Dimension Thresholds for Concavity of Normalized Mixed Moments

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October 24, 2026

Abstract

For a nonnegative coordinatewise nondecreasing integrable weight on a finite d-dimensional box, we prove joint log-concavity of its mixed moments divided by $\prod_i \Gamma(\alpha_i + 1)$ whenever $\alpha_i \geq d/4$, $d \geq 2$. The weight need not be log-concave, independent across coordinates, or positively associated. For finite product grids with ordered log-concave coordinate values and a weight nondecreasing in the coordinate indices, the corresponding sufficient orthant is $\alpha_i \geq d/4 + 4$. Increasing upper-corner densities and finite-grid weights give counterexamples at common exponent $d/4 - O(1)$. Thus the universal dimension thresholds in the two classes both have optimal leading constant $1/4$. The proof combines conditional logarithmic-deficit bounds with a signed-block covariance estimate and elementary trigamma inequalities. Explicit rational finite-grid and continuous counterexamples accompany the asymptotic construction. The same method gives spatial product-power smoothing of arbitrary decreasing densities and arrays at powers $d/4 + 4$ and $d/4 + 7$, respectively, again with optimal leading coefficient $1/4$. A classical matroid log-concavity input yields a dimension-free binary endpoint, whereas unweighted hereditary binary supports already require dimension-linear powers.

1 Two joint moment theorems

The variables of concavity in the first three theorems are the *exponents* of mixed moments. The weight and its underlying box or grid remain fixed. For $\alpha \in \Omega_d = (0, \infty)^d$, let

$$G(\alpha) = \frac{C(\alpha)}{\prod_{i=1}^d \Gamma(\alpha_i + 1)}. \tag{1}$$

Log-concavity means concavity of $\log G$ in the full vector α, rather than separate concavity in each coordinate.

Theorem 1.1 (Continuous boxes)**.** *Let $d \geq 2$, $L_i > 0$, and let w be a measurable nonnegative coordinatewise nondecreasing function on $B = \prod_{i=1}^d [0, L_i]$, with $0 < \int_B w(x)\, dx < \infty$. For*

$$C(\alpha) = \int_B w(x) \prod_{i=1}^d x_i^{\alpha_i}\, dx,$$

the function G in (1) is positive and jointly log-concave on the convex orthant

$$\alpha_i \geq d/4 \quad (1 \leq i \leq d). \tag{2}$$

No log-concavity, product structure, or association assumption on w is required.

For the discrete theorem, prescribe for each coordinate an ordered finite list

$$0 \le x_{i,0} \le \cdots \le x_{i,k_i}, \qquad k_i \ge 1, \quad x_{i,k_i} > 0, \tag{3}$$

satisfying $x_{i,j}^2 \ge x_{i,j-1}x_{i,j+1}$ at interior indices. Its positive nodes therefore form a suffix; positive values may be repeated. A constant positive list is allowed. The weight is an array on the *indices*, and repeated values retain their multiplicity.

Theorem 1.2 (Finite grids)**.** *Suppose the coordinate lists satisfy (3), and $w(j_1, \dots, j_d)$ is nonnegative, not identically zero, and nondecreasing in each index. For $d \ge 2$ put*

$$C(\alpha) = \sum_{j_1=0}^{k_1} \cdots \sum_{j_d=0}^{k_d} w(j_1, \dots, j_d) \prod_{i=1}^{d} x_{i,j_i}^{\alpha_i}.$$

Then G is positive and jointly log-concave on

$$\alpha_i \ge d/4 + 4 \quad (1 \le i \le d). \tag{4}$$

In particular this applies to ordinary finite arithmetic grids with nonnegative nodes. Terms containing a zero node vanish because the varied exponents are strictly positive.

The two additive constants are sufficient. The different finite-dimensional bounds reflect a continuous survival-contraction property that is not available for the atomic logarithmic-node laws. Their common leading coefficient is optimal.

For each $d \ge 2$, define τ_d^{box} to be the infimum of $t \ge 0$ such that *every* continuous weight in Theorem 1.1 gives log-concavity on $\Omega_d \cap [t, \infty)^d$. Define τ_d^{grid} analogously for Theorem 1.2.

Theorem 1.3 (Optimal leading dimension constant)**.** *For each of these two separate classes,*

$$\tau_d = \frac{d}{4} + O(1), \qquad \lim_{d\to\infty} \frac{\tau_d}{d} = \frac{1}{4}. \tag{5}$$

The lower bounds already hold for increasing bounded step densities on $[0,1]^d$ and for actual finite arithmetic grids, respectively.

Classical context and scope

Normalized one-variable Mellin concavity is classical. The Barlow–Marshall–Proschan and Borell results [1, 2], with an explicit proof in Section 5 of Nayar–Oleszkiewicz [3], establish the scalar ancestry. A coordinatewise increasing density on a bounded interval has a concave, hence log-concave, survival function, so its scalar normalized-moment consequences must not be counted as new here. Separate scalar concavity does not control the mixed covariance entries of the exponent Hessian. Theorems 1.1– 1.3 address that joint issue and show that its uniform exponent cost has leading order $d/4$.

Common, Martin and Stubbe [4] give a classical scalar normalized-Mellin theorem when the input has a concave or convex logarithm. The comparison here uses its indexed statement and secondary mathematical expositions; the full primary proof was not available during preparation. Related normalized-transform and variance arguments for concavity-shaped measures appear in Fradelizi–Li–Madiman [5]. Those shape assumptions are not imposed on the arbitrary monotone weights considered here.

