Dimension-dependent holomorphic certificate degrees on zero-sum hyperplanes

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Luminary Oddities

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Abstract

On the complex zero-sum hyperplane in $\mathbb{C}^n$, consider the symmetric quartics $A = n\sum_i |v_i|^4 - |\sum_i v_i^2|^2$ and $Q = \|v\|^4 - |\sum_i v_i^2|^2$. For every odd $n \geq 3$ and every $\alpha, \beta > 0$, we prove that a holomorphic squared-norm representation of $\|v\|^{2m}(\alpha A + \beta Q)$ requires $m \geq n-2$. For each fixed odd n, equality holds for all sufficiently small positive β/α, whereas at $\beta = 0$ the least exponent is zero. The lower bound uses an explicit Fischer annihilator and a positive Hamming-distance correlation identity on the odd-dimensional sign cube. The upper bound uses the classical quadratic complete intersection associated with the zero-sum hyperplane. For $n = 5$, we determine the entire first-degree window: the least exponent is three exactly when $0 < \beta/\alpha \leq \rho_*$, where $305\rho_*^3 + 786\rho_*^2 + 351\rho_* - 162 = 0$. An explicit four-by-four pencil and harmonic decomposition give this sharp endpoint, including its four-dimensional kernel. We also derive the quartic as a coefficient in a centered permanent deficit, which motivates the family without implying the full permanent inequality.

1 The family and its certificate degree

Let $n \geq 3$ and give

$$H_n = \{v \in \mathbb{C}^n : \textstyle\sum_i v_i = 0\}$$

the Hermitian metric inherited from $\mathbb{C}^n$. Write

$$N(v) = \sum_i |v_i|^2, \qquad p_k(v) = \sum_i v_i^k,$$

$$A(v) = n\sum_i |v_i|^4 - |p_2(v)|^2 = \sum_{i<j} |v_i^2 - v_j^2|^2, \tag{1}$$

$$Q(v) = N(v)^2 - |p_2(v)|^2 = \sum_{i<j} |v_i\overline{v}_j - v_j\overline{v}_i|^2. \tag{2}$$

Thus A is already a squared norm of holomorphic quadrics; Q is nonnegative, but its displayed squares are not holomorphic. For $\rho \geq 0$ put $q_\rho = A + \rho Q$ and define

$$\mu_n(\rho) = \min\{m \in \mathbb{Z}_{\geq 0} : N^m q_\rho = \textstyle\sum_\nu |h_\nu|^2 \text{ on } H_n,\ h_\nu \text{ holomorphic homogeneous of degree } m+2\}. \tag{3}$$

If the set is empty its minimum is $+\infty$. Multiplication by a positive scalar does not change the minimum, so this also treats $\alpha A + \beta Q$ with $\alpha > 0$ and $\rho = \beta/\alpha$.

When n is odd, $A(v) > 0$ for $v \neq 0$ in H_n: equality would force $v_i = \pm a$ for all i, incompatible with an odd number of terms summing to zero unless $a = 0$. The classical stabilization theorem of

Quillen [1] and Catlin–D'Angelo [2] therefore guarantees $\mu_n(\rho) < \infty$ for each $\rho \geq 0$. The question here is the *least* norm power, not eventual existence.

Theorem 1.1 (Odd-coordinate degree barrier and sharp small-ratio degree)**.** *For every odd $n \geq 3$ and every $\rho > 0$,*

$$\mu_n(\rho) \geq n - 2. \tag{4}$$

For each fixed odd n, there exists $\varepsilon_n > 0$ such that

$$\mu_n(\rho) = n - 2 \qquad (0 < \rho \leq \varepsilon_n). \tag{5}$$

At $\rho = 0$ one has $\mu_n(0) = 0$. More precisely, for each odd n there is a finite threshold $\rho_n^* > 0$ such that

$$\mu_n(\rho) = n - 2 \quad \Longleftrightarrow \quad 0 < \rho \leq \rho_n^*. \tag{6}$$

The general threshold is characterized spectrally below; an explicit value is obtained here for $n = 5$.

