Effective coefficient reserves for homogeneous real-stable polynomials

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Abstract

For homogeneous real-stable polynomials of degree n in n variables, with nonnegative coefficients and value one at the all-ones vector, we give explicit positive rational constants in every fixed degree for a global prescribed-gradient coefficient reserve. On an explicitly specified closed neighborhood of the balanced gradient, the squarefree coefficient deficit dominates both centered-root dispersion and the coefficient ℓ^1 distance from the corresponding power of a linear form. The proof replaces the compactness step in the earlier qualitative theorem by a radial extension to a degree-drop boundary and a positive contraction dichotomy. An explicit positive map then corrects the gradient while preserving the centered factors. The bounds include singular dispersion and stable coefficient boundaries. We also obtain an explicit Frobenius-distance reserve for Hermitian mixed discriminants. The constants are deliberately coarse.

1 The effective prescribed-gradient problem

A nonzero real polynomial is *real stable* if it has no zero when every variable has strictly positive imaginary part. Throughout, p is homogeneous of degree n in n variables, has nonnegative coefficients, is real stable, and satisfies $p(\mathbf{1}) = 1$. Put

$$t = \nabla p(\mathbf{1}), \quad L_t(x) = \frac{t^T x}{n}, \quad \mathcal{A}(p) = [x_1 \cdots x_n]p, \quad F(t) = \frac{n!}{n^n} \prod_i t_i.$$

Euler's identity gives $\sum_i t_i = n$. The coefficient norm is $\|q\|_{\mathrm{coef},1} = \sum_\alpha |[x^\alpha]q|$.

The earlier paper [1] proves, for each fixed n, a global nearbalanced-gradient minimum and a linear coefficient-distance reserve, with qualitative general-degree constants. It answers the prescribed-gradient question of Gurvits–Leake [3, Question 1.8]. The contribution here is an explicit all-degree mechanism for that reserve. The qualitative minimum is inherited from the cited prior work. The general estimates below are deliberately conservative and are not designed to optimize low-degree specializations.

The classical starting point is Gurvits's coefficient–capacity inequality and its degree refinement [2]. Positive directional derivatives give the generalized Friedland–Tverberg coefficient sums; the derivative/product argument is related to the positive-hyperbolic method in [4]. We retain a quadratic Bernstein payment and join it to the strict gap at a pure-vertex degree drop. Convex hyperbolicity cones [5] justify the radial extension even when its matrix has negative entries. A positive variable substitution, rather than a polynomial average, performs the gradient correction.

Existing productization bounds [6] and the recent minimizer results [3] concern closely related capacity questions; they do not by themselves give the explicit prescribed-gradient coefficient-distance estimate established here. In the product-of-row-forms subclass, Lavi [7, Section 2] obtains stronger quantitative balanced

permanent bounds using a column-capacity and entropy argument. That product representation is not an assumption of our theorem.

1.1 Explicit constants and the main theorem

Fix $n \geq 2$. All constants in this display are positive rational numbers:

$$v_n = \frac{n!}{n^n}, \qquad M_n = \frac{n(n-1)}{2}, \qquad h_n = \frac{(n-2)!}{n^{n-2}}, \tag{1}$$

$$\kappa_2 = 1, \qquad \kappa_n = \frac{(n-1)!(n-2)^{n-2}}{(n-1)^{2n-3}} \quad (n \geq 3), \qquad g_n = \kappa_n - v_n, \tag{2}$$

$$\beta_n = \min\left\{ \frac{g_n}{2M_n},\ h_n \left(\frac{g_n}{4n}\right)^{n-2} \right\}, \tag{3}$$

$$K_{n,k} = \binom{n}{k} \frac{2^k n^{2k-2} k^k}{k!}, \qquad W_n = \sum_{k=2}^{n} K_{n,k}, \tag{4}$$

$$\varepsilon_n = \min\left\{ \frac{1}{2}, \frac{\beta_n}{8nW_n} \right\}, \qquad c_n = \frac{\beta_n}{8W_n}. \tag{5}$$

As usual, a zeroth power equals one, including at an endpoint.

