Degree laws and edge correlations for random forests

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Abstract

For the unrooted spanning-forest measure on any finite simple graph with arbitrary positive edge weights, we prove real stability of the full incident-edge marginal at every vertex. Thus each star is strongly Rayleigh, incident edges are nonpositively correlated, and every vertex degree has a Poisson-binomial law. The proof identifies a classical optional-component-mark forest polynomial with a stable determinant differential operator applied to a signed matching polynomial. We also prove strict pairwise negative edge correlation on every complete graph K_N, $N \geq 3$, and complete bipartite graph $K_{m,n}$, $m, n \geq 2$, at every common positive fugacity. Their disjoint-edge proofs use exact Hermite and Laguerre counts, continued fractions and scalar crossing barriers, with complete exact polynomial certificates. Quantitative bipartite critical and supercritical deficit asymptotics are retained. Empty forests and isolated vertices are included.

1 Model, results and context

For a finite simple graph G with positive edge weights w_e, give an acyclic edge subset F weight $w_F = \prod_{e \in F} w_e$ and normalize by $Z_G = \sum_F w_F$. Isolated vertices and the empty edge set are included. These forests are unrooted: there is no product of component sizes in their weights. For an edge e write $X_e = \mathbf{1}_{\{e \in F\}}$.

Theorem 1.1 (Arbitrary weighted stars)**.** *For every vertex v of every finite simple graph with arbitrary positive edge weights, the polynomial*

$$S_v(z) = \sum_{F \text{ forest}} w_F \prod_{vi \in F} z_i$$

is real stable. The normalized full star marginal is strongly Rayleigh, hence negatively associated. In particular every pair of incident edge indicators has nonpositive covariance. Equality is allowed. The proof is in Section 2.

For the retained complete and complete-bipartite results, all edge weights equal a common $\lambda > 0$. In particular

$$Z_{m,n}(\lambda) = \sum_{F \in \mathcal{F}_{m,n}} \lambda^{|F|}, \qquad \mathbb{P}_\lambda(F) = \lambda^{|F|} / Z_{m,n}(\lambda).$$

Theorem 1.2 (Both graph families at every common fugacity)**.** *Let $G = K_N$ with $N \geq 3$, or $G = K_{m,n}$ with $m, n \geq 2$. For every $\lambda > 0$ and distinct edges e, f of G,*

$$\mathbb{P}_\lambda(e, f \in F) < \mathbb{P}_\lambda(e \in F)\mathbb{P}_\lambda(f \in F).$$

For $K_{1,n}$ the edge indicators are independent. The graphs K_1, K_2 have no distinct pair of edges.

Corollary 1.3 (Degree laws)**.** *For every vertex of any finite simple graph with positive edge weights, the degree polynomial has only strictly negative real zeros, with the constant polynomial allowed. Its degree law is Poisson-binomial. In the common-fugacity families $K_{m,n}$, $m, n \geq 2$, and K_N, $N \geq 3$, the incident covariance is strictly negative, as also follows from Theorem 1.2.*

Proposition 1.4. *Let $m, n \geq 2$. Every pair of distinct edges of $K_{m,n}$ satisfies*

$$\mathbb{P}_\lambda(e, f \in F) < \mathbb{P}_\lambda(e \in F)\mathbb{P}_\lambda(f \in F)$$

whenever $\lambda \geq 1$ or $0 < \lambda \leq 1/\sqrt{mn}$. In particular, this holds for the uniform forest measure.

Proposition 1.5. *For every m, n with $2 \leq \min(m, n) \leq 6$, every pair of distinct edges of $K_{m,n}$ is strictly negatively correlated for every $\lambda > 0$.*

Theorem 1.6 (Supercritical scaling)**.** *Fix $\Gamma \geq 2$, $\eta > 0$ and $T > 1+\eta$. Put $r = \sqrt{mn}$, $\gamma = (m+n)/r$ and $\lambda = t/r$. Uniformly over integer $m, n \geq 2$ with $\gamma \leq \Gamma$ and $1 + \eta \leq t \leq T$, as $r \to \infty$, disjoint edges satisfy*

$$\mathbb{P}_\lambda(e \in F)\mathbb{P}_\lambda(f \in F) - \mathbb{P}_\lambda(e, f \in F) = \frac{K_\gamma(t)}{r^3} + O_{\eta,T,\Gamma}(r^{-4}), \tag{1}$$

where

$$K_\gamma(t) = \frac{(t^2-1)(3\gamma t^2 + \gamma + 2t^3 + 6t)}{t^2(t^2 + \gamma t + 1)^2} > 0.$$

Consequently every pair is strictly negatively correlated for all sufficiently large hosts throughout this parameter range.

Theorem 1.7 (Outside the critical window)**.** *Fix $\Gamma \geq 2$ and $T > 1$. There are constants $C_0, C_1, r_0 > 0$, depending only on Γ, T, such that for the same r, γ, t, λ, if $r \geq r_0$, $\gamma \leq \Gamma$, $1 < t \leq T$ and $r(t-1)^3 \geq C_0$, the probability deficit in* (1) *equals*

$$\frac{K_\gamma(t)}{r^3}(1+\theta), \qquad |\theta| \leq \frac{C_1}{r(t-1)^3} < \frac{1}{2}.$$

In particular it is strictly positive. For any sequence with bounded γ, t and $r(t-1)^3 \to \infty$, the ratio to $K_\gamma(t)/r^3$ tends to one.

If one part has size one, the host is a tree and its edges are independent Bernoulli variables with parameter $\lambda/(1+\lambda)$. The star theorem permits arbitrary individual weights.

The forest conjecture and its random-cluster context were studied by Grimmett and Winkler [14], Semple and Welsh [15], and Wagner and subsequent authors. Series-parallel and two-sum methods already cover substantial classes [15, 16]. Earlier exact finite verifications also cover some small complete bipartite graphs: Erickson's thesis proves the stronger arbitrary-weight property for graphs on at most seven vertices, including $K_{3,4}$ [17, Section 5.3]; the unit-weight census in [14] includes $K_{3,n}$ for $n \leq 6$. These finite cases are not counted as separate new results here.

Stark developed bipartite forest enumeration and adjacent-edge asymptotics [19]. In the available 2011 author manuscript, the asymptotic correlation theorem requires both part sizes to grow and leaves nonadjacent pairs untreated. Our statements have no large-size restriction in the specified fugacity regimes. Huang [20, Theorem2.1] proves incident negative correlation on arbitrary finite connected hosts for sufficiently large common fugacity, using the leading spanning-tree coefficient and electrical currents. Theorem 1.1 removes that activity and common-weight restriction and treats the whole star. Its proof uses classical multivariate matching stability [3], Lagrange–Good inversion [2], Hermitian determinant stability and Lieb–Sokal/Borcea–Branden contraction [4, 5].

The component-marked identity is already contained in [1, Theorem5.4]; the stability identification and its original unrooted star interpretation are the point here. Strongly-Rayleigh consequences are classical [6]. Ciceksiz and Ravichandran [21, Theorem 1.5 and Corollary 1.7] identify the first two large-fugacity Rayleigh coefficients by electrical formulas and obtain graph-dependent large-fugacity conclusions. Their complete-graph random-cluster theorem uses common edge weights at least one. Its fixed-weight $q \to 0$ limit is a connected-subgraph measure; obtaining forests instead requires weights scaled by q. These results do not supply the all-host, all-common-fugacity bipartite conclusion of Theorem 1.2. For methodological context, Bauerschmidt et al. [7, Section 3] use stationary-phase expansions for the complete-graph mean-field partition function and vertex connectivity. Those observables differ from the smaller disjoint-edge deficit determined here. Critical forest and coalescing-saddle asymptotics, including the Airy function and its critical power, are substantial classical ancestry [9, 10]. Gupta [11] proves pairwise negative correlation for the uniform forest of every complete graph. Its unit-fugacity theorem is prior work; the complete-graph contribution here is every common positive activity. Stark [18] obtained eventual complete-graph uniform correlation; the fixed-common-weight adaptation recorded in [21] has a threshold depending on that weight. Huang [20] gives additional size and small/large-activity regimes. These quantifiers do not imply every finite host at every activity. We use classical generating-function and orthogonal-polynomial tools; our conclusions concern the assembled degree laws and correlation estimates stated above, with the stated hypotheses.