Theorem 1.2 (Sharp five-coordinate window)**.** *Let ρ_* be the unique positive root of*

$$305\rho^3 + 786\rho^2 + 351\rho - 162 = 0. \tag{7}$$

Then

$$\mu_5(\rho) = 3 \quad \Longleftrightarrow \quad 0 < \rho \leq \rho_*. \tag{8}$$

Moreover $0.274637 < \rho_* < 0.274638$. The Hermitian coefficient matrix of $N^3 q_\rho$ is positive definite for $0 \leq \rho < \rho_*$, is positive semidefinite with kernel dimension four at ρ_*, and is indefinite for $\rho > \rho_*$. At $\rho = 0$, its positive definiteness does not change the smaller minimum $\mu_5(0) = 0$.

The dimension-growing obstruction in Theorem 1.1 holds at fixed bidegree $(2, 2)$ and is independent of the positive perturbation size. It is the additional conclusion beyond eventual stabilization. The five-coordinate result identifies the full parameter window for the first admissible degree; the conclusion concerns that window. Its endpoint kernel is a coefficient-space kernel: q_{ρ_*} remains strictly positive at every nonzero point of H_5.

1.1 Relation to previous work

The stabilization theorem and its quantitative versions are classical; see [1, 2, 3]. The $n = 3$ member of our family is exactly the two-variable quartic example in Catlin–D'Angelo [2, Example 2] after a unitary change of coordinates. We give its parameter specialization in Section 7 as a comparison, not as a new family. Halfpap–Lebl [4] prove signature constraints and strict hierarchy examples, including examples that persist under arbitrarily small perturbations. Their high-depth construction increases the input degree. The present conclusion concerns a fixed quartic family with increasing dimension and a uniform-in-$\rho > 0$ lower bound. The general phenomenon of high certificate degrees and perturbative hierarchy jumps has the cited predecessors.

The quadratic ideal used for the upper bound is also classical: it is the $r = 2$ type-A ideal in Chmutova–Etingof [5]. Its Hilbert series supplies the full-span conclusion needed for the small-ratio plateau. It does not supply the negative Fischer energy needed for the lower bound; that sign is established in Sections 3–4. The real invariant-quartic geometry on the same hyperplane is studied in Debus–Goel–Kuhlmann–Riener [6]; the present obstruction concerns holomorphic squared norms rather than real sums of squares.

The family was suggested by permanent-deficit calculations related to the occupation and cofactor framework of [7]. Section 8 derives the relevant coefficient here and states the boundary of that connection.

2 Coefficient matrices and Fischer forms

Choose real orthonormal coordinates $z = (z_1, \dots, z_{n-1})$ on H_n. The Fischer Hermitian product on holomorphic polynomials is defined by

$$\langle z^a, z^b \rangle_F = a!\, \mathbf{1}_{a=b}.$$

For homogeneous degree ℓ, use the orthonormal basis $e_a = z^a/\sqrt{a!}$. A real-valued Hermitian polynomial of bidegree (ℓ, ℓ) has a unique Hermitian coefficient matrix C in this basis. It is a holomorphic squared norm exactly when $C \succeq 0$. We identify C with an operator on the Fischer space $\mathcal{P}_\ell$. The matrix for N^ℓ is $\ell! I$.

Multiplication by a holomorphic polynomial h has Fischer adjoint $\overline{h}(\partial)$, where the bar conjugates its coefficients. In particular, with $p = p_2 = \sum_{a=1}^{n-1} z_a^2$, the adjoint of multiplication by p is $\Delta = \sum_a \partial_{z_a}^2$. Therefore the coefficient operator of $N^{\ell-2}|p|^2$ is $(\ell-2)! p\Delta$, and the operator $C_{Q,\ell}$ of $N^{\ell-2}Q$ satisfies

$$\frac{\langle f, C_{Q,\ell} f \rangle_F}{\ell! \langle f, f \rangle_F} = 1 - \frac{\|\Delta f\|_F^2}{\ell(\ell-1)\|f\|_F^2}. \tag{9}$$

Here and below either consistent convention for the conjugate-linear argument of the Hermitian product gives the same quadratic forms.

Let $P = I - \mathbf{1}\mathbf{1}^T/n$ be the real orthogonal projection onto $H_n \cap \mathbb{R}^n$, and write ∂_i for differentiation in the tangent direction Pe_i. A polynomial on H_n may equivalently be extended to $\mathbb{C}^n$ by composing with P. This extension is translation invariant in the $\mathbf{1}$ direction; on it the ambient derivatives agree with these tangent derivatives and the ambient Laplacian is Δ.