The arguments arose from mixed-moment questions in the author's prefix-sum and toric investigations [12, 13]. Those unpublished companion manuscripts concern different variables and conclusions; their kernel formulas and geometric classifications are not reproduced here. The covariance identity, Cauchy–Schwarz inequality, conditional tail integration and gamma identities are classical tools. The contribution asserted is their joint arbitrary-monotone application with the matching leading dimension obstruction. Section 6 applies the same mechanism to spatial truncation, with a separate sharp-leading power threshold and explicit decreasing-input counterexamples.

2 Logarithmic deficits under power tilts

Normalize either moment to a probability law proportional to its summand or integrand, and call the sampled coordinate values U_i. In the continuous case $U_i > 0$ almost surely. In the grid case positive exponents remove every zero-node term. In both settings,

$$\nabla^2 \log C(\alpha) = \operatorname{Cov}(\log U_1, \dots, \log U_d), \qquad \nabla^2 \log G = \nabla^2 \log C - \operatorname{diag}_i \psi_1(\alpha_i + 1), \tag{6}$$

where $\psi_1 = (\log \Gamma)''$ is the trigamma function. For finite grids differentiation is finite. For continuous boxes, locally positive exponents make all products of the required logarithmic factors times a small positive power uniformly bounded on the box; an integrable multiple of w dominates the derivatives. The moments are positive by positive mass in the continuous case and positive weight at the grid's top corner in the discrete case.

Lemma 2.1 (Continuous conditional deficits)**.** *In the continuous setting, put $c_i = \alpha_i + 1$ and $Z_i = c_i \log(L_i/U_i)$. Then*

$$0 \le \mathbb{E} Z_i \le 1, \qquad \operatorname{Var} Z_i \le 1, \qquad \mathbb{E}[Z_i \mid Z_j] \le 1 \quad (i \ne j) \tag{7}$$

almost surely in the conditioning variable.

Proof. After fixing any other coordinates of positive conditional mass and integrating those not fixed, the marginal of U_i is proportional to $u^{\alpha_i} W_i(u)$, where W_i is nonnegative and nondecreasing. All such statements hold almost everywhere for the usual conditional densities. Set $T = \log(L_i/U_i)$ and $c = c_i$. Its density is proportional to

$$e^{-ct} g(t), \qquad g(t) = W_i(L_i e^{-t}), \qquad t \ge 0,$$

with g nonincreasing. If S is its survival function, shifting the integral gives

$$S(t + h) \le e^{-ch} S(t), \qquad t, h \ge 0. \tag{8}$$

Hence $\mu = \mathbb{E} T \le 1/c$ and $\int_t^\infty S(s)\, ds \le S(t)/c$. Tonelli's theorem gives

$$\mathbb{E} T^2 = 2 \int_0^\infty \int_t^\infty S(s)\, ds\, dt \le \frac{2\mu}{c}, \qquad \operatorname{Var} T \le \frac{2\mu}{c} - \mu^2 \le \frac{1}{c^2}.$$

The same mean bound applies after conditioning on U_j, and conditioning on Z_j is equivalent because that coordinate change is invertible on the positive interval. Scaling proves the lemma. □

Lemma 2.2 (Discrete conditional deficits)**.** *In the grid setting put*

$$M_i = x_{i,k_i}, \qquad Z_i = \alpha_i \log(M_i/U_i).$$

Then

$$0 \le \mathbb{E} Z_i \le 1, \qquad \operatorname{Var} Z_i \le 2, \qquad \mathbb{E}[Z_i \mid Z_j] \le 1 \quad (i \ne j) \tag{9}$$

almost surely.

Proof. It suffices to prove the tail estimate for one coordinate, including after fixing an index in another coordinate. Its marginal weights have the form $W_l x_l^\alpha$, W_l nondecreasing. On the positive suffix, log-concavity of the nodes gives

$$\frac{x_l}{x_j} \le \frac{x_{l+k-j}}{x_k} \quad (l \le j). \tag{10}$$

Indeed the logarithmic increments of the node sequence are nonincreasing, and the two compared increments span the same number of positions, with the latter interval shifted to the right. Zeros on the initial prefix contribute no tilted mass. The index injection $l \mapsto l+k-j$, and monotonicity of W, imply

$$\sum_{l \le j} W_l (x_l/x_j)^\alpha \le \sum_{l=0}^{k} W_l (x_l/x_k)^\alpha.$$

For $t > 0$, choose the last index with $x_j \le x_k e^{-t}$. If no positive such node exists the event has probability zero; otherwise division gives

$$\Pr\{\log(M_i/U_i) \ge t\} \le e^{-\alpha_i t}.$$

Repeated values cause no difficulty, and the assertion at $t = 0$ is trivial. Thus $\mathbb{E}Z_i \le 1$ and $\mathbb{E}Z_i^2 \le 2$, proving the variance bound. The injection applies to every conditional index marginal. When repeated values make Z_j a coarser observation of that index, the conditional mean bound persists by averaging.

□

3 Signed covariances and trigamma

The following elementary matrix estimate pays for mixed signs without an entrywise lower bound on the covariances.

Lemma 3.1. *Let $Z_1, \dots, Z_d \ge 0$ have finite second moments, and suppose $\mathbb{E}[Z_i \mid Z_j] \le 1$ for $i \ne j$, $\mathbb{E}Z_i \le 1$, and $\operatorname{Var} Z_i \le V$, where $V \ge 1/4$. Then*

$$\operatorname{Cov}(Z) \preceq \kappa_d(V) I, \qquad \kappa_d(V) = \min\left\{ dV, \frac{d + 8V - 2}{4} \right\}. \tag{11}$$

Here $\preceq$ denotes the order of symmetric quadratic forms.