2 Optional component marks and arbitrary weighted stars

A real polynomial is *real stable* if it does not vanish when all its variables have strictly positive imaginary parts. Limits, real specialization and positive affine substitution preserve stability, with the identically zero polynomial allowed as a limit. These elementary facts follow from the multivariate Hurwitz theorem; see [4].

Theorem 2.1 (Stable optional component marks)**.** *Let H be a finite simple graph with nonnegative edge weights $C_{ij} = C_{ji}$, with $C_{ii} = 0$. Its polynomial*

$$\Phi_C(b) = \sum_{A \text{ forest of } H} \prod_{e \in A} C_e \prod_{B \in \mathcal{C}(A)} \left(1 + \sum_{i \in B} b_i\right) \tag{2}$$

is real stable. Here $\mathcal{C}(A)$ includes singleton components. A component has either no marked vertex or exactly one; its unmarked contribution is one.

The component-marked forest identity itself is classical: it is the graph specialization of Caracciolo–Sokal–Sportiello [1, Theorem5.4] with $t_i = 1 + b_i$ and edge parameter 1, since a tree has one more vertex than edge. We reconstruct it in a form suitable for stability. The matching and differential-preserver tools below are also classical.

2.1 An exact individual-vertex coefficient identity

Write $V(H) = \{1, \dots, d\}$. For $d = 0$ set $\Phi_C = 1$. Otherwise introduce formal rooted tree series

$$T_i = x_i \exp\left(\sum_j C_{ij} T_j\right), \qquad \mathcal{U} = \sum_i T_i - \frac{1}{2} \sum_{i,j} C_{ij} T_i T_j.$$

They are exponential series for labeled, colored trees; colors may repeat in the auxiliary series. The squarefree coefficient $[x_1 \cdots x_d]$ uses each color exactly once, so it describes the fixed weighted host without repeated vertices. Each unrooted tree appears once per vertex in the first sum and once per edge in the second. Thus $\mathcal{U}$ counts it once. The set decomposition into components gives

$$\Phi_C(b) = [x_1 \cdots x_d] \exp\left(\mathcal{U} + \sum_i b_i T_i\right). \tag{3}$$

Indeed each tree is either unmarked or rooted at its single marked vertex. All coefficient factorials are one in this squarefree individual-color specialization.

The formal substitution $x_i = t_i \exp(-\sum_j C_{ij} t_j)$ has identity linear part. Its logarithmic Jacobian is

$$D_C(t) = \det(I - C \operatorname{diag} t),$$

because the (i,j) entry of $\partial \log x_i / \partial \log t_j$ is $\delta_{ij} - C_{ij} t_j$. Substituting in the formal coefficient residue, each x_i^{-1} contributes $\exp(\sum_j C_{ij} t_j)$. Consequently the multivariate Lagrange–Good change of variables [2] gives, directly,

$$\Phi_C(b) = [t_1 \cdots t_d] D_C(t) \exp\left(\sum_i a_i t_i - \sum_{i<j} C_{ij} t_i t_j\right), \qquad a_i = 1 + \sum_j C_{ij} + b_i. \tag{4}$$

This is a formal identity and uses no analytic convergence assumption. Define the signed unmatched-vertex matching polynomial

$$Q_C(a) = \sum_{M \text{ matching}} (-1)^{|M|} \prod_{ij \in M} C_{ij} \prod_{i \notin V(M)} a_i.$$

It equals the squarefree coefficient of the exponential in (4): selected quadratic terms must be disjoint pairs. Multiplication by t_i before extraction is differentiation in a_i. Since D_C is multiaffine, we obtain the exact identity

$$P_C(a) := D_C(\partial_a) Q_C(a), \qquad \Phi_C(b) = P_C(1 + \deg_C + b), \quad (\deg_C)_i = \sum_j C_{ij}. \tag{5}$$

Neither Q_C nor D_C separately is the unrooted forest partition.

2.2 Classical stability and contraction

For completeness, the weighted multivariate matching stability of Heilmann–Lieb [3] has the following elementary induction in this normalization. For an induced vertex set J and $v \in J$, matching deletion gives

$$\frac{Q_J}{Q_{J-v}} = a_v - \sum_{u \in J-v} \frac{C_{vu}}{Q_{J-v}/Q_{J-v-u}}.$$

Assume all a_i lie in the upper half-plane. Inductively the ratios in every denominator lie there, so their reciprocals have negative imaginary part. Since $C_{vu} \geq 0$, the displayed ratio has strictly positive imaginary part. Starting with $Q_\varnothing = 1$, this proves every denominator and every Q_J nonzero. Thus Q_C is real stable.

The determinant D_C is also real stable. If the real symmetric matrix C is invertible, then

$$D_C(t) = \det(C) \det(C^{-1} - \operatorname{diag} t).$$

For upper-half-plane t, the second matrix has strictly negative definite imaginary part. A nonzero vector in its kernel would contradict this after multiplication by its conjugate transpose. For singular C, approximate by invertible real symmetric $C + \epsilon I$ and pass to the polynomial limit. Symmetry suffices for this auxiliary analytic argument; the approximating diagonals need not describe the original graph. Since $D_C(0) = 1$, the limit is not zero. This is standard Hermitian determinant stability.

We use the precise multiaffine Lieb–Sokal lemma [4, Lemma3.2]. If $g(a) + yf(a)$ is stable and $\deg_{a_i} f \leq 1$, then $g - \partial_i f$ is stable or zero. Here is its short proof, including the sign. The leading coefficient f and specialization g are stable or zero. If $f = 0$ or $\partial_i f = 0$, the assertion follows. Otherwise

$$yf(a_i - 1/y, a_{\neq i}) = yf(a) - \partial_i f(a)$$

is stable, because $-1/y$ lies in the upper half-plane. For upper-half-plane a, nonvanishing of $g + yf$ for every such y implies $\operatorname{Im}(g/f) \geq 0$; the preceding shifted polynomial gives $\operatorname{Im}(-\partial_i f/f) \geq 0$. Adding proves stability of $g - \partial_i f + yf$, and specializing $y = 0$ proves the lemma.

As D_C has real coefficients, $D_C(-y)$ is stable: simultaneous sign reversal takes the upper to the lower half-plane, where a real stable polynomial is likewise nonzero. The product $D_C(-y)Q_C(a)$ is stable in disjoint variable sets. Successively apply the lemma, replacing y_i by $-\partial_{a_i}$. Every intermediate polynomial remains multiaffine in each remaining a_i, y_i, so all degree hypotheses hold. The result is exactly $D_C(\partial_a)Q_C(a) = P_C(a)$, stable or zero. Its coefficient of $a_1 \cdots a_d$ is one, which excludes zero. This also follows from the Borcea–Branden Weyl-algebra symbol criterion [5, Corollary4.1]: the symbol here is $D_C(-w)$, with its sign retained. Finally the real translation in (5) proves Theorem 2.1.