Lemma 2.1 (Ideal kernel)**.** *Let I be the ideal of $\mathbb{C}[H_n]$ generated by $v_i^2 - v_j^2$. If $f \in \mathcal{P}_\ell$ is annihilated by every $\partial_i^2 - \partial_j^2$, then f is in the kernel of the coefficient operator of $N^{\ell-2}A$.*

Proof. Expanding $N^{\ell-2}$ gives a positive weighted sum of all monomial squares of that degree. By (1), the resulting holomorphic factors span I_ℓ. The Fischer adjoint identity makes f orthogonal to every one of these factors, and hence to the range of their coefficient matrix. □

3 An explicit annihilator

Fix an odd $n \geq 3$, and put $d = n - 1$. For a sign vector $\epsilon \in \{1, -1\}^n$, set

$$c(\epsilon) = \frac{\prod_i \epsilon_i}{\sum_i \epsilon_i}, \qquad G(x) = \sum_{\epsilon \in \{1,-1\}^n} c(\epsilon)(\epsilon \cdot x)^d. \tag{10}$$

The denominators are nonzero because n is odd. In the next proof, ∂_i temporarily denotes the ambient coordinate derivative $\partial/\partial x_i$. After translation invariance has been established, these derivatives agree with the tangent derivatives used elsewhere in the paper.

Lemma 3.1. *The polynomial G is nonzero, translation invariant in the $\mathbf{1}$ direction, and satisfies $(\partial_i^2 - \partial_j^2)G = 0$ for all i, j.*

Proof. Equality of the second derivatives follows from $\epsilon_i^2 = 1$. Furthermore

$$\sum_i \partial_i G = d \sum_\epsilon \Big(\prod_i \epsilon_i \Big) (\epsilon \cdot x)^{d-1} = 0.$$

Indeed a monomial of degree $d-1=n-2$ cannot contain every one of the n sign coordinates to an odd power. Summing over signs therefore kills every term. This proves translation invariance. The coefficient of $x_1\cdots x_{n-1}$ is

$$d!\sum_{\epsilon}\frac{\epsilon_n}{\sum_i \epsilon_i}=\frac{d!\,2^n}{n}\neq 0.$$

For the equality, the n coordinate sums are equal by permutation symmetry and add to 2^n. Thus $G\neq 0$, also when regarded as a polynomial on H_n. □

For even $k\geq 0$, define

$$S_k=\sum_{\epsilon,\eta} c(\epsilon)c(\eta)(\epsilon\cdot\eta)^k. \tag{11}$$

The Fischer identity $\langle(\epsilon\cdot x)^d,(\eta\cdot x)^d\rangle_F=d!(\epsilon\cdot\eta)^d$ gives

$$\|G\|_F^2=d!S_d,\qquad \|\Delta G\|_F^2=[d(d-1)n]^2(d-2)!S_{d-2}. \tag{12}$$

These ambient calculations give the intrinsic norms because G is translation invariant. In particular $S_d>0$. Equations (9) and (12) show that its Q Rayleigh quotient relative to N^d is

$$1-\frac{n^2S_{d-2}}{S_d}. \tag{13}$$

We next prove that this is negative in every odd dimension.

4 A positive sign-cube correlation

For $0\leq k\leq n$ write

$$W_k=\sum_{\mathrm{Ham}(\epsilon,\eta)=k} c(\epsilon)c(\eta). \tag{14}$$

Individual summands can have either sign.

Proposition 4.1. *For odd n, every W_k is strictly positive.*

Proof. Use the normalized Walsh coefficients

$$a_S=2^{-n}\sum_{\epsilon} c(\epsilon)\prod_{i\in S}\epsilon_i.$$

They depend only on $j=|S|$; denote the value by a_j. Since $c(-\epsilon)=c(\epsilon)$, odd a_j vanish. Taking Walsh coefficients in the identity $(\sum_i\epsilon_i)c(\epsilon)=\prod_i\epsilon_i$ gives

$$ja_{j-1}+(n-j)a_{j+1}=0\quad(j<n\text{ odd}),\qquad na_{n-1}=1. \tag{15}$$

Consequently $a_0\neq 0$ and

$$\frac{a_{j+2}}{a_j}=-\frac{j+1}{n-j-1}\qquad(j<n-1\text{ even}). \tag{16}$$

Let T have the symmetric probability density $b_n(1+t^2)^{-(n+2)/2}$ on $\mathbb{R}$, with normalizing constant $b_n>0$. Its absolute moments through degree n are finite. Integration by parts, applied to $t^{j+1}(1+t^2)^{-n/2}$, gives

$$\frac{\mathbb{E}T^{j+2}}{\mathbb{E}T^j}=\frac{j+1}{n-j-1}\qquad(j<n-1\text{ even}). \tag{17}$$