Proof. Put $\mu_i = \mathbb{E}Z_i \in [0,1]$. Multiplying the conditional bound by nonnegative Z_j and taking expectations gives $\mathbb{E}Z_iZ_j \le \min(\mu_i, \mu_j)$. If $\mu_i \le \mu_j$, then

$$\operatorname{Cov}(Z_i, Z_j) \le \mu_i(1-\mu_j) \le \mu_i(1-\mu_i) \le 1/4.$$

The other order is identical. This is only an upper bound.

For a vector supported on p coordinates with one sign,

$$\begin{aligned} \operatorname{Var}\left(\sum_i v_i Z_i\right) &\le V \sum_i v_i^2 + \frac{1}{2} \sum_{i<j} |v_i v_j| \\ &= \left(V - \frac{1}{4}\right) \sum_i v_i^2 + \frac{1}{4} \left(\sum_i |v_i|\right)^2 \le \frac{p + 4V - 1}{4} \|v\|_2^2. \end{aligned}$$

For a general vector split positive and negative supports into sizes p, q. When both are nonempty, write the linear form as $P - Q$. Variance Cauchy–Schwarz and then two-dimensional Cauchy–Schwarz give

$$\begin{aligned}\operatorname{Var}(P-Q) &\le (\sqrt{\operatorname{Var} P} + \sqrt{\operatorname{Var} Q})^2 \\ &\le \left(\sqrt{\frac{p+4V-1}{4}}\|v_+\|_2 + \sqrt{\frac{q+4V-1}{4}}\|v_-\|_2\right)^2 \\ &\le \frac{p+q+8V-2}{4}\|v\|_2^2 \le \frac{d+8V-2}{4}\|v\|_2^2.\end{aligned}$$

A one-sign vector has the stronger bound already obtained; $V \ge 1/4$ ensures it is no larger than the displayed general bound. Independently, marginal Cauchy–Schwarz gives $\operatorname{Var}(\sum_i v_i Z_i) \le V(\sum_i |v_i|)^2 \le dV\|v\|_2^2$. Taking the minimum proves (11). □

Lemma 3.2. *For $c > 0$ and $a > 0$,*

$$\psi_1(c) \ge \frac{1}{c} + \frac{1}{2c^2}, \qquad \psi_1(a+1) \ge \frac{1}{a} - \frac{1}{2a^2}. \tag{12}$$

Proof. Use the standard series $\psi_1(c) = \sum_{n=0}^{\infty}(c+n)^{-2}$ and recurrence $\psi_1(c+1) = \psi_1(c) - c^{-2}$ [11]. Convexity of $f(t) = (c+t)^{-2}$ gives

$$\int_n^{n+1} f(t)\,dt \le \frac{f(n) + f(n+1)}{2}.$$

Summing yields $1/c \le \psi_1(c) - 1/(2c^2)$. The recurrence gives the second assertion, with the indicated inequality direction unchanged. □

Proof of Theorem 1.1. By Lemmas 2.1 and 3.1, $\operatorname{Cov}(Z) \preceq \kappa_d(1) I$. Since $\log U_i = \log L_i - Z_i/c_i$, (6) gives

$$\nabla^2 \log G \preceq \operatorname{diag}_i\left\{\frac{\kappa_d(1)}{c_i^2} - \psi_1(c_i)\right\} \preceq \operatorname{diag}_i \frac{\kappa_d(1) - c_i - 1/2}{c_i^2}.$$

For $d \ge 2$, $\kappa_d(1) = (d+6)/4$. The condition $\alpha_i \ge d/4$ is exactly $c_i \ge \kappa_d(1) - 1/2$. Thus the log-Hessian is negative semidefinite everywhere on the stated orthant. It is a convex subset of the positive differentiation domain, so this proves joint log-concavity, including its threshold boundary. □

Proof of Theorem 1.2. Lemma 2.2 gives $\operatorname{Cov}(Z) \preceq \kappa_d(2) I$. Here $\log U_i = \log M_i - Z_i/\alpha_i$, so

$$\nabla^2 \log G \preceq \operatorname{diag}_i\left\{\frac{\kappa_d(2)}{\alpha_i^2} - \psi_1(\alpha_i + 1)\right\} \preceq \operatorname{diag}_i \frac{\kappa_d(2) + 1/2 - \alpha_i}{\alpha_i^2}.$$

For $d \ge 2$, $\kappa_d(2) = (d+14)/4$. Therefore $\alpha_i \ge d/4 + 4 = \kappa_d(2) + 1/2$ makes every coefficient nonpositive, proving the assertion as above. □

4 Corner mixtures and necessary dimension growth

We construct weights in precisely the same monotonicity classes as the positive theorems. The atomic measure below is only a convenient limit; actual continuous counterexamples are honest bounded densities.

Fix an integer $c \geq 2$ and a dimension $d \geq 2$. Mix uniform probability on $[0,1]^d$ with an atom at $\mathbf{1}$, choosing original atom mass

$$q = \frac{\eta}{\eta + (1-\eta)c^d}, \qquad \eta = \frac{d-1}{2d}. \tag{13}$$

At $\alpha = (c-1, \ldots, c-1)$ the power-tilted mass of the atom is exactly η. The baseline conditional coordinates are independent Beta$(c,1)$, whose logarithms have mean $-1/c$ and variance $1/c^2$. At the corner all logarithms vanish. The covariance of the tilted log coordinates is therefore

$$\frac{1-\eta}{c^2} I + \frac{\eta(1-\eta)}{c^2} \mathbf{1}\mathbf{1}^T. \tag{14}$$

Its eigenvalue on $\mathbf{1}$ is

$$\lambda = \frac{(1-\eta)(1+d\eta)}{c^2} = \frac{(d+1)^2}{4dc^2}. \tag{15}$$

The chosen η maximizes this expression over mixture masses. The normalized log-Hessian subtracts $\psi_1(c)I$.

