# A discrete Jensen inequality for alcoved lattice counts and planar integer flows

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**Abstract**

Leake and Mohammadi Yekta asked whether lattice-point counts of totally unimodular fibers satisfy a discrete Jensen inequality, $\log N(\mathbb{E}a) \geq \mathbb{E}\log N(a)$, for every distribution of right-hand sides with integral mean. Two-point midpoint inequalities give only log-concavity along lattice lines, and a general distribution need not contain any pair with an integral midpoint. We prove the full Jensen inequality for type-A alcoved polytopes, $x_u - x_v \leq h_{uv}$, in every dimension, with every directed difference bound allowed to vary simultaneously and for every integrable distribution of bounds. This settles the type-$A_4$ branch of the dimension-four reduction of the question, and covers all type-A systems in any dimension. The proof balances finitely many quantiles with the entropy midpoint inequality of Gozlan, Roberto, Samson and Tetali, lifts the balance to several coordinates by a conditional quantile coupling, and rounds all coordinates with one common offset, which preserves every integral directed difference. The same argument proves Jensen inequalities for partition functions of integer height functions and of planar integer flows with convex edge energies, with conservation data and edge capacities varying together. For the remaining branch, $3 \times 3$ transportation tables, we give an explicit pair of inputs showing that no rounding scheme that is causal in a fixed coordinate order can succeed.

**Keywords.** Alcoved polytopes, unimodular fibers, discrete Jensen inequality, entropy, integer flows.
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## 1 Introduction

Let $M$ be a unimodular integer matrix. Leake and Mohammadi Yekta [5, Question 1.7] ask whether the number of nonnegative integral solutions to $My = a$ satisfies a discrete Jensen inequality for every distribution of $a$ with integral mean. This question extends an exact log-concavity question of Barvinok for contingency tables [1, (2.3)]. Leake and Mohammadi Yekta prove the integral two-point midpoint inequality for unimodular systems by comparing centered fibers of boxes [5, Theorems 1.4 and 3.1]. That proof gives log-concavity along lattice lines. An arbitrary distribution with integral mean may involve no pair of distinct support points with integral midpoint.

The earlier paper [7] proves a stronger power inequality when the central fiber has dimension at most three. It also reduces the four-dimensional case to two normal systems. The first is type $A_4$, with arbitrary directed differences among five vertex coordinates. The second comes from cell-bounded $3 \times 3$ transportation tables. We answer the logarithmic Jensen question for the first system. The same proof works in every dimension and for all the bond bounds at once. A second consequence is a partition-function inequality for planar integer flows, where the edge capacities and divergences may vary together.

Our transport proof starts with the entropy midpoint theorem of Gozlan, Roberto, Samson and Tetali [2, Theorem 8]. Slomka [8, Theorem 1.4 and §3.1] proved the two-law lattice entropy inequality for more general complementing operations using conditional monotone coupling. Halikias, Klartag and Slomka [3, Theorems 1.2 and 1.3] likewise give two-input functional inequalities for complementary operations, including weighted midpoint operations. We use a repeated scalar balance to treat an arbitrary finite list of input laws and then use the same conditional coupling strategy. This gives an entropy inequality averaged over one common rounding offset for arbitrary real weights.

There is also a distinct marginal approach. Speyer [9, Theorem 5.1] proves that summing over coordinates preserves $L$-log-concavity for $L$-convex sets. This applies to alcoved counts with moving bounds adjacent to the anchor and fixed internal bounds [7, Theorem 1.8]. In the problem here an internal bound $h_{uv}$ is itself a parameter in $x_u - x_v \le h_{uv}$. The corresponding joint system in the parameters and the lattice point need not be $L$-convex. Our proof rounds the lattice points after taking their weighted average at the integral mean of the parameters.

## 2 Alcoved lattice counts

Fix vertices $0, \dots, d$ and a finite labeled collection $\mathcal{A}$ of ordered pairs $e = (u, v)$. Labels allow multiple constraints on the same directed difference. For $h \in \mathbb{Z}^{\mathcal{A}}$ write

$$P(h) = \{x \in \mathbb{R}^{d+1} : x_0 = 0,\ x_u - x_v \le h_e \text{ for } e = (u, v) \in \mathcal{A}\}, \qquad N(h) = |P(h) \cap \mathbb{Z}^{d+1}|.$$

With all directed differences present this is the anchored type-$A_d$ description of an alcoved polytope [4].