Theorem 1.1. *Let p satisfy the standing hypotheses, with $n \geq 2$, and set*

$$B = \frac{n-1}{2n} tt^T - \frac{1}{2}\nabla^2 p(\mathbf{1}), \qquad d = \operatorname{Tr} B.$$

Then $B \geq 0$. If $\|t - \mathbf{1}\|_1 \leq \varepsilon_n$, then

$$\mathcal{A}(p) - F(t) \ \geq\ \frac{\beta_n}{8} d \ \geq\ c_n \left\| p - L_t^n \right\|_{\mathrm{coef},1}. \tag{6}$$

In particular, L_t^n is the unique minimizer of $\mathcal{A}$ on the entire feasible class with that prescribed gradient. The closed endpoint and singular B are included.

The result is global over each feasible nearbalanced-gradient class, not only a coefficient neighborhood of L_t^n. For $n = 1$ the only normalized polynomial is x_1, so the corresponding assertion is trivial. All constants depend on n. For illustration, the present quartic constants are

$$\beta_4 = \frac{169}{13759414272}, \qquad \varepsilon_4 = \frac{169}{324362303925387264}, \qquad c_4 = \frac{169}{81090575981346816}.$$

Their small size is a limitation of this proof, not an estimate of the largest possible neighborhood.

2 Centered roots and an explicit coefficient budget

We recall the intrinsic expansion from [1], including its short proof to fix the normalization.

Lemma 2.1. *There are homogeneous polynomials q_k of degree k, invariant under $x \mapsto x + a\mathbf{1}$, such that*

$$p(x) = L_t(x)^n + \sum_{k=2}^{n} L_t(x)^{n-k} q_k(x), \qquad q_2(x) = -x^T B x. \tag{7}$$

The matrix B in Theorem 1.1 is PSD, $B\mathbf{1} = 0$, and

$$|q_k(x)| \le \binom{n}{k}(2x^T Bx)^{k/2} \le \binom{n}{k}(2d\|x\|_2^2)^{k/2}. \tag{8}$$

Moreover, $d = 0$ exactly when $p = L_t^n$, and $0 \le d \le M_n$.

Proof. Put $y = x - L_t(x)\mathbf{1}$. The polynomial $p(z\mathbf{1} + y)$ is monic of degree n, with real roots and zero z^{n-1} coefficient. Reality of the roots follows from stability and complex conjugation. Denote its roots by λ_j. Their sum is zero, and its quadratic coefficient is $-\frac{1}{2}\sum_j \lambda_j^2 \le 0$. Expanding in z and substituting $z = L_t(x)$ gives (7), translation invariance, and a PSD quadratic form $q_2 = -x^T Bx$. Differentiating twice at $\mathbf{1}$ gives the stated formula for B. Translation invariance gives $B\mathbf{1} = 0$. Each root has absolute value at most $(2x^T Bx)^{1/2}$; bounding every elementary symmetric term gives (8). If $d = 0$, PSD forces $B = 0$ and all higher q_k vanish. The converse is immediate. Finally, every diagonal Hessian entry is nonnegative and $\sum_i t_i = n$, so $d \le (n-1)\sum_i t_i^2/(2n) \le M_n$. □

Lemma 2.2. *For every polynomial in the standing class,*

$$\|q_k\|_{\mathrm{coef},1} \le K_{n,k} d, \qquad \left\|p - L_t^n\right\|_{\mathrm{coef},1} \le W_n d. \tag{9}$$

Proof. Let T_k be the symmetric k-linear polar form of q_k. The real polarization identity expresses $T_k(e_{i_1}, \dots, e_{i_k})$ as a sum of 2^k evaluations at signed sums of coordinate vectors, divided by $2^k k!$. Their Euclidean norms are at most k, so

$$|T_k(e_{i_1}, \dots, e_{i_k})| \le \binom{n}{k}(2d)^{k/2}\frac{k^k}{k!}.$$

Summing the n^k ordered tensor coefficients bounds the polynomial coefficient norm. Since $d \le n^2$ and $k \ge 2$, $d^{k/2} \le dn^{k-2}$; also $2^{k/2} \le 2^k$. These give exactly the first estimate with (4). The coefficients of L_t are nonnegative and sum to one, so $\|L_t\|_{\mathrm{coef},1} = 1$. Summing the intrinsic expansion proves the second estimate. At $d = 0$ use Lemma 2.1 directly. □