Thus $a_{2r} = a_0(-1)^r \mathbb{E}T^{2r}$. For an independent copy U of T, this implies

$$a_S^2 = a_0^2 \mathbb{E}(TU)^{|S|}. \tag{18}$$

For odd $|S|$ both sides are zero.

Fix a sign vector σ with k negative entries. Walsh orthogonality and (18) yield

$$\begin{aligned}\sum_{\epsilon} c(\epsilon)c(\epsilon\sigma) &= 2^n \sum_S a_S^2 \prod_{i\in S} \sigma_i \\ &= 2^n a_0^2 \mathbb{E}\big[(1+TU)^{n-k}(1-TU)^k\big]. \end{aligned} \tag{19}$$

Multiplying by $\binom{n}{k}$ gives W_k.

To determine the sign of this expectation, put $R = |TU|$ and let h be its density on $(0,\infty)$. The density g of $|T|$ satisfies

$$\frac{g(1/t)}{t^2} = t^n g(t).$$

Inverting both independent factors gives the corresponding product identity

$$\frac{h(1/r)}{r^2} = r^n h(r). \tag{20}$$

For example, substitute $t = 1/u$ in $h(r) = \int_0^\infty g(t)g(r/t)\,\mathrm{d}t/t$ to verify the identity directly. The sign of TU is uniform and independent of R. Set

$$F_k(r) = (1+r)^{n-k}(1-r)^k + (1-r)^{n-k}(1+r)^k.$$

The expectation in (19) equals $\frac{1}{2}\int_0^\infty F_k(r)h(r)\,\mathrm{d}r$. Splitting at one and using (20), the integral without the prefactor $1/2$ is

$$\int_1^\infty \big[F_k(r) + r^n F_k(1/r)\big]h(r)\,\mathrm{d}r. \tag{21}$$

For $r > 1$, write $a = (1+r)^{n-k}(r-1)^k$ and $b = (r-1)^{n-k}(1+r)^k$. Since n is odd,

$$F_k(r) = (-1)^k(a-b), \qquad r^n F_k(1/r) = a + b.$$

Their sum is $2a$ for even k and $2b$ for odd k, strictly positive in both cases. All integrals converge: the integrands are polynomials of degree at most n in TU, and $\mathbb{E}|TU|^n < \infty$. Also $h(r) > 0$ for $r > 0$. Thus (21), the expectation in (19), and W_k are all positive. □

Proof of the lower bound in Theorem 1.1. By (11) and (14),

$$n^2 S_{d-2} - S_d = \sum_{k=0}^{n} W_k\big[n^2 - (n-2k)^2\big](n-2k)^{d-2} > 0. \tag{22}$$

Every term is nonnegative, and every interior k gives a strictly positive term: $d-2$ is even, $n-2k \neq 0$, and $n^2 - (n-2k)^2 = 4k(n-k) > 0$. Thus (13) is negative. Lemmas 2.1 and 3.1 put G in the kernel of the A operator at degree d. The coefficient operator of $N^{n-3}(A + \rho Q)$ is therefore indefinite for every $\rho > 0$. In particular it is not a holomorphic squared norm. If a smaller nonnegative power worked, multiplication by the remaining power of N, itself a holomorphic squared norm, would give a certificate at $n-3$. This is impossible, so $\mu_n(\rho) \geq n-2$. □

5 The first degree for small positive ratios

We use a classical complete-intersection fact, spelling out its application. The ideal I in Lemma 2.1 is generated by the $n-1$ quadrics

$$v_i^2 - v_n^2 \qquad (1 \le i < n).$$

For odd n they have no nonzero common affine zero on H_n, by the same balanced-sign argument used in Section 1. They form a homogeneous system of parameters in the polynomial ring on H_n. The standard complete-intersection criterion for a polynomial ring therefore makes them a regular sequence. Equivalently, this is the $r = 2$ specialization of the type-A ideal in [5, Theorem 2.3, Proposition 3.1, Theorem 3.2 and Corollary 3.3]. The generators there specialize to $v_i^2 - p_2/n$, which span the same quadratic ideal. Its Hilbert series is

$$\mathrm{Hilb}(\mathbb{C}[H_n]/I, t) = \frac{(1-t^2)^{n-1}}{(1-t)^{n-1}} = (1+t)^{n-1}. \tag{23}$$

In particular $I_n = \mathcal{P}_n$.