Take $d = 4c+3$. The elementary integral-test upper bound is

$$\psi_1(c) \leq \frac{1}{c^2} + \int_c^\infty \frac{dt}{t^2} = \frac{c+1}{c^2}.$$

On the other hand

$$(d+1)^2 - 4d(c+1) = 4c + 4 > 0.$$

Thus (15) is strictly larger than $\psi_1(c)$; the normalized log-Hessian has a positive eigenvalue.

Realization by continuous weights

For $0 < \varepsilon < 1$, define the probability density

$$w_\varepsilon(u) = (1-q) + q\varepsilon^{-d} \mathbf{1}_{[1-\varepsilon,1]^d}(u), \qquad u \in [0,1]^d. \tag{16}$$

For each fixed ε it is bounded, nonnegative and coordinatewise nondecreasing. Its moments and first two logarithmic exponent derivatives at the specified positive exponents converge to those of the mixture. The uniform component is fixed; the upper-box logarithms are uniformly bounded and tend to zero. Consequently the strict positive eigenvalue persists for every sufficiently small positive ε, including a rational choice. This is a counterexample among actual finite-mass densities.

Realization by finite arithmetic grids

On $\{0, 1/n, \ldots, 1\}^d$ prescribe index weights

$$w_n(j) = \frac{1-q}{(n+1)^d} + q\mathbf{1}_{j=(n,\ldots,n)}. \tag{17}$$

They are nonnegative and coordinatewise nondecreasing. At the strictly positive exponent $c-1$, the functions x^{c-1}, $x^{c-1}\log x$ and $x^{c-1}(\log x)^2$ extend continuously to zero by zero. Riemann sums

show convergence of the moments and their first two logarithmic derivatives to the same mixture quantities. Hence a finite integer n gives a strict failure. Rescaling nodes by n changes $\log G$ only by an affine function of α, so this also gives a counterexample on an integer arithmetic grid.

Proof of Theorem 1.3. Write $p = c - 1 \geq 1$. The constructions give a counterexample at the common exponent p in dimension $d_0 = 4p + 7$. Such a counterexample embeds into every larger dimension: append independent uniform box coordinates, or grid coordinates with values $\{0, 1\}$, and let the weight be independent of the new indices. The normalized moment factors, leaving the old bad Hessian block unchanged. For $d \geq 11$ choose $p = \lfloor (d-7)/4 \rfloor$. Thus

$$\left\lfloor \frac{d-7}{4} \right\rfloor \leq \tau_d^{\mathrm{box}} \leq \frac{d}{4}, \qquad \left\lfloor \frac{d-7}{4} \right\rfloor \leq \tau_d^{\mathrm{grid}} \leq \frac{d}{4} + 4. \tag{18}$$

The strict Hessian violation rules out concavity on the relevant orthant, even when the common exponent is its boundary, by continuity and a short segment in the positive all-one direction. The bounds prove (5). No optimal additive constant or exact value of either threshold is inferred. □

5 Explicit rational counterexamples

The limiting construction above is not needed to obtain concrete finite witnesses. The following two have dimension ten and common exponent one, below both sufficient orthants.

An integer-grid witness

On $\{0, \ldots, 100\}^{10}$ let the weight be one everywhere and increase it at the top corner by

$$M = \left(\frac{101}{2}\right)^{10}.$$

At exponent $\alpha = \mathbf{1}$, the tilted law is a half mixture of independent coordinates $\Pr(I = i) = i/5050$, $1 \leq i \leq 100$, and the deterministic top corner. For a baseline coordinate put $X = \log(100/I)$, $\mu = \mathbb{E}X$ and $\tau = \mathbb{E}X^2$. The all-one covariance eigenvalue is

$$\lambda = \frac{\tau}{2} + \frac{10-2}{4}\mu^2 = \frac{\tau}{2} + 2\mu^2. \tag{19}$$

There is a short exact certificate without evaluating logarithms. For $z_i = (100 - i)/(100 + i)$,

$$\log(100/i) = 2 \operatorname{arctanh} z_i \geq \ell_i = 2(z_i + z_i^3/3) \geq 0.$$

Exact rational summation gives

$$\sum_{i=1}^{100} \frac{i}{5050}\ell_i > \frac{12}{25}, \qquad \sum_{i=1}^{100} \frac{i}{5050}\ell_i^2 > \frac{7}{16}. \tag{20}$$

Hence $\lambda > 7/32 + 288/625 = 13591/20000$. A four-term integral bound gives

$$\psi_1(2) \leq \sum_{j=2}^{5} \frac{1}{j^2} + \frac{1}{5} = \frac{2389}{3600} < \frac{2}{3}.$$

Thus the normalized log-Hessian has a positive eigenvalue exceeding

$$\frac{13591}{20000} - \frac{2}{3} = \frac{773}{60000}. \tag{21}$$

The two finite rational inequalities (20) are reconstructed by the supplied exact checker. Their role is an exact Hessian witness, not an experimental eigenvalue calculation.