3 Balanced derivatives and Bernstein payment

Call a normalized polynomial *balanced* when its gradient at $\mathbf{1}$ equals $\mathbf{1}$. We use Gurvits's coefficient–capacity theorem and its variable-degree refinement [2, Theorem 2.4 and Corollary 2.5]. If p is balanced, coefficient-weighted AM–GM gives $p(x) \ge \prod_i x_i$ for $x_i > 0$. Thus $\mathrm{Cap}(p) = 1$ and $\mathcal{A}(p) \ge v_n$.

Lemma 3.1. *If q is balanced and satisfies $q(e_i) = 0$ for some coordinate vertex, then $\mathcal{A}(q) \ge \kappa_n = v_n + g_n$, with $g_n > 0$.*

Proof. Nonnegative coefficients and $q(e_i) = 0$ imply degree at most $n-1$ in that variable. Eliminate it first in the degree-refined capacity induction. Its first factor is $((n-2)/(n-1))^{n-2}$; the remaining factors multiply to $(n-1)!/(n-1)^{n-1}$. Their product is (2). For $n = 2$ the first factor is one, giving $\kappa_2 = 1$. The function $G(m) = ((m-1)/m)^{m-1}$ strictly decreases for integers $m \ge 2$, with $G(1) = 1$. Replacing the degree-n first factor by $G(n-1) > G(n)$ therefore gives $g_n > 0$. The capacity theorem includes zero coefficients and degree drops. □

For a balanced q, define

$$E_k(q) = \sum_{|S|=k} \partial_S q(\mathbf{1}), \qquad E_0(q) = 1, \qquad T_s x = (1-s)L_{\mathbf{1}}(x)\mathbf{1} + sx.$$

Lemma 3.2. *For* $0 \le k \le n$,

$$E_k(q) \ge \binom{n}{k} \frac{n!}{(n-k)!n^k}. \tag{10}$$

For $0 \le s \le 1$ *one has the exact identity*

$$\mathcal{A}(q \circ T_s) = \sum_{k=0}^{n} s^k (1-s)^{n-k} \frac{(n-k)!}{n^{n-k}} E_k(q), \tag{11}$$

and consequently

$$\mathcal{A}(q \circ T_s) - v_n \ge h_n s^2 (1-s)^{n-2} d(q). \tag{12}$$

Proof. Write $D_{\mathbf{1}} = \sum_i \partial_i$ and set $r_k = (k!/n!) D_{\mathbf{1}}^{n-k} q$. Directional differentiation preserves stability; Euler's identities give $r_k(\mathbf{1}) = 1$ and $\nabla r_k(\mathbf{1}) = (k/n)\mathbf{1}$. Therefore $r_k L_{\mathbf{1}}^{n-k}$ is a balanced stable polynomial of degree n. For every squarefree S of size k, homogeneity gives

$$\partial_S r_k(0) = \frac{k!(n-k)!}{n!} \partial_S q(\mathbf{1}).$$

Its product with $L_{\mathbf{1}}^{n-k}$ has squarefree coefficient $\frac{(n-k)!}{n^{n-k}} E_k(q) / \binom{n}{k}$. Applying the balanced coefficient bound proves (10), including $k = 0, 1$. This is a general stable-polynomial derivative consequence, with no factorization into linear forms assumed.