Expanding $N^{n-2}A$ as in Lemma 2.1 now gives a positive weighted sum of squares whose holomorphic factors span all of $\mathcal{P}_n$. Its coefficient operator M_A is consequently positive definite. Let M_Q be the operator of $N^{n-2}Q$ in the same orthonormal Fischer basis. For example, the choice

$$\varepsilon_n = \frac{\lambda_{\min}(M_A)}{2(1 + \|M_Q\|_{\mathrm{op}})} > 0 \tag{24}$$

makes $M_A + \rho M_Q$ positive definite for $0 \le \rho \le \varepsilon_n$. Combined with the lower bound, this proves (5). Finally (1) gives $\mu_n(0) = 0$, completing the small-ratio and zero-parameter assertions. The Hilbert-series input is classical; the independent negative Q energy in the ideal kernel is what supplies the lower bound.

To obtain the finite first-degree window in (6), let h_1 be any nonzero real linear polynomial on H_n, and set $f = p_2^{(n-1)/2} h_1 \in \mathcal{P}_n$. The elementary identity for its Laplacian gives

$$p_2 \Delta f = 2(n-1)^2 f.$$

Indeed in $d = n-1$ orthonormal coordinates, $\Delta(p^j h_1) = 2j(d+2j)p^{j-1}h_1$, with $j = (n-1)/2$. By (9), the M_Q Rayleigh quotient on f relative to N^n is $1 - 2(n-1)/n = -(n-2)/n < 0$. Thus the Hermitian matrix $M_A^{-1/2} M_Q M_A^{-1/2}$ has a negative least eigenvalue λ_n. The finite positive number

$$\rho_n^* = -\lambda_n^{-1} \tag{25}$$

is exactly the last nonnegative ρ for which $M_A + \rho M_Q \succeq 0$. This follows by diagonalizing the congruent matrix $I + \rho M_A^{-1/2} M_Q M_A^{-1/2}$. Combining with the lower bound proves (6). This is an implicit spectral characterization, not an explicit all-dimensional threshold formula.

6 The sharp four-by-four pencil for five coordinates

In this section $n = 5$. Write M_A and M_Q for the coefficient operators of $N^3 A$ and $N^3 Q$ on $\mathcal{P}_5(H_5)$, of dimension 56. The preceding ideal argument proves $M_A \succ 0$.

6.1 The negative index of the Q operator

Let $p = p_2$. By Section 2,

$$\frac{1}{120} M_Q = I - \frac{1}{20} p\Delta. \tag{26}$$

The Fischer harmonic decomposition in four real orthonormal coordinates is

$$\mathcal{P}_5 = \mathcal{H}_5 \oplus p\mathcal{H}_3 \oplus p^2\mathcal{H}_1, \qquad \mathcal{H}_k = \ker(\Delta|_{\mathcal{P}_k}). \tag{27}$$

The summands are orthogonal. For completeness, multiplication by p is adjoint to Δ, and repeated splitting by their range and kernel gives this decomposition; the following identity also verifies its directness:

$$\Delta(p^j h_k) = 2j(2k + 4 + 2j - 2)p^{j-1}h_k \qquad (h_k \in \mathcal{H}_k).$$

Thus $p\Delta$ has eigenvalues $0, 20, 32$ on the three summands of (27). Their dimensions are $36, 16, 4$, respectively, from $\dim \mathcal{H}_k = \binom{k+3}{3} - \binom{k+1}{3}$. Consequently M_Q has 36 positive, 16 zero, and exactly four negative eigenvalues. Relative to the coefficient operator $120I$ of N^5, the three eigenvalues are $1, 0, -3/5$.