A bounded increasing box density

On $[0,1]^{10}$ set

$$w(u) = 1 + \left(\frac{10000}{199}\right)^{10} \mathbf{1}_{[99/100,1]^{10}}(u). \tag{22}$$

At exponent $\mathbf{1}$ the two tilted components have equal mass. The baseline coordinates are Beta$(2,1)$, whose logarithmic deficits have mean $1/2$ and variance $1/4$. In the upper-box component the coordinate deficits have mean $\mu_1 \le \log(100/99) < 1/99$ and nonnegative variance. Conditional coordinates are independent within each component, so the all-one covariance eigenvalue is at least

$$\frac{1}{8} + \frac{5}{2}\left(\frac{1}{2} - \frac{1}{99}\right)^2 = \frac{28423}{39204} > \frac{2}{3} > \psi_1(2).$$

The normalized log-Hessian therefore has a positive eigenvalue exceeding $2287/39204$. No singular measure, limiting density, irrational weight or numerical eigentest is used in this example. A positive eigenvalue of the smooth exponent log-Hessian gives an actual local two-point log-concavity violation.

6 Spatial power smoothing of decreasing weights

The same conditional-covariance mechanism controls a different parameter: the spatial truncation point of a one-sided product-power convolution. There is no gamma normalization in this section. The input μ and powers p_i are fixed while A varies.

Theorem 6.1 (Spatial smoothing)**.** *Let $d \ge 1$ and $A \in (0,\infty)^d$.*

1. *Suppose μ is a measurable nonnegative coordinatewise nonincreasing function on $[0,\infty)^d$, integrable on every finite box and with positive integral on at least one such box. Then*

$$F(A) = \int_{[0,\infty)^d} \mu(I) \prod_i (A_i - I_i)_+^{p_i}\, dI \tag{23}$$

   is jointly log-concave whenever $p_i \ge d/4 + 4$ for every i.

2. *Suppose $\mu : \mathbb{Z}^d_{\ge 0} \to [0,\infty)$ is finite-valued, nonzero and coordinatewise nonincreasing. Then*

$$F(A) = \sum_{n \in \mathbb{Z}^d_{\ge 0}} \mu(n) \prod_i (A_i - n_i)_+^{p_i} \tag{24}$$

   is jointly log-concave whenever $p_i \ge d/4 + 7$ for every i.

Both functions are finite, positive and C^2 on the positive orthant under the stated power bounds. Infinite total continuous mass and infinite global array support are permitted: every active integral is on a bounded box and every active sum is finite.

Here $x_+ = \max(x, 0)$. For the continuous input, monotonicity and positive box mass imply positive mass in a smaller lower box below any positive A. For the array, $0 < \mu(0) < \infty$. These facts give positivity. Powers greater than two make the kernels C^2; local integrability, or a locally finite set of summands, justifies original-coordinate differentiation on compact positive A-domains. These observations also cover all support walls.

At a fixed A, put $z_i = A_i - I_i$ and normalize by $F(A)$. The continuous residual density is proportional to $\prod_i z_i^{p_i}\mu(A-z)$ on $[0,A]$. Its residual weight is coordinatewise nondecreasing. The analogous finite active grid has that same monotonicity after reversing its coordinate index lists. These representations bound the law at a fixed A; one must *not* differentiate their A-dependent residual weights as though they were fixed. Differentiation of the original fixed-input formulas gives

$$\nabla_A^2 \log F = \operatorname{Cov}\left(\frac{p_i}{z_i}\right)_{i=1}^{d} - \operatorname{diag}_i \mathbb{E}\frac{p_i}{z_i^2}. \tag{25}$$

Zero residuals have zero tilted mass. The power bounds justify all inverse-residual moments in this identity, including on grid walls.

Continuous residual moments

For a conditional coordinate law, put $T = \log(A_i/z_i)$ and $W = e^T - 1$. Lemma 2.1, now at exponent p_i, gives conditional survival contraction of T at rate $p_i + 1$. Writing $p = p_i$, the survival of W consequently obeys

$$S_W(s+h) \le \left(\frac{1+s}{1+s+h}\right)^{p+1} S_W(s). \tag{26}$$

Thus $m = \mathbb{E}W \le 1/p$ and $\int_s^\infty S_W(t)\,dt \le (1+s)S_W(s)/p$. Integrating once more gives

$$\frac{\mathbb{E}W^2}{2} \le \frac{\mathbb{E}W + \mathbb{E}W^2/2}{p}, \qquad \mathbb{E}W^2 \le \frac{2m}{p-1}.$$

The quadratic $2m/(p-1) - m^2$ is increasing on $0 \le m \le 1/p$. Therefore $Z_i = p_i W_i$ has conditional mean at most one and

$$\operatorname{Var} Z_i \le \frac{p_i + 1}{p_i - 1}. \tag{27}$$

The conditioning may be on any other coordinate, because its remaining residual weight is still nondecreasing.

Continuous part of Theorem 6.1. Let $P = \min_i p_i$ and $V = (P+1)/(P-1)$. Lemma 3.1 gives $\operatorname{Cov}(Z) \preceq \kappa I$, with

$$\kappa \le \frac{d}{4} + 2V - \frac{1}{2} = \frac{d}{4} + \frac{3}{2} + \frac{4}{P-1}.$$

Since $p_i/z_i = (p_i + Z_i)/A_i$ and $z_i \le A_i$, (25) gives

$$\nabla_A^2 \log F \preceq \operatorname{diag}_i \frac{\kappa - p_i}{A_i^2}.$$

At $P \ge d/4 + 4$, the difference $P - \kappa$ is bounded below by $5/2 - 4/(d/4+3) > 0$. Thus the Hessian is negative semidefinite throughout the convex positive orthant. □

Discrete residual moments

For each active coordinate, reverse the indices $n_i < A_i$. Its positive residuals form an arithmetic list $t, t+1, \dots, A_i$ with $0 < t \le 1$. Reversed weights are nondecreasing. The conditional injection in Lemma 2.2 therefore gives

$$\Pr\{W_i > s \mid \text{another coordinate}\} \le (1+s)^{-p_i}, \qquad W_i = A_i/z_i - 1.$$

For $p_i > 2$, tail integration yields

$$\mathbb{E}W_i \le \frac{1}{p_i - 1}, \qquad \mathbb{E}W_i^2 \le \frac{2}{(p_i - 1)(p_i - 2)}.$$

Consequently $Z_i = (p_i - 1)W_i$ has conditional mean at most one and $\operatorname{Var} Z_i \le 2(p_i - 1)/(p_i - 2)$.