Taylor expansion at $(1-s)L_{\mathbf{1}}(x)\mathbf{1}$ and squarefree extraction give (11). By (10) each displayed coefficient is at least $\binom{n}{k} v_n$. Subtract the binomial expansion of v_n; all resulting Bernstein deficit coefficients are nonnegative. The total Hessian sum is $n(n-1)$, whereas its trace is $(n-1) - 2d(q)$. Hence

$$E_2(q) = \frac{(n-1)^2}{2} + d(q).$$

The baseline $k = 2$ value is $(n-1)^2/2$. Retaining this single deficit proves (12). For $n = 2$ its zeroth power is one; all endpoints are included by direct evaluation or continuity. □

4 A radial degree-drop boundary and global balanced reserve

We use the following classical hyperbolicity-cone facts [5, 2]. A homogeneous real-stable polynomial with nonnegative coefficients is hyperbolic in $\mathbf{1}$. Its cone C containing $\mathbf{1}$ is open and convex, it is hyperbolic in every direction in C, and its closed cone consists of the vectors with nonnegative hyperbolic eigenvalues. Positive combinations of directions in the closed cone, with an interior summand, are interior. Pullback along closed-cone directions whose sum is interior is hyperbolic in every strictly positive direction, hence is real stable and has nonnegative coefficients. Equivalently, the coefficient signs follow from the standard mixed-directional-derivative positivity on that cone. These facts do not require strictly hyperbolic roots or a pointed cone.

Lemma 4.1. *If* p *is balanced and* $d(p) > 0$*, there is a finite* $D \ge 1$ *such that* $q = p \circ T_D$ *is balanced, stable, normalized, has nonnegative coefficients, and satisfies* $q(e_i) = 0$ *for some* i*. Moreover,*

$$p = q \circ T_{1/D}, \qquad d(q) = D^2 d(p). \tag{13}$$

Proof. Let a_{ij} be the roots in z of $p(z\mathbf{1} - e_i)$. They are nonnegative because $e_i \in \overline{C}$, and balance gives $\sum_j a_{ij} = 1$. Let $a = \min_{i,j} a_{ij}$. If $a = 1/n$, every coordinate's eigenvalues equal $1/n$. Its centered quadratic

variance is zero, so every $B_{ii} = 0$ and PSD forces $d = 0$, a contradiction. Thus $0 \le a < 1/n$. Set $D = (1 - na)^{-1}$.

The eigenvalues of $T_D e_i$ are $(1-D)/n + Da_{ij} \ge 0$, with at least one zero. Thus every $T_D e_i$ lies in $\overline{C}$ and their sum is $\mathbf{1} \in C$. For $x > 0$,

$$T_D x = (\min_i x_i)\mathbf{1} + \sum_i (x_i - \min_j x_j) T_D e_i \in C.$$

The cone facts just stated prove the stability and coefficient signs of q. Also $q(\mathbf{1}) = 1$ and $T_D^T \mathbf{1} = \mathbf{1}$, so q is balanced. The zero eigenvalue gives $q(e_i) = 0$ at some vertex. The identities $T_D T_s = T_{Ds}$ and the scaling by D on centered directions give (13). If $a = 0$, then $D = 1$ and p already has a degree-drop vertex. Repeated roots cause no difficulty. The case $D > 1$ is justified by the cone argument, not by entrywise positivity: T_D has negative off-diagonal entries. □

Proposition 4.2. *Every balanced polynomial in the standing class satisfies*

$$\mathcal{A}(p) - v_n \ge \beta_n d(p). \tag{14}$$

Proof. The case $d = 0$ follows from Lemma 2.1. Otherwise take q, D from Lemma 4.1, put $s = 1/D$ and $\theta = 1 - s$. The positive contraction T_s has row forms of coefficient norm one, whose distances from the corresponding coordinate forms are at most 2θ. Since the nonnegative coefficients of q sum to one, telescoping each degree-n monomial product gives

$$\|q - q \circ T_s\|_{\mathrm{coef},1} \le 2n\theta. \tag{15}$$

If $\theta \le g_n/(4n)$, the boundary gap gives

$$\mathcal{A}(p) - v_n \ge g_n - 2n\theta \ge g_n/2 \ge \frac{g_n}{2M_n} d(p).$$

If $\theta > g_n/(4n)$, apply (12) to q and use $s^2 d(q) = d(p)$:

$$\mathcal{A}(p) - v_n \ge h_n \theta^{n-2} d(p) \ge h_n \left(\frac{g_n}{4n}\right)^{n-2} d(p).$$