6.2 An invariant four-dimensional slice

Consider the four degree-five polynomials

$$\begin{aligned} f_1 &= p_4 v_5, \qquad & f_2 &= p_2^2 v_5, \\ f_3 &= p_3(v_5^2 - p_2/5), \qquad & f_4 &= p_2(v_5^3 - p_3/5). \end{aligned} \tag{28}$$

Their M_A and M_Q forms are explicit:

$$\left(\langle f_i, M_A f_j\rangle_F\right)_{ij} = \frac{1152}{125}\mathsf{A}, \qquad \left(\langle f_i, M_Q f_j\rangle_F\right)_{ij} = \frac{1152}{125}\mathsf{Q}, \tag{29}$$

where

$$\mathsf{A} = \begin{pmatrix} 1932 & 2280 & 756 & 1332 \\ 2280 & 3200 & 840 & 1980 \\ 756 & 840 & 1368 & 756 \\ 1332 & 1980 & 756 & 1377 \end{pmatrix}, \qquad \mathsf{Q} = \begin{pmatrix} -24 & -960 & 18 & -384 \\ -960 & -2400 & -180 & -960 \\ 18 & -180 & 459 & -72 \\ -384 & -960 & -72 & -384 \end{pmatrix}. \tag{30}$$

The differentiation formulas in Appendix A provide a direct rational computation of every entry. The leading principal minors of A are

$$1932, \quad 984000, \quad 1049760000, \quad 62985600000. \tag{31}$$

In particular the f_i are independent. Direct determinant expansion gives

$$\det(\mathsf{A} + \rho\mathsf{Q}) = -388800000(305\rho^3 + 786\rho^2 + 351\rho - 162). \tag{32}$$

Let R average over the permutations fixing coordinate 5, and S average over all five coordinates. These are orthogonal projections for the Fischer product and $RS = SR = S$. The space

$$\mathcal{W} = \operatorname{im} R \cap \ker S$$

has dimension $6 - 2 = 4$. Indeed, invariants of the subgroup on four independent coordinates have generators of degrees $1, 2, 3, 4$; there are six monomials of weighted degree five. Full symmetric invariants on H_5 have generators of degrees $2, 3, 4, 5$; their degree-five component has dimension two, with possibilities 5 and $3 + 2$. Each f_i is fixed by R, and replacing its distinguished index 5 by every j and summing gives zero. Thus $Sf_i = 0$, and their independence shows that they form a basis of $\mathcal{W}$.

6.3 Why the slice gives the full endpoint

Both coefficient operators commute with the real orthogonal permutation action. Therefore $T = M_A^{-1} M_Q$ commutes with that action, preserves $\mathcal{W}$, and is self-adjoint in the positive M_A inner product. Its restriction to $\mathcal{W}$ is the generalized eigenvalue problem for the pencil (30).

The cubic in (7) is strictly increasing on $[0, \infty)$, negative at zero, and positive for large ρ. By (31), $\mathsf{A} \succ 0$. Hence the pencil remains positive definite until its unique positive singularity ρ_*. That determinant root is simple, so the endpoint kernel on $\mathcal{W}$ has dimension one. In particular T has a nonzero eigenvector $h \in \mathcal{W}$ for $\lambda_- = -1/\rho_* < 0$.

Because h is fixed by the subgroup fixing 5, its permutation orbit consists of at most five vectors h_j. Their sum is $5Sh = 0$. The map from the zero-sum permutation representation in $\mathbb{C}^5$ to their span is nonzero and equivariant. That representation is irreducible: subtracting the image of a transposition from a nonconstant coordinate vector yields a nonzero coordinate difference, whose permutations span the zero-sum space. The orbit span therefore has dimension four. Since T commutes with the group, it is entirely a λ_- eigenspace.

The operator T is similar to the Hermitian matrix $M_A^{-1/2} M_Q M_A^{-1/2}$. By Sylvester inertia it has exactly four negative eigenvalues, counted with multiplicity, because M_Q does. They are therefore precisely the four copies of λ_- just exhibited. Every other eigenvalue of T is nonnegative. By congruence, for $\rho \geq 0$,

$$M_A + \rho M_Q \succeq 0 \quad \Longleftrightarrow \quad 1 + \rho\lambda_- \geq 0 \quad \Longleftrightarrow \quad \rho \leq \rho_*.$$

All the remaining eigenvalues of the congruent matrix are strictly positive at ρ_*. This proves the endpoint kernel dimension and the full positivity assertions in Theorem 1.2. Theorem 1.1 excludes all exponents below three for $\rho > 0$; above ρ_* exponent three fails. The rational bracket follows by substituting $274637/10^6$ and $274638/10^6$ into the cubic, with opposite signs.