2.3 The original unrooted star law

Fix a vertex v of a weighted graph G and delete it. For a fixed forest A of $G - v$, each tree component can remain unattached to v, or receive exactly one edge from v. Two such edges to the same tree would create a cycle; every single choice is legal. Therefore the incident-edge-marked partition is exactly

$$S_v(z) = \sum_{F \text{ forest of } G} w_F \prod_{vi \in F} z_i = \Phi_C(b), \qquad b_i = w_{vi} z_i, \tag{6}$$

where $w_{vi} = 0$ for nonneighbors and C is the weighted adjacency matrix of $G - v$. Positive scaling and real specialization preserve stability or zero; $S_v(\mathbf{1}) = Z_G > 0$ excludes zero. This proves Theorem 1.1. Conversely, attach a new cone vertex to every vertex of H with edge weight one: its marked star polynomial is precisely $\Phi_C(b)$. Thus the two formulations are equivalent, not separate independent generalizations.

A multiaffine stable probability generating polynomial defines a strongly Rayleigh law. Such laws are negatively associated, even under conditioning and external fields [6, Theorem4.9]. This classical implication is applied only to the star marginal. In particular

$$(\partial_i S_v)(\partial_j S_v) - S_v \partial_{ij} S_v \geq 0 \quad \text{at } z = \mathbf{1},$$

which is exactly the nonpositive incident covariance assertion. The conclusion concerns the incident-edge star marginal.

For a vertex with d_v neighbors, set all its star variables equal to z. Stability and real coefficients give a real-rooted degree polynomial. Every star-edge subset is itself a forest, so each coefficient from

degree zero through d_v is positive. All zeros are consequently strictly negative, and normalization gives

$$\frac{S_v(z,\dots,z)}{S_v(1,\dots,1)} = \prod_{j=1}^{d_v}(1-p_j+p_jz), \qquad 0<p_j<1.$$

An isolated vertex has the constant polynomial one. This proves the general degree assertion in Corollary 1.3. The actual incident indicators need not be independent.

Finally, (2) is not a rooted-forest determinant: the latter has component factors $\sum_{i\in B} b_i$, without the additional one. At root marks one it weights each forest by the product of its component sizes [6, Section3.4]. Optional marks with positive activities also tilt the induced forest marginal. Equation (6), rather than a rooted normalizer shortcut, identifies the original unrooted law.

Corollary 2.2 (Conditioning and a required component boundary)**.** *Condition on disjoint edge sets A, B being present and absent, respectively, with positive probability. The remaining incident-edge marginal at any fixed vertex is real stable. More generally the same holds for the remaining boundary edges of one actual connected component of the required forest A.*

Proof. The set A must be a forest. Delete B and contract A. Forests satisfying the conditioning correspond, up to the constant weight $\prod_{e\in A} w_e$, to forests of the resulting multigraph. Loops are forbidden and removed. A forest chooses at most one edge from each parallel class c. Replace that class by a single edge of weight $W_c = \sum_{e\in c} w_e$. For tracked marks, substitute

$$y_c = \frac{\sum_{e\in c \text{ tracked}} w_e z_e + \sum_{e\in c \text{ untracked}} w_e}{W_c}$$

in its simple-graph star polynomial. Thus $W_c y_c$ is exactly the original marked parallel sum, with no extra weight factor. A tracked class has an upper-half-plane affine image; untracked classes specialize to $y_c = 1$. The positive value at all marks one excludes zero. Required-present incident edges contribute monomial factors, while absent edges and loops are omitted. The boundary of one required component is exactly a star after contraction. This argument does not concern an arbitrary unconditioned cut. □

3 An exact coefficient formula

Use exponential generating functions in labeled left and right vertices. The rooted bipartite tree series satisfy

$$U = xe^{\lambda V}, \qquad V = ye^{\lambda U}. \tag{7}$$

The unrooted tree series is $T = U + V - \lambda UV$: a tree is counted once per vertex in the first two terms and once per edge in the last. The forest series is $E = e^T$.

The formal change of variables in (7) gives

$$[x^p y^n]H(U,V) = [u^p v^n]H(u,v)e^{\lambda pv+\lambda nu}(1-\lambda^2 uv). \tag{8}$$

For completeness, substitute $x = ue^{-\lambda v}$ and $y = ve^{-\lambda u}$ in the formal coefficient residue. The logarithmic Jacobian determinant is $1-\lambda^2 uv$, yielding (8).

For nonnegative integers p, n define

$$q_{p,n}(a,b,\lambda) = \sum_{k=0}^{\min(p,n)} (-\lambda)^k \binom{p}{k} (n)_k a^{p-k} b^{n-k}, \tag{9}$$

where $(n)_k = n(n-1)\cdots(n-k+1)$. It equals $p!n![u^p v^n]e^{au+bv-\lambda uv}$. Subscripts on q below denote partial derivatives in its auxiliary continuous variables, with its integer indices held fixed. Termwise differentiation gives

$$q_\lambda = -q_{ab}, \qquad \lambda q_{ab} = bq_b - nq, \qquad aq_a - bq_b = (p-n)q. \tag{10}$$

Set $h = \min(p,n)$ and $\alpha = |n-p|$. The standard coefficient formula for associated Laguerre polynomials gives

$$q_{p,n} = (-\lambda)^h h! a^{p-h} b^{n-h} L_h^{(\alpha)}(ab/\lambda). \tag{11}$$

We use their usual facts: for $\alpha \geq 0$ all zeros are distinct and positive, and their three-term recurrence and derivative identity follow from the same coefficient formula; see also [13, Chapter 18].

4 Bipartite degree counts and strict incident correlation

Fix a left vertex and put $p = m-1$. Its component consists of this vertex and an unordered collection of right-rooted branches, each attached by one edge. Other components contribute E. Therefore its unnormalized degree polynomial is exactly

$$G(z) = p!n![x^p y^n]E(x,y)e^{\lambda z V(x,y)}. \tag{12}$$

In particular every forest is counted once, not once for each choice of roots in its other components. Applying (8), put $a = 1+n\lambda$ and $b = 1+(p+z)\lambda$. Then

$$G(z) = q_{p,n} - \lambda^2 (q_{p,n})_{ab} = aq_{p,n} - \lambda b (q_{p,n})_b. \tag{13}$$

The last identity uses (10) and $a = 1+n\lambda$.

Lemma 4.1. *Suppose $Q(b) = b^k Q_1(b)$ has degree n, where Q_1 has $h = n-k$ distinct positive zeros. For $a = 1+n\lambda$ and $\lambda > 0$, the polynomial $aQ - \lambda bQ'$ has k zeros at zero and h distinct positive zeros.*

Proof. The nonzero factor is $(1+h\lambda)Q_1 - \lambda bQ_1'$. Put $P(t) = t^h Q_1(1/t)$; this factor is $b^h(P(t) + \lambda t P'(t))$, $t = 1/b$. If $0 < t_1 < \cdots < t_h$ are the zeros of P, then

$$\frac{P(t)+\lambda t P'(t)}{P(t)} = 1 + \lambda \sum_{i=1}^{h} \frac{t}{t-t_i}, \qquad \frac{d}{dt}\left(1 + \lambda \sum_i \frac{t}{t-t_i}\right) = -\lambda \sum_i \frac{t_i}{(t-t_i)^2} < 0.$$

There is one zero in $(0, t_1)$ and one in each (t_i, t_{i+1}). These are all h zeros. The case $h = 0$ is immediate. □

By (11), the hypotheses of Lemma 4.1 hold for $Q = q_{p,n}$. Hence all zeros of (13) are real after the affine change from b to z. Its forest interpretation gives nonnegative coefficients, a positive constant term and a positive leading coefficient. It has no nonnegative zero, so all its zeros are strictly negative.