**Theorem 1** (Discrete Jensen for directed difference bounds)**.** *Let $A$ be an integrable $\mathbb{Z}^{\mathcal{A}}$-valued random vector and let $b = \mathbb{E}A \in \mathbb{Z}^{\mathcal{A}}$. If $N(A) > 0$ almost surely and $N(b)$ is finite, then the central count is positive, all counts at support atoms are finite, and*

$$\log N(b) \ \ge\ \mathbb{E} \log N(A).$$

The proof first develops the entropy inequality for a finite family. We then use it on uniform distributions over the endpoint lattice points. An exact-mean approximation gives the countable-support case.

## 3 Entropy under simultaneous rounding

For a finitely supported random variable $X$, write $\mathsf{H}(X) = -\sum_x \Pr(X = x) \log \Pr(X = x)$ for its Shannon entropy. For a probability law $\mu$ on $\mathbb{Z}$, its quantile at $u \in (0, 1)$ is the least integer $t$ with $\mu((-\infty, t]) \ge u$.

**Lemma 2** (Balancing scalar quantiles)**.** *Let $U$ be uniform on $(0, 1)$, let $f_1, \dots, f_m$ be finite-valued nondecreasing functions from $(0, 1)$ to $\mathbb{Z}$, and put $S = \sum_i f_i(U)$. Then*

$$\sum_{r=0}^{m-1} \mathsf{H}\left( \left\lfloor \frac{S + r}{m} \right\rfloor \right) \ \ge\ \sum_{i=1}^{m} \mathsf{H}(f_i(U)).$$

*Proof.* For two such functions, [2, Theorem 8, equation (9)] says that replacing their values by the floor and ceiling of their mean can only increase the sum of their Shannon entropies. Indeed, the

common uniform variable gives the monotone coupling in that theorem, and its relative entropy with respect to counting measure has the opposite sign.

Ignore the endpoints of the finite common interval partition on which all $f_i$ are constant. Whenever a pair differs by at least two on an interval, replace the pair everywhere by the floor and ceiling of its mean. The functions remain nondecreasing and constant on the same partition. Each replacement preserves the pointwise sum and does not decrease total entropy. The sum of the squared values over the functions and the positive-length intervals is a nonnegative integer. Balancing a pair never increases that sum and strictly decreases it on every interval where the pair differs by at least two. Thus the procedure ends with all values at each point differing by at most one.

Sort the resulting functions pointwise using a fixed finite sequence of pair comparisons, such as a sorting network of adjacent comparisons. Each comparison is another permitted entropy operation: for two integers at distance at most one the floor and ceiling of the mean are their minimum and maximum. The ordering at every point is therefore attained without decreasing total entropy. If $S = mq + k$ with $0 \le k < m$, the sorted list has $m - k$ entries $q$ and $k$ entries $q + 1$. Its $r$-th entry, indexed from zero, is $\lfloor (S + r)/m \rfloor$. □

For finitely supported laws $\mu_1, \dots, \mu_m$ on $\mathbb{Z}^d$, construct a joint law of $X_1, \dots, X_m$ by using a common independent uniform variable $U_j$ for each coordinate $j$. At coordinate $j$, take $X_{i,j}$ to be the $\mu_i$-conditional quantile at $U_j$ given its own preceding coordinates. This is the coordinatewise Knothe construction used for two laws in [8, §3.1]. Each $X_i$ has law $\mu_i$.

**Theorem 3** (Entropy of rounded weighted averages)**.** *Use this coupling for finitely supported laws $\mu_1, \dots, \mu_m$ on $\mathbb{Z}^d$. Suppose $\lambda_i \ge 0$ and $\sum_i \lambda_i = 1$. Write $W = \sum_i \lambda_i X_i$, and for $0 \le s < 1$ set $Y(s) = \lfloor W + s\mathbf{1} \rfloor$, using the same offset in each coordinate. Then*

$$\int_0^1 \mathsf{H}(Y(s))\, ds \;\ge\; \sum_i \lambda_i \mathsf{H}(X_i).$$

*Here the entropy in the integrand is for a fixed offset $s$.*

*Proof.* For equal weights set $Y_r = \lfloor W + (r/m)\mathbf{1} \rfloor$ for $0 \le r < m$. At coordinate $j$, condition on the full input prefix $B_j = (X_{i,\ell})_{i,\, \ell < j}$. Since this prefix depends only on the earlier uniforms, $U_j$ remains independent of $B_j$. The conditional coordinates $X_{i,j}$ are nondecreasing functions of $U_j$, so Lemma 2 gives, after averaging over the prefix,

$$\sum_r \mathsf{H}(Y_{r,j} \mid B_j) \;\ge\; \sum_i \mathsf{H}(X_{i,j} \mid B_j).$$