The minimum of the two constants is β_n. This includes $n = 2$ and equality at the dividing threshold. No compactness minimum or norm estimate for the negative-entry extension T_D is used. □

5 An explicit positive gradient correction

Proof of Theorem 1.1. PSD of B is Lemma 2.1. Put $e = \|t - \mathbf{1}\|_1$. When $e = 0$, Proposition 4.2 is stronger than the required estimate. Suppose $0 < e \le \varepsilon_n \le 1/2$, and define

$$z_i = \frac{1 - (1-e)t_i}{ne}, \qquad Sx = (1-e)x + e(z^T x)\mathbf{1}. \tag{16}$$

The vector z is a probability vector. Indeed its entries sum to one. For $t_i \le 1$ nonnegativity is immediate; for $t_i > 1$, $t_i - 1 \le e$ and $et_i - (t_i - 1) \ge e - (t_i - 1) \ge 0$. The matrix S is nonnegative, has positive diagonal, and satisfies $S\mathbf{1} = \mathbf{1}$ and $S^T t = \mathbf{1}$. Thus $r = p \circ S$ is normalized, balanced, stable and has nonnegative coefficients. Stability follows directly because S maps strictly positive imaginary parts to strictly positive imaginary parts.

The correction preserves the intrinsic factors exactly:

$$L_t(Sx) = L_{\mathbf{1}}(x), \quad q_k(Sx) = (1-e)^k q_k(x), \quad d(r) = (1-e)^2 d(p). \tag{17}$$

The middle identity uses translation invariance and homogeneity, not a loss estimate or inverse-dispersion bound. The balanced theorem therefore gives

$$\mathcal{A}(r) - v_n \geq \beta_n (1-e)^2 d \geq \beta_n d/4.$$

Now $\|L_t\|_{\mathrm{coef},1} = \|L_{\mathbf{1}}\|_{\mathrm{coef},1} = 1$ and $\|L_t - L_{\mathbf{1}}\|_{\mathrm{coef},1} = e/n$. Telescoping powers and $1-(1-e)^k \leq ke$ yield

$$\left\| L_t^{n-k} - (1-e)^k L_{\mathbf{1}}^{n-k} \right\|_{\mathrm{coef},1} \leq \frac{n-k}{n} e + ke \leq ne.$$

Subtract the two intrinsic expansions after removing their respective linear-form powers. Lemma 2.2 bounds the resulting coefficient norm by $neW_n d$. Since the squarefree coefficient is bounded in absolute value by that norm,

$$\mathcal{A}(p) - F(t) \geq \beta_n d/4 - neW_n d \geq \beta_n d/8.$$

The final inequality uses (5). Applying $\left\| p - L_t^n \right\|_{\mathrm{coef},1} \leq W_n d$ proves (6). If the deficit is zero, $d = 0$ and $p = L_t^n$, proving uniqueness. □

6 Matrix and probability recipients

Corollary 6.1. *Let $X_1, \dots, X_n$ be complex Hermitian PSD matrices of order n, with $\sum_i X_i = I$. Put $t_i = \operatorname{Tr} X_i$ and use the normalized mixed discriminant*

$$\mathrm{MD}(X_1, \dots, X_n) = \frac{1}{n!} [x_1 \cdots x_n] \det\left(\sum_i x_i X_i \right).$$

If $\|t - \mathbf{1}\|_1 \leq \varepsilon_n$, then

$$\mathrm{MD}(X_1, \dots, X_n) - \frac{\prod_i t_i}{n^n} \geq \frac{\beta_n}{16 n!} \sum_i \left\| X_i - \frac{t_i}{n} I \right\|_F^2. \tag{18}$$

Individual X_i may be singular and have complex entries.