Discrete part of Theorem 6.1. Take $V = 2(P-1)/(P-2)$, where $P = \min_i p_i$. The matrix lemma gives

$$\kappa \le \frac{d}{4} + \frac{7}{2} + \frac{4}{P-2}.$$

Now $p_i/z_i = p_i/A_i + [p_i/(p_i-1)]Z_i/A_i$, so

$$\nabla_A^2 \log F \preceq \operatorname{diag}_i \frac{\kappa[p_i/(p_i-1)]^2 - p_i}{A_i^2}. \tag{28}$$

The factor $p_i/(p_i-1)$ must be retained. It suffices that $\kappa \le (p_i-1)^2/p_i = p_i - 2 + 1/p_i$, an increasing function for $p_i > 1$. For $P \ge d/4 + 7$,

$$P - 2 + 1/P - \kappa \ge \frac{3}{2} + \frac{1}{P} - \frac{4}{P-2} > 0,$$

since $P \ge 7$. This proves (28) is negative semidefinite everywhere, including support walls by C^2 regularity. □

The spatial leading constant is also sharp

Define σ_d^{box} as the infimum of $t \ge 2$ such that for every continuous input in Theorem 6.1, every common power $p > 2$ satisfying $p \ge t$ gives spatial log-concavity. Define σ_d^{grid} in the same way for decreasing arrays. The condition $p > 2$ is part of this definition, also when $t = 2$.

Proposition 6.2. *Both separate common-power thresholds satisfy $\sigma_d = d/4 + O(1)$, and $\sigma_d/d \to 1/4$.*

Proof. At $A = \mathbf{1}$ consider, as an auxiliary measure, uniform volume on $[0,1]^d$ plus an origin atom of mass $M = (p+1)^{-d}$. Their tilted masses are equal. The uniform residuals have independent $\mathrm{Beta}(p+1, 1)$ laws. Their scores p/z_i have mean $p+1$ and variance $(p+1)/(p-1)$; at the origin the scores equal p. The all-one covariance eigenvalue is $(p+1)/[2(p-1)] + d/4$. The expected curvature in (25) is the diagonal scalar $p^2/(p-1)$. Thus the all-one log-Hessian eigenvalue is exactly

$$\frac{d}{4} - p - \frac{1}{2}. \tag{29}$$

For $d > 11$ take $p = (d-3)/4 > 2$; the value is $1/4 > 0$.

Actual decreasing densities replace the atom by $M\varepsilon^{-d}\mathbf{1}_{[0,\varepsilon]^d}$ and keep the background $\mathbf{1}_{[0,1]^d}$. Their functions and first two spatial derivatives converge to the auxiliary expressions: for $p > 2$ all required kernels are continuous on the compact integration box. Strict Hessian failure therefore persists for a positive ε.

For an actual decreasing array use a constant background on $\{0, \dots, N\}^d$ and an extra origin weight balancing its tilted mass with that background at $A = (N, \dots, N)$. After rescaling by N, Riemann sums for z^p, z^{p-1}, z^{p-2} converge to the same expressions. Their zero-endpoint extensions are continuous because $p > 2$. A finite N thus gives a genuine counterexample, not merely a continuous analogy. It follows that $\sigma_d \ge (d-3)/4$ for sufficiently large d. The upper bounds $d/4 + 4$ and $d/4 + 7$ from Theorem 6.1 finish the proof. □

Two explicit spatial witnesses

These examples use $p = 3$, $d = 15$ and involve neither a singular input nor an approximation limit. For the array, let $N = 100$, set $\mu(n) = 1$ on $\{0, \dots, N\}^{15}$ and zero outside, and add an origin weight

$$M = \left(\frac{10201}{400}\right)^{15}.$$

At $A_i = N$ the tilted components have equal mass. The background scaled scores $Np/(N - n_i)$ have mean $402/101$. Summing the first three integer powers gives the all-one log-Hessian eigenvalue λ exactly as

$$N^2\lambda = \frac{5001}{40804} > 0. \tag{30}$$

The terms with $n_i = N$ have zero tilted mass and zero first and second kernel derivatives at $p = 3$; the finite sum is C^2 there. The checker reconstructs the score moments and curvature directly from the power sums, including the factor N^2.

For a continuous input take $\varepsilon = 1/1000$ and

$$\mu(I) = \mathbf{1}_{[0,1]^{15}}(I) + [1 - (1 - \varepsilon)^4]^{-15}\mathbf{1}_{[0,\varepsilon]^{15}}(I).$$

At $A = \mathbf{1}$ the tilted masses are again equal. Baseline scores have mean four, variance two and expected curvature six. The origin-box scores have mean at most $3/(1 - \varepsilon) < 4$, nonnegative variance and expected curvature at most $3/(1 - \varepsilon)^2$. Hence its all-one log-Hessian eigenvalue is at least

$$1 + \frac{15}{4}\left(4 - \frac{3}{1 - \varepsilon}\right)^2 - \frac{6 + 3/(1 - \varepsilon)^2}{2} = \frac{74686}{332667} > 0. \tag{31}$$

Both weights are nonincreasing in every coordinate. These are counterexamples for the *general decreasing-input class*; they are not counterexamples for a uniform simplex indicator.