Normalize and factor

$$\frac{G(z)}{G(1)} = \prod_{i=1}^{n} (1 - p_i + p_i z), \qquad 0 < p_i < 1.$$

The actual incident edge indicators are exchangeable, though this does not make them independent. Writing $\bar{p} = n^{-1}\sum_i p_i$, their common pair covariance is

$$\mathrm{Cov}(X_e, X_f) = -\frac{\sum_i (p_i - \bar{p})^2}{n(n-1)}. \tag{14}$$

For $m, n \geq 2$, either $h \geq 2$, giving distinct nonzero b roots, or $h = 1$ and a zero b root also occurs. Thus the parameters are not all equal and (14) is strict. Swapping the parts gives strict incident correlation on both sides. Degree real-rootedness is already a corollary of the general star theorem; the formulas here are retained for exact counts and the strictness argument. The complete-graph count is given in Section 11.

5 The disjoint-edge quadratic

Here $q = q_{m,n}$ and

$$W(a, b, \lambda) = q - \lambda^2 q_{ab} = (1 + n\lambda)q - \lambda b q_b.$$

At $a = 1 + n\lambda, b = 1 + m\lambda$, $W = Z_{m,n}(\lambda)$ by (8). Write

$$D = n\partial_a + m\partial_b + \partial_\lambda.$$

Let $N_e = \sum_{F \ni e} \lambda^{|F|}$ and $N_{ef} = \sum_{F \supset \{e,f\}} \lambda^{|F|}$ for a disjoint pair. Edge transitivity gives

$$N_e = \frac{\lambda D W}{mn}. \tag{15}$$

The degree-marked count of a fixed left vertex is

$$\frac{1}{m} W_a(a, b - \lambda + \lambda z, \lambda)$$

at physical a, b: the coefficient in (12) has one fewer left label, and the partial derivative converts its factorial. Consequently the fixed left-adjacent and right-adjacent pair counts are

$$B_L = \frac{\lambda^2 W_{abb}}{mn(n-1)}, \qquad B_R = \frac{\lambda^2 W_{aab}}{nm(m-1)}.$$

The ordered edge-pair count is $\lambda^2 D^2 W$. Splitting ordered pairs into these two adjacent types and the $m(m-1)n(n-1)$ disjoint pairs yields

$$N_{ef} = \frac{\lambda^2 (D^2 W - W_{abb} - W_{aab})}{m(m-1)n(n-1)}. \tag{16}$$

All derivatives precede physical specialization. In addition to (10), differentiation gives

$$q_{bb} = -\frac{an}{\lambda b} q + \frac{ab - \lambda(m - n + 1)}{\lambda b} q_b. \tag{17}$$

These relations reduce every derivative in (15)–(16) to q, q_b.

Put $s = m + n$, $p = mn$ and $R = bq_b/q - n$. Exact substitution gives

$$\frac{N_e^2 - Z N_{ef}}{q^2} = F(R) = \frac{A_2 R^2 - A_1 R + A_0}{m^2 n^2 (m-1)(n-1)(1 + m\lambda)(1 + n\lambda)}, \tag{18}$$

where the explicit general coefficients are

$$\begin{aligned}
A_0 &= \lambda p^2\{\lambda^3 p + \lambda^2(2s-p) + 3\lambda + 1\},\\
A_1 &= p\{2\lambda^5 p^2 + \lambda^4(3sp - p^2 + 2p) + \lambda^3(2s+6p) + \lambda^2(3s+2) + 4\lambda + 1\},\\
A_2 &= \lambda\{\lambda^5 p^3 + \lambda^4(s+2)p^2 + \lambda^3[p(s+1) + 4p^2]\\
&\qquad + \lambda^2(4sp + s - s^2 - p^2 + 2p) + \lambda(5p + 1 - s) + p\}.
\end{aligned} \tag{19}$$

Equations (10),(17) and the displayed finite polynomial define an exact way to check (18). The supplied verifier reconstructs the differential reduction independently of these closed coefficients and checks their identity.

6 Exterior Laguerre ratios and continued fractions

Let $h = \min(m,n)$, $\alpha = |m-n|$ and

$$x = \frac{(1+n\lambda)(1+m\lambda)}{\lambda} = m + n + mn\lambda + \lambda^{-1}.$$

For the positive Laguerre zeros c_i in (11),

$$R = R_{m,n}(x) = \sum_{i=1}^{h} \frac{c_i}{x - c_i}. \tag{20}$$

The Jacobi matrix for $L_h^{(\alpha)}$ has diagonal $2j+\alpha+1$ and off-diagonal $\sqrt{j(j+\alpha)}$. Its maximum absolute row sum is strictly below $m+n-1+2\sqrt{mn}$. In contrast, $x \geq m+n+2\sqrt{mn}$ by AM–GM. Thus x is beyond every root, $q > 0$, and the same holds for all lower-index matrices below.

The Laguerre recurrence and derivative identity give

$$R_{m,n}(x) = \frac{mn}{x - (m+n-1) - R_{m-1,n-1}(x)}, \qquad R_{0,n} = R_{m,0} = 0. \tag{21}$$

The argument x is fixed when the indices decrease. Each denominator is positive, being a ratio of consecutive exterior Laguerre polynomials, or a positive Jacobi Schur complement.

For $m, n \geq 3$ put $d_j = x - (m+n-2j-1)$ and

$$L_3 = \cfrac{mn}{d_0 - \cfrac{(m-1)(n-1)}{d_1 - \cfrac{(m-2)(n-2)}{d_2}}}. \tag{22}$$

Replacing the remaining nonnegative tail by zero and using the increasing map $t \mapsto c/(d-t)$ gives $R \geq L_3$. All truncated denominators stay positive in this exterior region.

We also need an upper bound. For $c, d > 0$ below the original indices, let $\nu_- < \nu_+$ solve $x = c + d + cd\nu + 1/\nu$. The positive forest enumerator and its coefficient identity imply

$$0 < \frac{Z_{c,d}(\nu_+)}{q_{c,d}(\nu_+)} = 1 - \nu_+ R_{c,d}(x), \quad R_{c,d}(x) < \nu_+^{-1} = cd\nu_- < cd\lambda.$$

The last inequality follows because $x > c + d + cd\lambda + 1/\lambda$, placing λ strictly between the two roots. This uses positivity of a partition function, not a correlation assertion. A zero index gives a zero

tail. Hence with $T = (m-3)(n-3)\lambda$,

$$R \le U_3 = \cfrac{mn}{d_0 - \cfrac{(m-1)(n-1)}{d_1 - \cfrac{(m-2)(n-2)}{d_2 - T}}}. \tag{23}$$

Indeed $d_2 - T > 1/\lambda$, so its fraction is at most $(m-2)(n-2)\lambda$; repeating this observation outward proves denominator positivity and allows every monotone substitution.

7 Exact positivity certificates and the two regimes

The following finite polynomial identities have complete exact coefficient data supplied in the accompanying file `certificates.json`. They are not parameter-grid experiments.

Lemma 7.1. *For $m, n \ge 3$:*

1. *If $\lambda \ge 1$, then $F(L_3) > 0$, $F'(L_3) > 0$ and $F'' > 0$.*
2. *If $0 < \lambda \le 1/\sqrt{mn}$, then $F(U_3) > 0$ and $F'(0) < 0$.*

Proof. For the first assertion substitute $m = 3 + u, n = 3 + v, \lambda = 1 + w$. The reduced rational functions $F(L_3), F'(L_3), F''$ have respectively 208/960, 396/441, 112/75 numerator/denominator monomials. Every coefficient is nonnegative and each constant is positive.