7 The classical three-coordinate example

The endpoint $n = 3$ of Theorem 1.1 is consistent with the familiar example of [2]. To specify the comparison, take real orthonormal coordinates z_1, z_2 on H_3, then put $w_\pm = (z_1 \pm i z_2)/\sqrt{2}$. One has

$$N = |w_+|^2 + |w_-|^2, \qquad p_2 = 2w_+ w_-, \qquad A = N^2 - \tfrac{1}{2}|p_2|^2.$$

Thus $q_\rho = (1+\rho)N^2 - (2+4\rho)|w_+ w_-|^2$. At $\ell = m + 2$ its stabilized coefficient matrix is diagonal, with entries in the ordinary monomial basis $w_+^j w_-^{\ell - j}$ given by

$$(1+\rho)\binom{\ell}{j} - (2+4\rho)\binom{\ell-2}{j-1}.$$

Their signs are controlled by the maximum of $j(\ell - j)$. For even ℓ all entries are nonnegative exactly when $\ell \geq 2 + 2\rho$; for odd ℓ exactly when $\ell \geq 1 + 2\rho$. Consequently

$$\mu_3(\rho) = 2\lceil \rho \rceil - 1 \qquad (\rho > 0). \tag{33}$$

This records the semidefinite endpoint of the same classical diagonal example; the all-odd dimensional obstruction and the five-coordinate pencil are the additional conclusions of the present paper.

8 A coefficient of the centered permanent deficit

This section explains the origin of the family. It is not a reduction of a full permanent inequality to the preceding certificate-degree results. Let K be Hermitian positive semidefinite of order n with $K\mathbf{1} = 0$, let $J = \mathbf{1}\mathbf{1}^*$, and put $s = \sqrt{2}$. Define

$$F_K(u) = \operatorname{per}(K + uJ), \quad P_n = F_K(1) + F_K'(1), \quad D_n(K) = P_n^2 - \operatorname{per}\big(|K + sJ|^2\big). \tag{34}$$

The absolute values and square in the last term are entrywise. This deficit is related to the occupation/cofactor framework in [7]. Write $[D_n]_3$ for its homogeneous degree-three part in K, and use

$$S_3 = \sum_{a,b} K_{ab}^3, \qquad B_3 = \sum_{a,b} K_{ab}|K_{ab}|^2.$$

These are real for Hermitian K; S_3 is not the spectral trace $\operatorname{tr} K^3$.

Proposition 8.1. *For $n \geq 3$,*

$$\begin{aligned} [D_n]_3 &= c_n S_3 - e_n B_3, \\ c_n &= \frac{4}{3}(n-3)!\big[(n+1)(n-2)n! - s^{2n-3}\big], \\ e_n &= 2n(n-3)!s^{2n-3}. \end{aligned} \tag{35}$$

For $v \in H_n$ and $W = nI - J$, the coefficient of $t^2 a$ in $D_n(tvv^* + aW)$ is

$$\alpha_n A(v) + \beta_n Q(v), \qquad \alpha_n = 3(c_n - e_n) > 0, \quad \beta_n = 2e_n > 0. \tag{36}$$

Proof. The degree-k residual term of $F_K(u)$ is $(n-k)!u^{n-k}/k!$ times the ordered sum of k entries over distinct rows and columns. Enforce distinctness by partition-lattice inclusion–exclusion. By centering, a singleton row or column block vanishes. For $k = 2, 3$ the only surviving partitions without singletons are the single blocks, with Möbius factors -1 and 2. The linear term is zero, and the next terms are

$$[F_K(u)]_2 = \tfrac{1}{2}(n-2)!u^{n-2}\sum_{a,b} K_{ab}^2, \quad [F_K(u)]_3 = \tfrac{2}{3}(n-3)!u^{n-3}S_3.$$

Since the constant P_n is $(n+1)n!$, its squared cubic contribution is $\frac{4}{3}(n+1)(n-2)n!(n-3)!S_3$.