The output prefix $Y_r^{<j}$ is determined by $B_j$. Conditioning on $Y_r^{<j}$ therefore yields entropy at least $\mathsf{H}(Y_{r,j} \mid B_j)$. Each input coordinate uses only its own prefix and $U_j$, so $\mathsf{H}(X_{i,j} \mid B_j) = \mathsf{H}(X_{i,j} \mid X_i^{<j})$. The entropy chain rule gives $\sum_r \mathsf{H}(Y_r) \ge \sum_i \mathsf{H}(X_i)$. Since $mW$ is integral, $Y(s) = Y_r$ for $r/m \le s < (r+1)/m$. Dividing the last inequality by $m$ proves the equal-weight case.

For rational weights, repeat each law according to its numerator over a common denominator. Repeated copies of any one law use the same uniforms and the same conditional quantiles, so they agree almost surely in every coordinate. Thus the equal-weight result applies to the weighted average in the stated coupling. For real weights, approximate them by rational weights in the simplex and retain the same input random variables. Their joint array has finite support. For almost every fixed offset, none of its finitely many limiting weighted coordinates lands at a rounding boundary. At these offsets the rounded laws eventually agree with the limiting rounded law. The rounded variables lie in one fixed finite box, so dominated convergence proves the stated entropy inequality. □

*Proof of Theorem 1.* When $P(h)$ is feasible, the directed graph of its constraints, with the arc $v \to u$ of length $h_e$, has no negative directed cycle. Each simple cycle length at $b$ is the mean of nonnegative simple cycle lengths at the support atoms. A new source with zero-length arcs to all the vertices then gives integral shortest-path potentials at $b$. Subtracting the potential of vertex 0 gives a central lattice point. If any nonempty endpoint were unbounded, the common rational recession cone would contain a nonzero integral ray. Adding its nonnegative integer multiples to the central lattice point would contradict the finiteness of $N(b)$.

First let the bound distribution be finite, with positive weights $\lambda_i$ at $h_i$. Use uniform laws on the endpoint lattice fibers in Theorem 3. For each realization, $W = \sum_i \lambda_i X_i$ belongs to $P(b)$ by linearity, with its anchor coordinate zero. For $e = (u, v)$, the integral upper bound gives

$$W_u \le W_v + b_e \quad \Longrightarrow \quad \lfloor W_u + s \rfloor \le \lfloor W_v + s \rfloor + b_e.$$

Thus every $Y(s)$ belongs to the central fiber. Since a law supported on at most $N(b)$ points has entropy at most $\log N(b)$, Theorem 3 gives the result.

For countable support, write the distinct atoms as $h_i$ with positive masses $p_i$. Choose an affine basis $c_0, \dots, c_q$ of their affine span from among the atoms. Truncate the law after $n$ atoms for $n$ large enough to retain the basis. The omitted joint mass and moment $\sum_{i>n} p_i(1, h_i)$ tend to zero by integrability. Their unique coordinates $\delta_{n,k}$ in the basis $(1, c_k)$ also tend to zero. Add $\delta_{n,k}$ to the retained mass of $c_k$. These masses stay positive for large $n$ and the finite corrected law has mass one and mean $b$. Finite Jensen applies. The correction to the expectation of $\log N$ tends to zero, while the uncorrected partial sums increase because every positive lattice count is at least one. This establishes the inequality and the finiteness of its right side. □

**Corollary 4** (Graphic kernel normals)**.** *Suppose a family of unimodular fibers has a lattice parameterization in which each fiber is given by fixed signed type-A roots with integer-affine right-hand sides in the demand. Its counts satisfy the discrete Jensen inequality for every integrable demand law with an integral mean and finite central count, provided the endpoint counts are positive almost surely. This includes the type-$A_4$ branch of [7, Proposition 9.1].*

*Proof.* After adjoining the anchored vertex, every signed root is a directed difference. Keep the raw inequalities as labeled constraints, including parallel ones that share a root. Their right-hand sides commute with averaging. Any zero-normal constraints also hold at the integral mean by affinity. Apply Theorem 1 in the lattice coordinates. The fixed kernel parameterization in the active face used in [7, §5 and Proposition 9.1] has these properties in the type-$A_4$ branch. □

