_D \geq (2a)^{-2}$. By [3, Proposition 2.2] the essential spectrum is $[(2a)^{-2}, \infty)$. Hence the spectrum is $[(2a)^{-2}, \infty)$, and there are no isolated eigenvalues of finite multiplicity, because every point of the spectrum is in the essential spectrum. (b) Here $L = \ell_\infty$, and the same argument applies. □

Remark 5.2. For m rotated arms $r(\theta - 2\pi j/m)$ the angular slices have length $2\pi/m$, the angular inequality gains the factor m^2, and the same computation gives $\int |\nabla u|^2 \geq \int (\pi^2/\ell_m(\vartheta)^2 + (m^2 - 1)/(4\rho^2)) |u|^2$ on each component, with $\ell_m(\vartheta) = r(\vartheta) - r(\vartheta - 2\pi/m)$. For the m-armed Archimedean spiral this recovers the absence of spectrum below the threshold proved in [3, Proposition 4.1]. For a Fermat component it gives the lower bound π/b^2 for the ground state, since $\ell_2(\vartheta) = b(\sqrt{\vartheta} - \sqrt{(\vartheta - \pi)_+}) \leq b\sqrt{\pi}$.

6 Comparison with the computed spectrum

This section is numerical and is not used in the proofs. We computed the eigenvalues of a Fermat component with $b = 1$ in the coordinates of Lemma 2.2, reparametrizing $\sigma = F(t)$ by $\eta = \int_0^t \sqrt{1 + 4s^4} \, \mathrm{d}s$, which makes the ratio of the longitudinal kinetic coefficient to the mass density equal to one on the arm $\psi = 0$. Two discretizations were used, both with K sine modes in ψ and Dirichlet conditions at the origin and at a truncation $\eta_{\max}$ beyond all turning points: a conforming Galerkin scheme with continuous piecewise-linear elements in η, whose trial functions lie in H^1_0 of the truncated component, so that in exact arithmetic its eigenvalues would lie above the true ones; and a second-order finite-difference scheme in η. The element integrals were computed by Gauss quadrature and the computations were done in floating point, so the results below are numerical approximations, not certified bounds. Eigenvalues were located by bisection on the inertia of the discrete pencils, computed by block LDL^{T} factorization. The scripts are `fermat_solver.py` and `fermat_numerics.py`.

Table 1 concerns the four eigenvalues displayed in [3, Figure 5]. The two schemes, each extrapolated from the mesh sizes $h = 0.02$ and $h = 0.01$, agree to 10^{-5}. The conforming computations at $h = 0.01$ place at least 7, 15, 28 and 44 eigenvalues below the four values of [3], and the extrapolated eigenvalues number 7, 15, 28 and 44 match these values to within $6 \cdot 10^{-3}$, while the neighbouring computed eigenvalues are about 0.5 away. The computed eigenvalues numbered 28 and 44 thus match the values labelled E_{27} and E_{42} in [3, Figure 5]. With these indices the ratio of the count to the one-mode estimate (1.1) at the four energies is 1.173, 1.198, 1.190 and 1.181.

k	conforming, $h = 0.01$	extrapolated	value in [3, Fig. 5]	$\zeta(3)E^2/64$ at that value
7	19.54611	19.54594	$E_7 = 19.5462$	7.18
15	28.31112	28.31068	$E_{15} = 28.3118$	15.05
28	38.80407	38.80338	$E_{27} = 38.8062$	28.28
44	48.83224	48.83077	$E_{42} = 48.8367$	44.80

Table 1: Eigenvalue number k of a Fermat component with $b = 1$ ($K = 16$ sine modes). The conforming values approximate from above (up to quadrature and rounding); the extrapolated values of the conforming and the finite-difference schemes coincide to the digits shown.

Table 2 lists the counting function. The finite-difference scheme ($h = 0.01$, $K = 20$) and the conforming scheme ($h = 0.01$, $K = 16$), whose count approximates $N(E)$ from below up to quadrature and rounding, give identical counts at all twelve energies. The ratio $N(E)/(E^2/64)$ approaches $\zeta(3) = 1.2020\ldots$, and in this range $|N(E) - \zeta(3)E^2/64| \le 1.5$, far below the size $E \log E$ of the proved remainder.