For the paired permanent, write $|K + sJ|^2 = s^2 J + X + Y$ with $X = s(K + \overline{K})$ and $Y = |K|^2$. The matrix X is centered. The cubic contribution of three X entries is

$$\tfrac{4}{3}(n-3)!s^{2n-3}(S_3 + 3B_3).$$

For one X and one Y, the ordered distinct-row/column sum reduces by centering to $\sum_{a,b} X_{ab}Y_{ab} = 2sB_3$. Its contribution is $2(n-2)!s^{2n-3}B_3$. Combining gives the cubic paired term $\frac{4}{3}(n-3)!s^{2n-3}S_3 + 2n(n-3)!s^{2n-3}B_3$, and subtraction proves (35).

In $K = tvv^* + aW$, direct expansion gives $[t^2a]S_3 = 3A$ and $[t^2a]B_3 = 3A - 2Q$. This proves (36). Positivity of β_n is immediate. For α_n, write

$$c_n - e_n = \tfrac{2}{3}(n-3)![2(n+1)(n-2)n! - (3n+2)s^{2n-3}].$$

Use $n! \geq 3 \cdot 2^{n-2}$, $s^{2n-3} = 2^{n-2}s$, and $s < 2$. After division by 2^{n-2} the bracket is at least $6(n+1)(n-2) - (3n+2)s > 6(n-1)^2 - 22 > 0$ for $n \geq 3$. □

For $n = 5$, the coefficients are $\alpha_5 = 17280 - 544\sqrt{2}$ and $\beta_5 = 320\sqrt{2}$. One has $0 < \beta_5/\alpha_5 < 1/12 < \rho_*$, so the least norm exponent for this quartic is three. The first inequality follows from $4384\sqrt{2} < 17280$; the second follows by evaluating the increasing cubic at $1/12$, where it is negative. This concerns a single permanent-energy coefficient.

A Reproducing the small pencil

All entries of (30) can be checked by short rational differentiation, independently of the spectral argument. For the polynomials in (28), one has

$$\begin{aligned}\langle f, M_A g\rangle_F &= 6\sum_{i<j}\langle(\partial_i^2-\partial_j^2)f,(\partial_i^2-\partial_j^2)g\rangle_F,\\ \langle f, M_Q g\rangle_F &= 120\langle f,g\rangle_F - 6\langle\Delta f,\Delta g\rangle_F.\end{aligned}\tag{37}$$

These follow directly from multiplication–differentiation adjointness. Replace v_i by $x_i-(\sum_j x_j)/5$, expand in five ambient variables, and use $\langle x^a,x^b\rangle_F = a!\mathbf{1}_{a=b}$. This is an entirely rational recipe. Equivalently, the pairing of products of linear forms is

$$\left\langle\prod_{r=1}^{\ell}(a_r\cdot x),\prod_{r=1}^{\ell}(b_r\cdot x)\right\rangle_F = \sum_{\sigma\in S_\ell}\prod_{r=1}^{\ell}(\overline{a}_r\cdot b_{\sigma(r)}).$$

For the physical coordinates the relevant linear-form products are $P_{ij}=\delta_{ij}-1/5$.

As a third normalization check, the matrix $(120\langle f_i,f_j\rangle_F)_{ij}$ is $(1152/125)\mathsf{N}$, where

$$\mathsf{N}=\begin{pmatrix}1040 & 1600 & 270 & 740\\ 1600 & 4000 & 300 & 1600\\ 270 & 300 & 585 & 270\\ 740 & 1600 & 270 & 890\end{pmatrix}.$$

Its leading principal minors are $1040, 1600000, 810000000, 162000000000$. For instance $f_2=p^2v_5$ is in $p^2\mathcal{H}_1$, so its Q form is $-3/5$ of its N^5 form: the $(2,2)$ entries indeed give $-2400/4000=-3/5$. The four-by-four determinant (32), its principal minors, and the two rational bracket signs then involve only integer arithmetic.

The accompanying `verify_certificates.py` reconstructs the physical Fischer tables and independently checks them against a direct coefficient expansion in four independent coordinates. It also verifies the full coefficient-matrix inertia, permutation covariance, a cyclic relation and finite sign-cube identities as redundant checks. These computations support the displayed finite identities; the all-odd theorem is established by the analytic proof, not by finite sampling.

Assistance and reproducibility

AI-assisted tools were used in mathematical exploration and manuscript preparation. The proofs and their classical dependencies are stated in the paper. The accompanying exact-arithmetic verifier uses only the Python standard library and does not require numerical eigenvalue tolerances.

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