## 4 Partition functions of heights and planar flows

Let $E_{\mathrm{en}}$ be a fixed finite list of directed bonds on the vertex set. For each bond $e = (u, v)$ let $\phi_e : \mathbb{Z} \to \mathbb{R}$ have nonnegative second differences, so its continuous piecewise-linear interpolation $\widetilde{\phi}_e$ is convex. Define

$$\mathcal{E}(x) = \sum_{e=(u,v)\in E_{\mathrm{en}}} \phi_e(x_u - x_v), \qquad Z(h) = \sum_{x \in P(h)\cap \mathbb{Z}^{d+1}} \exp[-\mathcal{E}(x)].$$

**Theorem 5** (Partition functions for integer heights)**.** *Let $h_i$ be a finite list of bound vectors with positive finite counts. Suppose their weighted mean $b = \sum_i \lambda_i h_i$ is integral, with positive weights summing to one. Then $Z(b) \ge \prod_i Z(h_i)^{\lambda_i}$.*

*Proof.* The recession cone is fixed across fibers and the endpoint fibers are nonempty and bounded, so the central fiber is finite. Use the endpoint Gibbs laws $e^{-\mathcal{E}}/Z(h_i)$ in Theorem 3. The proof of Theorem 1 shows that every $Y(s)$ is central. For a fixed input realization, the rounded bond $Y_u(s)-Y_v(s)$ is the floor or ceiling of $W_u-W_v$. For $t = W_u-W_v$, subtract $\lfloor W_v+s\rfloor$ inside $\lfloor W_u+s\rfloor$. The difference is then $\lfloor t+\{W_v+s\}\rfloor$, where $\{\,\cdot\,\}$ denotes the fractional part. Also $\int_0^1 \lfloor t+s\rfloor\, ds = t$ for each real $t$. It follows that the rounded bond has mean $W_u - W_v$ over offsets. The interpolation $\widetilde{\phi}_e$ is affine between those two neighboring integers, so

$$\int_0^1 \phi_e(Y_u(s) - Y_v(s))\, ds = \widetilde{\phi}_e(W_u - W_v) \le \sum_i \lambda_i \phi_e(X_{i,u} - X_{i,v}).$$

Summing over the bonds and taking expectations gives the energy comparison. On any finite fiber the relative entropy from its Gibbs law gives $\mathsf{H}(X) - \mathbb{E}\mathcal{E}(X) \le \log Z$, with equality for that Gibbs law. Apply this inequality to each $Y(s)$ at the central fiber and integrate. Theorem 3 and the energy comparison give the conclusion. □

For example, impose $|x_u - x_v| \le 1$ along every edge of a finite connected graph and take $\phi_e(k) = -|k| \log t$ with $0 < t \le 1$. This is the integer Lipschitz height weight $t^{\#\{e:x_u \ne x_v\}}$. Prescribed boundary values are imposed by two opposite constraints to the anchor. Theorem 5 applies to any finite mixture of boundary values and bond limits whose averages are integral. OpenAI [6, Theorem 3.3 and Lemma 3.4] use the two-input midpoint energy comparison for this height model.

**Corollary 6** (Planar integer flows)**.** *Let a fixed connected plane graph have oriented edges. For an integer divergence vector $a$ of total sum zero and integer edge bounds $\ell \le u$, define*

$$Z(a, \ell, u) = \sum_{\substack{x \in \mathbb{Z}^E \\ \operatorname{div} x = a,\ \ell \le x \le u}} \exp\left[-\sum_{e \in E} \phi_e(x_e)\right],$$

*with fixed discrete convex edge energies $\phi_e$. For every finite mixture of feasible data $(a_i, \ell_i, u_i)$ with an integral mean $(\bar{a}, \bar{\ell}, \bar{u})$,*

$$Z(\bar{a}, \bar{\ell}, \bar{u}) \;\ge\; \prod_i Z(a_i, \ell_i, u_i)^{\lambda_i}.$$

*In particular, the lattice counts satisfy this inequality by taking zero energy. The count statement holds also for countable integrable laws on bounded data fibers. For nonnegative flows without upper capacities, it holds for finite mixtures of finite fibers.*

*Proof.* Choose a spanning tree. There is an integer-linear particular flow $p(a)$ supported on the tree, since each tree-edge flow is the signed sum of the divergences on one side of its cut. Every circulation has the form

$$x_e - p_e(a) = z_{f_+(e)} - z_{f_-(e)},$$

where $f_+(e)$ and $f_-(e)$ are the faces on the two sides of the oriented edge, in the order determined by the signed face boundaries. Anchor the outer face at zero. The face-boundary map takes values in the circulation space and is injective after anchoring because the dual graph is connected. Euler's formula gives equal dimensions for the two real spaces. Integral circulations correspond to integral potentials by propagating along a dual spanning tree. A bridge has the same face on both sides and contributes a zero normal.