For the second put $m = a^2, n = b^2$, $\lambda = [ab(1+w)]^{-1}$ and $a = \sqrt{3} + u, b = \sqrt{3} + v$, where $u, v, w \ge 0$. This covers every real $m, n \ge 3$ in the claimed interval, and does not require integer sizes to be perfect squares. The reduced rational functions $F(U_3)$ and $-F'(0)$ have respectively 844/2468 and 93/454 numerator/denominator monomials. Every coefficient and constant is strictly positive in $\mathbb{Q}(\sqrt{3})$. The verifier stores coefficients as pairs of exact rationals and checks positivity by rational signs and squared comparisons, without floating point. □

For $\lambda \ge 1$, Lemma 7.1 makes F increasing on $[L_3, \infty)$, so $F(R) \ge F(L_3) > 0$. The smaller regime requires a discriminant barrier: the assertion $F'(U_3) < 0$ need not hold throughout it. Direct algebra gives

$$A_1^2 - 4A_2A_0 = p^2 LB, \qquad L = \lambda^2(p-s) - 2\lambda - 1, \tag{24}$$

where

$$\begin{aligned} B = {} & \lambda^6 p^2(p-s) + \lambda^5(6p^2 - 8sp + 4p) \\ & + \lambda^4(-3p^2 + 10sp - 8s^2 - 8p + 4s) + \lambda^3(8p - 16s + 4) \\ & + \lambda^2(3p - 5s - 8) - 6\lambda - 1. \end{aligned}$$

On $0 < \lambda \le 1/\sqrt{p}$ we have $L = p\lambda^2 - 1 - s\lambda^2 - 2\lambda < 0$. Under the same a, b, w substitution, let

$$\widetilde{B}(w) = [ab(1+w)]^6 B.$$

It is a polynomial of degree six in w. The negatives of its coefficients of w^j, $j = 1, \dots, 6$, have respectively $39, 46, 49, 49, 49, 49$ strictly positive coefficients in u, v over $\mathbb{Q}(\sqrt{3})$, with positive constants. These complete lists and (24) are also checked by the verifier. No sign is required for the

constant coefficient of $\widetilde{B}$. Thus $\widetilde{B}$ is strictly decreasing on $w \geq 0$ and tends to $-\infty$. As λ increases from zero to $1/\sqrt{p}$, w decreases from infinity to zero. Consequently

$$I = \{\lambda \in (0, 1/\sqrt{p}] : B(\lambda) \leq 0\}$$

is an initial interval containing all sufficiently small positive λ, possibly the whole interval. The discriminant of F is nonnegative on I and strictly negative on its complement.

Fix m, n and consider $H(\lambda) = -F'(U_3)$. It is continuous since the continued-fraction denominators are positive. As $\lambda \downarrow 0$,

$$U_3 = p\lambda + O(\lambda^2), \qquad F'(U_3) \longrightarrow -\frac{1}{mn(m-1)(n-1)} < 0.$$

If H had a first zero in I, the inequalities $F'(0) < 0$ and $U_3 > 0$, together with affinity of F', would force $A_2/D > 0$, where D is the positive denominator in (18). At that zero,

$$\operatorname{disc}(F) = F'(U_3)^2 - 4(A_2/D)F(U_3) < 0,$$

contrary to its nonnegative discriminant on I. Hence $F'(U_3) < 0$ on I. Both endpoint derivatives are negative, so F decreases on $[0, U_3]$ and $F(R) \geq F(U_3) > 0$ there. On the complement of I the quadratic has strictly negative discriminant and $F(U_3) > 0$, so it is positive on the whole real line. This proves $F(R) > 0$ throughout the smaller regime. Equation (18) and $q > 0$ now prove strict negative correlation for the disjoint orbit.

If $m = 2$, the continued fraction terminates after two levels; direct substitution gives, for every $\lambda > 0$,

$$F(R) = \frac{\lambda^4(\lambda+1)^2}{(4\lambda^4 n^2 + 8\lambda^3 n + 3\lambda^2 n + 4\lambda^2 + 4\lambda + 1)^2} > 0. \tag{25}$$

Swapping parts covers $n = 2$. Together with Theorem 1.3, this proves Proposition 1.4 and the width-two assertion.

7.1 The uniform specialization

At $\lambda = 1$ the lower fraction simplifies to

$$L_3 = \frac{mn(m^2n^2 + 9mn + 2m + 2n + 20)}{B},$$
$$B = m^3n^3 + 10m^2n^2 + 3m^2n + 3mn^2 + 31mn + 10m + 10n + 34.$$

In this case $F(L_3) = 2P/[(m+1)(n+1)B^2]$, where

$$P = m^3n^3 + 6m^3n^2 - 5m^3n + 2m^3 + 6m^2n^3 + 10m^2n^2 + 44m^2n - 8m^2 - 5mn^3 + 44mn^2 + 87mn + 50m + 2n^3 - 8n^2 + 50n + 132.$$

Its positivity can be seen without a coefficient file: group $6m^3n^2 - 5m^3n$, $6m^2n^3 - 5mn^3$, $44m^2n - 8m^2$ and $44mn^2 - 8n^2$. The remaining terms are positive. The derivative certificate has 36 positive coefficients after shifting m, n by three. This specialization is also checked independently by the verifier.

8 All fugacities on $K_{3,n}$

Process right vertices one at a time and retain the connectivity partition of the three left vertices. A selected neighbor set S is allowed precisely when its members lie in distinct existing blocks. Otherwise it creates a cycle through the new vertex; conversely any new cycle must do so. Merge the blocks meeting S. Choices with $|S| \le 1$ have total weight $A = 1+3\lambda$ and preserve the partition. The remaining transfer N strictly decreases block count, so $N^3 = 0$. Therefore

$$(AI + N)^r = A^r I + rA^{r-1}N + \binom{r}{2}A^{r-2}N^2.$$

This counts unrooted forests exactly, including isolated right vertices. Summing states gives

$$Z_{3,n} = A^n + n(3\lambda^2 + \lambda^3)A^{n-1} + 3n(n-1)\lambda^4 A^{n-2}.$$

There are three edge-pair orbits: L, sharing a left vertex; R, sharing a right vertex; and D, disjoint. Mark representatives $(0,0),(0,1)$, respectively $(0,0),(1,0)$ and $(0,0),(1,1)$, in the first two columns. Assign variables z_1, z_2 to their selection. The 64 possible first-two-column neighbor-set choices, followed by the displayed nilpotent transfer for the other $r = n-2$ columns, give exact marked polynomials. Differentiating these variables before setting them to one yields

$$N_e N_f - Z N_{ef} = c_O \lambda^4 A^{2n-6} Q_O(r,\lambda), \quad c_L = 2, \quad c_R = r+1, \quad c_D = 1. \tag{26}$$

The coefficient of λ^k in Q_O is as follows:

k	Q_L	Q_R	Q_D
0	1	1	1
1	12	12	12
2	$5r + 59$	$6r + 58$	$7r + 58$
3	$36r + 154$	$46r + 144$	$44r + 144$
4	$5r^2 + 100r + 230$	$12r^2 + 130r + 193$	$7r^2 + 101r + 193$
5	$18r^2 + 132r + 192$	$48r^2 + 162r + 132$	$18r^2 + 102r + 132$
6	$18r^2 + 72r + 72$	$9r^3 + 45r^2 + 72r + 36$	$9r^2 + 36r + 36$

The finite marked-transfer algebra behind (26) is reproduced in the verifier. Every coefficient is positive for $r \ge 0$. Since $A > 0$, including the harmless negative power at $n = 2$, equation (26) proves the width-three assertion of Proposition 1.5.

8.1 Exact widths four through six

Fix $m \in \{4,5,6\}$ and $n \ge m$; put $A = 1 + m\lambda$ and $e = n - m - 1$. The same connectivity-partition transfer works on the m left vertices. A column is legal exactly when its neighbor set meets each old block at most once: an old path between two chosen neighbors is equivalent to a new cycle. Empty and singleton sets have weight A and preserve the state. Every other legal set lowers block count, so $T = AI + N$ with $N^m = 0$. After marking a disjoint pair in the first two columns, the exact marked enumerator has the form $A^e P_m$, by

$$T^{n-2} = \sum_{j=0}^{m-1} \binom{n-2}{j} A^{n-2-j} N^j.$$

Thus its Rayleigh numerator is $A^{2e} D_m(n-m,\lambda)$. There are no component-root weights in this construction.