Proof. The polynomial $p(x) = \det(\sum_i x_i X_i)$ has nonnegative coefficients and $p(\mathbf{1}) = 1$. If all imaginary parts of x_i are positive, the pencil has positive-definite imaginary part, so is invertible; hence p is real stable. Differentiation at I gives $\partial_i p(\mathbf{1}) = t_i$ and $\partial_{ii} p(\mathbf{1}) = t_i^2 - \operatorname{Tr}(X_i^2)$. Therefore

$$d = \frac{1}{2} \sum_i \left(\operatorname{Tr}(X_i^2) - \frac{t_i^2}{n} \right) = \frac{1}{2} \sum_i \left\| X_i - \frac{t_i}{n} I \right\|_F^2.$$

Divide the dispersion reserve of Theorem 1.1 by $n!$. □

The coefficient probability distribution of p is a count law on $\{\alpha \in \mathbb{N}^n : |\alpha| = n\}$ with mean t. The coefficients of L_t^n give the multinomial law with probabilities t_i/n. Their coefficient distance is twice total variation. Consequently (6) also reads

$$\Pr_p\{\alpha = \mathbf{1}\} - F(t) \geq 2 c_n d_{\mathrm{TV}}(p, \mathrm{Mult}(n, t/n)).$$

This gives a direct probabilistic interpretation of the coefficient bound.

7 A failed shortcut, checks and limitations

Nonnegative Bernstein *deficits* in Lemma 3.2 do not mean every intrinsic q_k has nonnegative squarefree coefficient. For an exact quartic example, let J_4 have entries $1/4$ and put

$$E = \begin{pmatrix} -2 & -1 & 1 & 2 \\ 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -2 & 1 \end{pmatrix}, \qquad A = J_4 + E/20, \qquad p_A(x) = \prod_{i=1}^{4} \sum_j A_{ij} x_j .$$

The matrix A is positive and doubly stochastic, so p_A satisfies the balanced hypotheses. Its intrinsic degree-four term has squarefree coefficient $\mathrm{per}(E)/20^4 = -1/8000$, since $\mathrm{per}(E) = -20$. Thus intrinsic quartic positivity cannot replace the derivative/boundary argument.

The accompanying standard-library Python checker uses exact fractions to test the displayed constants, derivative sums, Bernstein identities, radial dispersion and positive correction identities on balanced product fixtures of degrees two through six. It also tests a positive balancing of the stable specialized Vámos quartic from [8], and the negative intrinsic example above. These are reproducibility controls, not a finite proof of the universal stable class or a stability-testing algorithm. The analytic proof does not depend on their execution.

Real-rooted line restrictions are essential; the theorem applies to the stated real-stable class. The constants are sufficient and coarse. Replacing the coefficient budget W_n or the boundary-gap threshold by a sharper dispersion-sensitive payment is a natural remaining quantitative problem.

Assistance disclosure

AI tools assisted with exploration, drafting, literature discovery, exact calculations and internal checking.

References

[1] S. Schubr, *Prescribed-gradient minimizers and coefficient stability for homogeneous real-stable polynomials*, Luminary Research Archive, edition 1 (2026). Public predecessor.

[2] L. Gurvits, Van der Waerden/Schrijver–Valiant like conjectures and stable (aka hyperbolic) homogeneous polynomials: one theorem for all, *Electronic Journal of Combinatorics* **15** (2008), R66. arXiv:0711.3496v2.

[3] L. Gurvits and J. Leake, Unique minimizers for permanents, mixed discriminants, and log-concave polynomials (2026). arXiv:2609.21083v2.

[4] S. Friedland and L. Gurvits, Lower bounds for partial matchings in regular bipartite graphs and applications to monomer-dimer entropy (2007), author manuscript. Author PDF.

[5] H. H. Bauschke, O. Güler, A. S. Lewis and H. S. Sendov, Hyperbolic polynomials and convex analysis, *Canadian Journal of Mathematics* **53** (2001), 470–488. doi:10.4153/CJM-2001-020-6.

[6] L. Gurvits and J. Leake, Capacity lower bounds via productization, *STOC 2021*, 847–858. arXiv:2007.08390.

[7] Y. Lavi, The Marcus–Minc transform inequality (2026). arXiv:2609.29262v1.

[8] M. Kummer, A note on the hyperbolicity cone of the specialized Vámos polynomial (2015). arXiv:1306.4483v2.