E	$N(E)$	$\zeta(3)E^2/64$	$E^2/64$	$N(E)/(E^2/64)$	$N(E) - \zeta(3)E^2/64$
10	2	1.88	1.56	1.280	0.12
20	7	7.51	6.25	1.120	−0.51
30	16	16.90	14.06	1.138	−0.90
40	29	30.05	25.00	1.160	−1.05
50	46	46.96	39.06	1.178	−0.96
60	67	67.62	56.25	1.191	−0.62
70	91	92.03	76.56	1.189	−1.03
80	119	120.21	100.00	1.190	−1.21
90	151	152.14	126.56	1.193	−1.14
100	187	187.82	156.25	1.197	−0.82
110	226	227.26	189.06	1.195	−1.26
120	269	270.46	225.00	1.196	−1.46

Table 2: Counting function of a Fermat component with $b = 1$, compared with Corollary 1.5 and with the one-mode estimate (1.1).

For the Archimedean spiral with $a = \frac{1}{2}$, the conforming scheme of `archimedes_check.py`, applied to the region truncated at radius R, gives ground-state values 1.1745, 1.0065 and 1.0010 for $R = 4, 8, 12$, decreasing towards the threshold $(2a)^{-2} = 1$ from above, in accordance with Corollary 1.8.

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References

[1] D. Barseghyan and P. Exner. *Spectral estimates for the Dirichlet Laplacian on spiral-shaped regions*. J. Spectr. Theory **13** (2023), no. 1, 243–261. https://doi.org/10.4171/JST/454. arXiv:2206.14058.

[2] J. Bory-Reyes, D. Barseghyan and B. Schneider. *Lieb–Thirring type estimates for Dirichlet Laplacians on spiral-shaped domains*. Rep. Math. Phys. **95** (2025), no. 2, 241–257. https://doi.org/10.1016/S0034-4877(25)00026-6. arXiv:2406.00580.

[3] P. Exner and M. Tater. *Spectral properties of spiral-shaped quantum waveguides*. J. Phys. A: Math. Theor. **53** (2020), no. 50, 505303. https://doi.org/10.1088/1751-8121/abc5d3. Section, proposition, equation and figure numbers refer to arXiv:2009.02730v1.

[4] M. Reed and B. Simon. *Methods of Modern Mathematical Physics. IV. Analysis of Operators*. Academic Press, New York, 1978.

[5] S. Schubr. *Non-classical Weyl laws for Dirichlet tubes with diverging twisting*, edition 1. Luminary Research Archive, November 2026. https://doi.org/10.5281/zenodo.24406128.

[6] B. Simon. *Nonclassical eigenvalue asymptotics*. J. Funct. Anal. **53** (1983), no. 1, 84–98. https://doi.org/10.1016/0022-1236(83)90047-2.

[7] M. van den Berg. *On the spectrum of the Dirichlet Laplacian for horn-shaped regions in $\mathbb{R}^n$ with infinite volume*. J. Funct. Anal. **58** (1984), no. 2, 150–156. https://doi.org/10.1016/0022-1236(84)90036-3.

[8] M. van den Berg. *Dirichlet–Neumann bracketing for horn-shaped regions*. J. Funct. Anal. **104** (1992), no. 1, 110–120. https://doi.org/10.1016/0022-1236(92)90092-W.

[9] M. van den Berg and J. T. Lewis. *On the heat equation and the spectrum of the Dirichlet Laplacian for spiral regions in $\mathbb{R}^2$*. Preprint DIAS-STP-84-15, Dublin Institute for Advanced Studies, 1984. https://dair.dias.ie/id/eprint/860/.

[10] M. van den Berg and M. Lianantonakis. *Asymptotics for the spectrum of the Dirichlet Laplacian on horn-shaped regions*. Indiana Univ. Math. J. **50** (2001), no. 1, 299–334. https://doi.org/10.1512/iumj.2001.50.1913.