Here is a second exact way to reconstruct and verify the supplied D_m, using only the earlier Good formula. With $a = 1 + n\lambda$, the polynomial $P = A^{-e}Z$ is

$$P = \sum_{k=0}^{m} (-\lambda)^k \binom{m}{k} (n)_k \big[a^{m-k} A^{m+1-k} \\ - \lambda^2 (m-k)(n-k) a^{m-k-1} A^{m-k} \big],$$

where the second term is omitted when $m = k$. Put

$$J = AP_\lambda + emP, \qquad T_2 = A^2 P_{\lambda\lambda} + 2emAP_\lambda + e(e-1)m^2 P, \qquad L_R = A^{2-e}(W_{abb} + W_{aab})$$

at physical a, b. The derivatives defining L_R are taken before specialization, using the finite q sum. Differentiating $Z = A^e P$ and using (16) gives the exact polynomial identity

$$\lambda^2[(m-1)(n-1)J^2 - mnP(T_2 - L_R)] = A^2 m^2 n^2 (m-1)(n-1) D_m(n-m, \lambda). \tag{27}$$

All normalized expressions are polynomials obtained from the fixed finite sum; zero falling-derivative terms are omitted before powers are interpreted. The public verifier reconstructs the entire left side independently from that sum and checks every coefficient of (27) in the indeterminates $n - m, \lambda$.

The complete exact D_m data have respectively $45, 77, 117$ nonzero monomials, all with positive integer coefficients. Their degrees in $(\lambda, n-m)$ are respectively $(16,4), (20,6), (24,8)$, and the coefficient of $\lambda^4 (n-m)^0$ is one. Thus $D_m > 0$ for every $n \geq m$ and $\lambda > 0$. At $n = m$ the factor $A^e = A^{-1}$ is harmless since $A > 0$; moreover the $j = m - 1$ transfer term has zero binomial coefficient, so every surviving marked P_m term contains A and $D_m(0, \lambda)$ contains A^2. The verifier also checks that divisibility exactly. This includes the endpoint and proves the remaining assertions of Proposition 1.5, by symmetry and the incident-edge result. The three finite-width identities are not extrapolated to width seven.

9 Quantitative scaling and uniform auxiliary estimates

This section proves the critical and supercritical quantitative results and records uniform auxiliary sign estimates. The exact global proof in Section 10 uses only its algebraic normalization and compactified polynomial, not these asymptotic sign estimates. We retain exact probability normalization throughout; an asymptotic numerator sign is transferred only through positive actual denominators.

9.1 A regular compactification

Put

$$r = \sqrt{mn}, \quad \gamma = (m+n)/r, \quad \zeta = (\gamma+2)^{-1}, \quad \varepsilon = \frac{\gamma+2}{r} = \left(\frac{1}{\sqrt{m}} + \frac{1}{\sqrt{n}} \right)^2, \quad b = 1 - 2\zeta.$$

Thus $0 < \zeta \leq 1/4$, $\varepsilon \leq 4/\min(m,n)$, $r^{-1} = \varepsilon\zeta$ and $\gamma = b/\zeta$. The endpoint $\zeta = 0$ is used only for compact uniform estimates; it is not an additional finite host. The symbol ζ is distinct from the matching polynomial q and the random-cluster parameter.

Write $t = \lambda r$, $y = t + t^{-1}$, $D = t^2 - 1$, $L = t + \zeta(t-1)^2$ and $Z_0 = L/t = 1 + \zeta(y-2)$. For the exterior root sum define $f(y) = R(r(\gamma + y))/r$. The Laguerre differential equation gives exactly

$$\varepsilon Z_0 f_y = yf - 1 - f^2, \qquad 0 < f < t^{-1}, \quad f_y < 0, \qquad t \geq 1. \tag{28}$$

Indeed $R = xP'/P - \min(m,n)$ for $P = L^{(|m-n|)}_{\min(m,n)}$; substitution into $xP'' + (|m-n|+1-x)P' + \min(m,n)P = 0$ gives $xR_x = (x-m-n)R - mn - R^2$ and hence (28). The upper bound follows from the exact partition ratio $Z/q = 1-\lambda R > 0$, on the larger physical root $\lambda = t/r$. The derivative sign follows from (20). These facts hold at $t = 1$, still strictly outside every finite Laguerre root.

Set

$$V = \frac{t^{-1} - f}{\varepsilon}, \qquad J_0 = 1 - \varepsilon b + \varepsilon^2\zeta^2 = (1-1/m)(1-1/n) > 0.$$

Also put

$$\mathcal{H} = \frac{\zeta N}{r^2}, \qquad N = A_2R^2 - A_1R + A_0.$$

The actual probability deficit is exactly

$$\mathrm{Def} := \mathbb{P}_\lambda(e \in F)\mathbb{P}_\lambda(f \in F) - \mathbb{P}_\lambda(e, f \in F) = \frac{\varepsilon^2\zeta^4\mathcal{H}}{J_0 L t^2 V^2}. \tag{29}$$

To check the normalization, the denominator in (18) is $r^6 J_0 L/\zeta$ and $Z/q = t\varepsilon V$. In particular $V > 0$ at every actual host.

Direct substitution into (19) cancels all negative powers of ζ and gives the regular polynomial

$$\mathcal{H} = \sum_{j=0}^{4} \varepsilon^j E_j(V), \tag{30}$$

where

$$\begin{aligned}
E_0 &= L - D^2V,\\
E_1 &= -b^2t - \zeta b(t^2+1) + [b(t^4-5t^2) - \zeta(2t^3+6t)]V + (t-t^3)V^2,\\
E_2 &= \zeta^2 L + [2\zeta bt + 2b^2t^2 - 2\zeta^2(t^2+t^4)]V\\
&\quad + [\zeta(5t^2+4t^4+t^6) + b(4t^3+t^5)]V^2,\\
E_3 &= [-2\zeta^3(t+t^3) - 2\zeta^2 bt^2]V\\
&\quad + [-\zeta bt^2 + (2\zeta^2 - b^2)t^3 + \zeta bt^4 + 2\zeta^2 t^5]V^2,\\
E_4 &= \zeta^2 t^2 L V^2.
\end{aligned}$$

These finite identities, including (29), are reconstructed in the verifier from the original quadratic coefficients.

9.2 A ratio estimate outside the critical window

Primes in the correction formulas mean y derivatives, so $d/dy = t^2 D^{-1} d/dt$. Define

$$\begin{aligned}
&f_0 = t^{-1}, \qquad \Delta = t - t^{-1}, \qquad G = -L/D^2,\\
&P = 3bt^3 + 2bt + \zeta(2t^4 + 7t^2 + 1), \qquad H = tLP/D^5,\\
&\mathcal{A} = \frac{L[b(3t^2+1) + \zeta(2t^3+6t)]}{D^3}.
\end{aligned}$$

They obey $\Delta G = Z_0 f_0'$ and $\Delta H = Z_0 G' + G^2$. For $f_{\mathrm{app}} = f_0 + \varepsilon G + \varepsilon^2 H$, the exact residual is

$$\mathcal{R} = \varepsilon Z_0 f_{\mathrm{app}}' - (y f_{\mathrm{app}} - 1 - f_{\mathrm{app}}^2) = \varepsilon^3(Z_0 H' + 2GH) + \varepsilon^4 H^2. \tag{31}$$

Fix $T > 1$ and write $s = t - 1$ for $1 < t \le T$. All following constants depend at most on T and are uniform on $0 \le \zeta \le 1/4$. Near one the rational formulas give $G = O(s^{-2})$, $H = O(s^{-5})$ and $Z_0 H' + 2GH = O(s^{-7})$. The terminal halfline also matters: for $t' \ge 2$ the same formulas give $G = O(t'^{-2})$, $H = O(t'^{-3})$, $Z_0 = O(t')$ and $Z_0 H' + 2GH = O(t'^{-3})$. Consequently, at every later $t' \ge t$,

$$H(t')/\Delta(t') \le C_T s^{-6}, \qquad |\mathcal{R}(t')|/\Delta(t') \le C_T \varepsilon^3 s^{-8}$$

provided εs^{-3} is sufficiently small.