Corollary 6.3 (High-power simplex kernels)**.** *For every integer $r \geq 0$ and every dimension $d \geq 1$,*

$$\sum_{\substack{n \in \mathbb{Z}^d_{\geq 0} \\ \sum_i n_i \leq r}} \prod_i (A_i - n_i)_+^{p_i}$$

is jointly log-concave on $A > 0$ whenever $p_i \geq d/4 + 7$.

Proof. The simplex indicator is a nonzero coordinatewise decreasing array, so Theorem 6.1 applies. □

The broad-class witnesses above do not refute that narrower question.

The application concerns a spatial vector, whereas Varošanec [6] studies scalar fractional-integration *order* at a fixed endpoint. Classical log-concave convolution preservation requires log-concavity of both inputs; see Saumard–Wellner [7], Proposition 3.5. That hypothesis is absent for our arbitrary decreasing μ, so it cannot be supplied by monotonicity alone. Fractional integration, Pareto moments and the score-covariance identity are classical tools; the assertion here is the arbitrary-monotone joint theorem with its leading dimension obstruction.

7 Binary structure and the quadratic endpoint

The dimension-dependent bounds above cannot be replaced by a fixed power for arbitrary hereditary binary supports. Matroid structure gives a contrasting dimension-free endpoint. The positive assertion below is an attributed consequence of the classical matroid log-concavity theorem.

Proposition 7.1 (Matroid binary supports)**.** *Let $\mathcal{I}$ be the independent sets of a finite matroid on $[d]$, and let $0 \le \lambda_i \le 1$. For independent real powers $p_i \ge 2$,*

$$F_p(A) = \sum_{S \in \mathcal{I}} \left(\prod_{i \in S} \lambda_i \right) \prod_{i=1}^{d} (A_i - \mathbf{1}_{i \in S})_+^{p_i}$$

is jointly log-concave on $A > 0$, in every dimension.

Proof. For a matroid on d elements, Anari–Liu–Oveis Gharan–Vinzant [8], Theorem 4.1, prove complete log-concavity of

$$g(y, z) = \sum_{S \in \mathcal{I}} y^{d-|S|} \prod_{i \in S} z_i.$$

The homogenizing degree is the *ground-set size*, not the rank. Consequently $Q(z) = g(1, z)$ is log-concave. Write $Q = R_i + z_i Q_i$, with $R_i = Q|_{z_i=0}$. Hereditary deletion gives $Q_i \le R_i$ coefficientwise, hence $0 \le \partial_i \log Q \le 1/(1 + z_i) \le 1$. Since $Q \ge 1$, these inequalities and its log-Hessian extend to zero coordinates.

For $p_i = 2$, factor $F_2 = \prod_i A_i^2 \, Q(\phi_1(A_1), \dots, \phi_d(A_d))$, where $\phi_i(a) = \lambda_i (1 - 1/a)_+^2$. On each chamber separated by $A_i = 1$,

$$\begin{aligned} \nabla^2 \log F_2 = \; & \operatorname{diag}_i \left[-\frac{2}{A_i^2} + \phi_i''(A_i) \partial_i \log Q \right] \\ & + \operatorname{diag}(\phi_i') \nabla^2 \log Q \, \operatorname{diag}(\phi_i'). \end{aligned}$$

The second term is negative semidefinite. For $a > 1$, $\phi_i''(a) = 2\lambda_i(3 - 2a)/a^4$; if this is positive it is at most $2/a^2$, since $\lambda_i \le 1$ and $a^2 - (3 - 2a) = (a - 1)(a + 3) \ge 0$. If it is nonpositive the diagonal bound is immediate. Inactive coordinates have $\phi_i'' = 0$. Thus the first term is also negative semidefinite. Positivity and C^1 regularity glue log-concavity across the chamber walls.

The zero extension of F_2 is coordinatewise nondecreasing and log-concave. For $p_i > 2$, convolve in coordinate i with $s^{\alpha_i - 1} \mathbf{1}_{s \ge 0}$, $\alpha_i = p_i - 2$, omitting a coordinate when $\alpha_i = 0$. The beta identity

$$\int_0^{z_+} (z - s)_+^2 s^{\alpha - 1} \, ds = B(3, \alpha) z_+^{\alpha + 2}$$

gives a positive constant times F_p. For $\alpha \ge 1$ the power kernel is log-concave. For $0 < \alpha < 1$, substitute $s = u^{1/\alpha}$; its weighted Jacobian is constant, and $A_i - u^{1/\alpha}$ is concave. Composition with a nondecreasing log-concave function remains log-concave. Prékopa marginalization [9] applies, also with different orders in different coordinates; all integration fibers are bounded. □

Remark 7.2. The quadratic calculation is a compensated composition rule. It only requires a positive C^2 function Q on the nonnegative orthant, with continuous boundary derivatives, concave logarithm, and $0 \le \partial_i \log Q \le 1$. Under those hypotheses the same transform $\prod_i A_i^2 Q((\lambda_i(1 - 1/A_i)_+^2)_i)$ is log-concave for $0 \le \lambda_i \le 1$. Matroid polynomials supply these hypotheses by the cited classical theorem and deletion bound.

The universal power endpoint cannot decrease: the one-element nonloop matroid with activity one gives $A^p+(A-1)_+^p$, whose midpoint product at $1\pm\varepsilon$ exceeds its value at one squared for small ε when $0<p<2$: the difference is $(1-\varepsilon^2)^p+\varepsilon^p(1-\varepsilon)^p-1\sim\varepsilon^p$. At $p=2$ an activity $\lambda>1$ gives right log-curvature numerator $2(\lambda-1)>0$ at one. These are universal class statements; loops or zero activities can remove an individual obstruction.