In these coordinates the lower and upper edge bounds give directed differences with integer-affine right-hand sides. This gives the count results by Theorem 1. For the weighted result,

use the conditional quantile coupling of endpoint Gibbs laws in the face coordinates. Because $p(\bar{a}) = \sum_i \lambda_i p(a_i)$ is integral, common-offset rounding of the face potentials makes each central edge flow a floor or ceiling of the mean input edge flow and preserves that mean over offsets. Therefore the energy and entropy argument of Theorem 5 applies with $\phi_e(x_e)$ on each full edge flow. If upper capacities are absent and the finitely many nonnegative endpoint and central flow fibers are finite, add a common cap above all their feasible lattice coordinates. This leaves their counts unchanged. □

## 5 A fixed-order obstruction for $3 \times 3$ tables

The second normal system in [7, Proposition 9.1] consists of the nine cell functionals for a $3 \times 3$ table in the four free corner coordinates $x_{11}, x_{12}, x_{21}, x_{22}$. The following example concerns rounding schemes that choose a floor or ceiling in each coordinate while preserving the two inputs' coordinate sum. Call such a scheme *causal in a fixed coordinate order* if a rounding decision can use the endpoint data and the already observed input coordinates, including the current one.

**Proposition 7.** *There is a single pair of $3 \times 3$ endpoint data with integral midpoint such that, for every fixed coordinate order, no causal rounding scheme can produce two feasible midpoint tables with probability one for each feasible input pair.*

*Proof.* At the first endpoint all margins are zero, so the only table is the zero table. At the second endpoint every margin is two. At both endpoints take lower bound zero and upper bound two on all cells. In particular the second endpoint contains

$$T^{(0)} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad T^{(1)} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

At the midpoint all row and column margins are one. First suppose the last observed coordinate is $x_{22}$. The two input pairs agree at each preceding coordinate. For $T^{(0)}$, both output last coordinates are zero. Each output has $x_{33} = \sum_{i,j \le 2} x_{ij} - 1 \ge 0$, so the two output corner sums, which add to two, both equal one. The up-rounds of $x_{12}$ and $x_{21}$ must go to different outputs. For $T^{(1)}$, the combined second corner-row sum is two and so is the combined second corner-column sum. Each such sum is at most one in each output, hence both equal one. The up-rounds of $x_{12}$ and $x_{21}$ must now go to the same output. The earlier prefix cannot distinguish these input pairs. The same proof applies to randomized decisions required to succeed almost surely because the events demanded at that prefix are disjoint. Finally, swapping the first two rows or columns preserves both endpoint data and places any prescribed last corner at $x_{22}$. □

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## References

[1] A. Barvinok, *Brunn–Minkowski inequalities for contingency tables and integer flows*, Adv. Math. 211 (2007), 105–122. doi:10.1016/j.aim.2006.07.012.

[2] N. Gozlan, C. Roberto, P.-M. Samson and P. Tetali, *Transport proofs of some discrete variants of the Prékopa–Leindler inequality*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 22 (2021), 1207–1232. doi:10.2422/2036-2145.201906_002. Theorem numbering cited from arXiv:1905.04038v1.

[3] D. Halikias, B. Klartag and B. A. Slomka, *Discrete variants of Brunn–Minkowski type inequalities*, Ann. Fac. Sci. Toulouse Math. (6) 30 (2021), 267–279. doi:10.5802/afst.1674.

[4] T. Lam and A. Postnikov, *Alcoved polytopes I*, Discrete Comput. Geom. 38 (2007), 453–478. arXiv:math/0501246.

[5] J. Leake and M. Mohammadi Yekta, *Log-concavity and approximate counting for totally unimodular polytopes*, 2026. arXiv:2609.39917v1.

[6] OpenAI, *Gaussian free-field limits of weighted integer Lipschitz heights*, preprint, 6 October 2026. OpenAI Mathematics research release.

[7] S. Schubr, *Discrete Jensen inequalities for unimodular fiber counts of dimension at most three*, Luminary Research Archive, edition 6, 20 November 2026. doi:10.5281/zenodo.24406132. Archive record.

[8] B. A. Slomka, *A remark on discrete Brunn–Minkowski type inequalities via transportation of measure*, Israel J. Math. 261 (2024), 791–807. doi:10.1007/s11856-023-2596-3. Theorem numbering cited from arXiv:2008.00738v2.

[9] D. E. Speyer, *L-log-concavity and a proof of the conjecture of Lam, Postnikov and Pylyavskyy*, 2026. arXiv:2601.05007v2.