For $e = f - f_{\mathrm{app}}$ subtraction gives the exact equation

$$\varepsilon Z_0 e' = Ke - \mathcal{R}, \qquad K = y - f - f_{\mathrm{app}} \ge \Delta - \varepsilon^2 H \ge \Delta/2.$$

On the late halfline K/Z_0 is bounded below by a positive absolute constant, while $e = O(t'^{-1})$. Thus the terminal homogeneous term vanishes when solving backward from infinity. The resulting kernel, written as

$$e(y) = \int_y^\infty \frac{\mathcal{R}(v)}{K(v)} \frac{K(v)}{\varepsilon Z_0(v)} \exp\Big[-\int_y^v \frac{K(w)}{\varepsilon Z_0(w)}\, dw\Big]\, dv,$$

has total weight one. Hence

$$V = V_0 - \varepsilon H + O_T(\varepsilon^2 s^{-8}), \quad V_0 = L/D^2, \quad |V - V_0| \le C_T \varepsilon s^{-5}, \quad V = O_T(s^{-2}). \tag{32}$$

This uses actual terminal damping, not an extrapolation of a bounded-t expansion.

The exact next cancellation is

$$D^2 H + E_1(V_0) = \mathcal{A}, \qquad c_T s^{-3} \le \mathcal{A} \le C_T s^{-3}.$$

At V_0, $E_1' = O_T(s^{-1})$: its quadratic coefficient $t - t^3$ vanishes linearly at one. Its second derivative is $O_T(s)$. Taylor expansion in (30), using (32), therefore gives

$$\mathcal{H} = \varepsilon \mathcal{A} + O_T(\varepsilon^2 s^{-6}) = \varepsilon \mathcal{A}[1 + O_T(\varepsilon s^{-3})]. \tag{33}$$

Explicitly the E_0 ratio error is $O(\varepsilon^2 s^{-6})$; the linear change in εE_1 has that size and its quadratic change is $O(\varepsilon^3 s^{-9})$, absorbed by the small parameter. The remaining E_j have bounded coefficients and degree at most two in V, and give no larger error. This proves the uniform outside sign for small $\varepsilon/(t-1)^3$, including arbitrarily small ζ.

For bounded γ and compact $t > 1$, this also proves the explicit expansion in Theorem 1.6. Indeed $\varepsilon = (\gamma + 2)/r$, $\zeta = (\gamma + 2)^{-1}$, $L = \zeta(t^2 + \gamma t + 1)$, $V_0 = \zeta u_0$ with $u_0 = (t^2 + \gamma t + 1)/(t^2 - 1)^2$, and $\mathcal{A} = \zeta^2 A_\gamma$, where

$$A_\gamma = \frac{(t^2 + \gamma t + 1)(3\gamma t^2 + \gamma + 2t^3 + 6t)}{(t^2 - 1)^3}.$$

Substitution into (29) gives exactly K_γ and its $O(r^{-4})$ remainder. The same relative estimates yield Theorem 1.7 with error $O([r(t-1)^3]^{-1})$.

9.3 The critical limit and its strict sign

Put $\delta = \varepsilon^{1/3}$, $y = 2 + \delta^2 \xi$ and $w_\varepsilon(\xi) = (1 - f(y))/\delta$. Then exactly

$$(1 + \zeta \delta^2 \xi) w_\varepsilon' = w_\varepsilon^2 - \xi + \delta \xi w_\varepsilon. \tag{34}$$

It is positive and increasing. The physical bound in (28) gives

$$w_\varepsilon(\xi) > \sqrt{\xi + \delta^2 \xi^2/4} - \delta \xi/2.$$

We justify the global branch and uniformity before using this ODE.

Choose a fixed sufficiently large A. Let $t = 1 + \delta v$ with $A = v^2/(1 + \delta v)$. The outside small parameter is v^{-3}, so (32) applies for this fixed A once ε is small. Since $L(1) = 1$ and $P(1) = 5$, every subsequential limit satisfies

$$w(A) = \sqrt{A} + \frac{1}{4A} - \frac{5}{32A^{5/2}} + O(A^{-4}). \tag{35}$$

In detail the three terms come from $(1 - f_0)/\delta$, $-\delta^2 G$ and $-\delta^5 H$; the last outside error divided by δ is $O(v^{-8}) = O(A^{-4})$. The constant is independent of A, from the single outside bound with $t \le 2$. First fix A, then let $\varepsilon \to 0$; only afterwards may A grow. No simultaneous matching limit is presumed.

At each fixed large A this also bounds $w_\varepsilon(A)$. Monotonicity bounds the functions on $[0, A]$, and (34) bounds their derivatives there. Arzela–Ascoli and diagonal extraction give locally C^1 limits satisfying

$$w' = w^2 - \xi, \qquad w \ge \sqrt{\xi},$$

and (35), even along arbitrary varying ζ. Two possible limits have difference $d' = (w_1 + w_2)d$ and hence

$$|d(\xi)| \le |d(A)| \exp[-4(A^{3/2} - \xi^{3/2})/3] \longrightarrow 0.$$

Thus the global limit is unique, and a subsequence contradiction gives uniform C^1 convergence on every compact ξ interval over all realized host aspect ratios.

The limit is the classical function $w_*(\xi) = -\operatorname{Ai}'(\xi)/\operatorname{Ai}(\xi)$. The Airy differential equation, absence of nonnegative zeros and positive-axis asymptotic are classical [8, Sections 9.2, 9.7.5–9.7.6 and 9.9(i)]. They give a global smooth logarithmic derivative and $w_* = \sqrt{\xi} + O(\xi^{-1})$ on the late halfline. To identify it with the extracted branch, the preceding backward difference argument needs a positive square-root lower growth only on that sufficiently late halfline. The finite initial integral is harmless; ordinary ODE uniqueness extends equality back to zero. No global Airy inequality is assumed in this step.

Define

$$B(u) = w_*(u^2) - u, \qquad \mathcal{C}(u) = -B'(u)/B(u)^2.$$

The limiting ODE and matching give

$$B' = 2uB(2u + B) - 1, \qquad B(u) = 1/(4u^2) + O(u^{-5}) > 0 \text{ eventually}, \qquad B(u) \to 0.$$

At any zero of B its derivative is -1; a negative region could therefore never return to the eventually positive tail. This also excludes a zero at zero, so $B > 0$ everywhere. At a stationary point, $B'' = 2B(4u + B) > 0$, while $B'(0) = -1$. Such a point could only change B' from negative to positive, with no later return; that would contradict positive decay to zero. Thus $B' < 0$ and $\mathcal{C} > 0$ on $[0, \infty)$. Their relevant compact minima are strictly positive.