For a uniform matroid the proposition treats all binary occupancies with a total-size bound. In particular, on the convex first box $0<A_i\le 2$ it also treats the full simplex kernel, since all higher occupancies vanish. It does not treat the other multioccupancy cells.

Proposition 7.3 (Uniform hereditary obstruction)**.** *Every fixed positive real power fails spatial log-concavity for some unweighted hereditary binary support. More quantitatively, for every real $p\ge 2$ there is such a failure in dimension $2\lceil 50p\rceil$.*

Proof. Take two disjoint blocks E,G of size m and allow all subsets of one block or the other, counting the empty set once. This is the classical independent-set family of $K_{m,m}$ [10]; it is hereditary but not a matroid when $m\ge 2$. With $\phi(a)=(1-1/a)_+^p$, its exact smoothing formula is

$$F(A)=\prod_i A_i^p\left[\prod_{i\in E}(1+\phi(A_i))+\prod_{i\in G}(1+\phi(A_i))-1\right].$$

At $A_i=a>1$, put $u=\phi(a)$ and $q=(1+u)^m$. Block symmetry gives a log-Hessian eigenvector $(\mathbf{1}_E,-\mathbf{1}_G)/\sqrt{2m}$ with eigenvalue

$$L=-\frac{p}{a^2}+\frac{q}{2q-1}\left[\frac{\phi''(a)}{1+u}+(m-1)\frac{\phi'(a)^2}{(1+u)^2}\right]. \tag{32}$$

Indeed the bracketed partition function has zero first variation in that direction; differentiating it twice yields the displayed formula after normalization by $2m$. For fixed $p>0$ and $a>1$, $\phi'(a)>0$ and $q/(2q-1)\to 1/2$, so $L>0$ for large m.

For the quantitative statement choose $a=p$, $m=\lceil 50p\rceil$, and $b=u/(1+u)$. The function $(1-1/p)^p$ increases for $p\ge 2$, so $u\ge 1/4$ and $1/5\le b<1$. Its logarithmic derivative is $\log(1-1/p)+1/(p-1)>0$: with $x=1/p$, its derivative in x is $x/(1-x)^2>0$ and its value at zero is zero. Also $\phi'=u/(p-1)$ and $\phi''=-u/[p(p-1)]$. The square bracket in (32) is at least

$$\frac{50p-1}{25(p-1)^2}-\frac{1}{p(p-1)}=\frac{50p^2-26p+25}{25p(p-1)^2}>0.$$

Using $q/(2q-1)>1/2$ now gives

$$L>-\frac{1}{p}+\frac{50p-1}{50(p-1)^2}-\frac{1}{2p(p-1)}=\frac{74p-25}{50p(p-1)^2}>0. \tag{33}$$

For a smaller exact quadratic witness take $p=2$, $m=26$, $a=3/2$. Then $u=1/9$, $\phi'=8/27$, $\phi''=0$, and

$$L=\frac{8\cdot 9^{25}}{2\cdot 10^{26}-9^{26}}>0. \tag{34}$$

This is a 52-dimensional unweighted binary counterexample. □

For precision let χ_d be the universal common-power threshold for nonempty unweighted hereditary binary supports, defined as in Proposition 6.2 using $p>2$ and $p\ge t$. Padding the two-block construction with inactive coordinates and using Theorem 6.1 gives, for $d\ge 200$,

$$\frac{\lfloor d/2\rfloor}{50}\le\chi_d\le\frac{d}{4}+7.$$

Choose $p = \lfloor d/2 \rfloor/50$ and $m = \lfloor d/2 \rfloor$; when $p = 2$ the lower bound just uses the definition $\chi_d \geq 2$. Thus $\chi_d = \Theta(d)$, but its optimal leading constant is not determined. The graph family and its partition formula are classical; the contrast here is the raw spatial smoothing threshold.

These supports even have a totally unimodular stable-set description: the cross-block inequalities $x_i + x_j \leq 1$ become a directed incidence transpose after a column sign change. At integer powers and endpoints, auxiliary integer ranges also realize the smoothing values as counts. The positive real log-Hessian witnesses do *not* refute integer midpoint log-concavity or global integer-mean Jensen for unimodular counts. Nor are they matroid or uniform-simplex counterexamples.

8 Scope, open constants and reproducibility

Theorems 1.1 and 1.2 give sufficient convex orthants; Theorem 1.3 determines the optimal leading coefficient of their universal dimension thresholds. Section 7 gives an attributed matroid endpoint and an order-d hereditary obstruction. The moment results concern joint exponent concavity on finite boxes and grids. Section 6 separately treats spatial product-power truncation with fixed decreasing inputs, allowing locally finite active sums and integrals despite infinite global support or mass. Neither conclusion requires ordinary log-concavity of its weight.

The file `verify_moments.py` uses only Python's standard library and exact rational arithmetic. It reconstructs the explicit finite-grid and continuous witnesses and the elementary polynomial identities used in the dimension comparison, including the explicit spatial witnesses and their coordinate scale, as well as the exact binary obstruction and its scalar algebra, with checks retained under optimization. Its finite checks do not replace the conditional-law, covariance, differentiation and approximation arguments proved in the text. The source, checker and citation metadata are supplied for independent inspection. Source comparison is bounded by actual accessible material; the access limitation for [4] remains explicit rather than being turned into a claim of absence of earlier work.

Disclosure. AI-assisted tools were used in development, source comparison, exact arithmetic and internal verification. Separate internal reconstructions checked the mathematical arguments and witnesses, with overlap in subsequent sharpening disclosed during review.

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