For $t = 1 + u\delta$, $0 \le u \le U$ fixed, the exact identities are

$$\delta^2 V = w_\varepsilon(\xi) - \frac{u}{1 + u\delta}, \qquad \xi = \frac{u^2}{1 + u\delta}.$$

Uniformly in ζ and compact u, $\delta^2 V \to B(u)$. In (30), $E_0 \to 1 - 4u^2 B$. The quadratic term of εE_1 contributes $-2uB^2$ at the same order; its linear and constant terms vanish. Every other term vanishes, the largest being $O(\varepsilon^2 V^2) = O(\delta^2)$. Consequently

$$\mathcal{H} \longrightarrow 1 - 4u^2 B - 2uB^2 = -B'(u) > 0, \qquad \frac{\text{Def}}{\varepsilon^{10/3}\zeta^4} \longrightarrow \mathcal{C}(u) > 0. \tag{36}$$

Both limits are uniform on each fixed compact u interval and across all actual aspect ratios. The second normalization uses (29), $J_0, L, t \to 1$ and $\delta^2 V \to B > 0$. Each finite host has $\zeta > 0$ even when that parameter tends to zero. For a fixed aspect parameter this gives the probability scale $r^{-10/3}$; its coefficient is $(\gamma+2)^{-2/3}\mathcal{C}((t-1)r^{1/3}/(\gamma+2)^{1/3})$ when the displayed critical coordinate stays bounded.

For each fixed $T > 1$, choose one finite U large enough that the outside sign (33) holds whenever $(t-1)/\varepsilon^{1/3} \geq U$. The critical compact limit gives the sign on $0 \leq u \leq U$ once ε is small. Together these cover $1 \leq t \leq T$. This uses a fixed-U limit and an overlapping uniform outside estimate, not a growing-u extrapolation.

9.4 A weighted estimate on the entire high-fugacity halfline

It remains to control $t \geq 2$ with no upper cutoff, uniformly in ζ. Set $h = \zeta + t^{-1}$. Absolute constants give

$$L \asymp t^2 h, \quad Z_0 \asymp th, \quad \mathcal{A} \asymp h^2/t > 0, \qquad 0 < h \leq 3/4.$$

Define the third correction

$$J = (Z_0 H' + 2GH)/\Delta = -t^2 L\mathcal{P}/D^8,$$

where

$$\begin{aligned}\mathcal{P} = {} & 15b^2t^6 + 35b^2t^4 + 10b^2t^2 + 20\zeta bt^7 + 112\zeta bt^5 + 96\zeta bt^3 + 12\zeta bt \\ & + \zeta^2(6t^8 + 63t^6 + 119t^4 + 49t^2 + 3).\end{aligned}$$

The explicit rational formulas yield, uniformly for $t \geq 2$,

$$|G| \leq Ch/t^2, \quad |H| \leq Ch^2/t^3, \quad |J| \leq Ch^3/t^4, \quad |J'| \leq Ch^3/t^5.$$

Indeed $P = O(t^4 h)$, $\mathcal{P} = O(t^8 h^2)$ and $D \asymp t^2$; differentiation loses at least one t power since $|h_t| = t^{-2} \leq h/t$ and d/dy is bounded. In particular h is not held constant when differentiating.

For $f_{\mathrm{app}} = f_0 + \varepsilon G + \varepsilon^2 H + \varepsilon^3 J$ the exact residual is

$$\mathcal{R} = \varepsilon^4(Z_0 J' + 2GJ + H^2) + 2\varepsilon^5 HJ + \varepsilon^6 J^2, \qquad |\mathcal{R}| \leq C\varepsilon^4 h^4/t^4.$$

For one absolute sufficiently small ε, $K = y - f - f_{\mathrm{app}} \geq \Delta - \varepsilon^2 H \geq ct$; here $G, J < 0$ and $H \geq 0$. Also $K/Z_0 \geq c/h \geq c$. The same terminal kernel has total weight one, and both h and $1/t$ decrease along its halfline. It follows that

$$|f - f_{\mathrm{app}}| \leq C\varepsilon^4 h^4/t^5, \qquad V = V_{\mathrm{app}} + O(\varepsilon^3 h^4/t^5), \quad V_{\mathrm{app}} = V_0 - \varepsilon H - \varepsilon^2 J, \quad |V| + |V_{\mathrm{app}}| \leq Ch/t^2. \tag{37}$$

The finite algebra required for a uniform high-t sign is

$$\mathcal{H}(V_{\mathrm{app}}) = \varepsilon\mathcal{A} + \varepsilon^2\zeta^3 J_0 + \sum_{j=2}^{8} \varepsilon^j R_j(t,\zeta), \qquad |R_j| \leq C\mathcal{A}. \tag{38}$$

We specify the exact calculation, not a sampled inequality. Put $H_N = tLP$, $J_N = -t^2 L\mathcal{P}$ and $V_N = LD^6 - \varepsilon H_N D^3 - \varepsilon^2 J_N$, so $V_{\mathrm{app}} = V_N/D^8$. Substitute this into the five displayed E_j, clear D^{16}, and subtract

$$\varepsilon\mathcal{A}D^{16} + (\varepsilon^2\zeta^3 - \varepsilon^3 b\zeta^3 + \varepsilon^4\zeta^5)D^{16}.$$

The powers zero and one in ε vanish. The following table gives every remaining nonzero ζ power and its maximum numerator degree in t:

j	ζ powers, in order	corresponding degrees in t
2	$0,1,2,3$	$29,30,31,31$
3	$0,1,2,3,4$	$29,30,31,31,31$
4	$0,1,2,3,4,5$	$27,28,29,30,31,31$
5	$0,1,2,3,4,5,6$	$25,26,27,28,29,30,31$
6	$0,1,2,3,4,5,6,7$	$23,24,25,26,27,28,29,30$
7	$0,1,2,3,4,5,6,7,8$	$21,22,23,24,25,26,27,28,29$
8	$2,3,4,5,6,7,8,9$	$21,22,23,24,25,26,27,28$

This is reproduced from the exact formulas by the verifier; all undisplayed coefficients are zero. For ζ power k, its degree is at most $29+\min(k,2)$. Since $D^{16} \asymp t^{32}$, and $\zeta^k \le \zeta^2$ for $k \ge 2$, each coefficient is bounded by $C(t^{-3}+\zeta t^{-2}+\zeta^2 t^{-1}) \le C'h^2/t \le C''\mathcal{A}$. This proves (38) with absolute constants. The retained nonnegative term $\varepsilon^2\zeta^3 J_0$ is essential near the spanning-tree endpoint; a fixed-ζ limit would not supply this weighted bound.

On the segment between V and V_{app}, the displayed E_j and (37) give $|\partial_V \mathcal{H}| \le Ct^4$ for $\varepsilon \le 1$: their individual derivative orders are at most t^4, t^4, t^4, t^3, t^2, respectively. Thus the actual ratio error changes $\mathcal{H}$ by at most $C\varepsilon^3 h^4/t \le C'\varepsilon^3\mathcal{A}$. Summing the finitely many terms in (38), for $\varepsilon \le 1/2$, gives

$$\mathcal{H} = \varepsilon\mathcal{A} + \varepsilon^2\zeta^3 J_0 + O(\varepsilon^2\mathcal{A}) > 0$$

once one absolute small threshold on ε is imposed. This transfers to the actual deficit through (29) without approximating its possibly tiny positive denominator.

The bounded-window argument with $T = 2$ covers $1 \le t \le 2$; the weighted argument covers $t \ge 2$; the earlier all-size theorem covers $0 < t \le 1$. Their finitely many absolute thresholds are independent of λ and of aspect ratio. Choose an integer k_0 with $4/k_0$ below their minimum. Then $m, n \ge k_0$ implies the required small ε. This auxiliary argument gives an eventual all-fugacity sign theorem without an explicit threshold. The exact barrier in Section 10 instead proves the stronger global sign directly, without using this eventual result.

10 An exact barrier for every host

We prove the complete-bipartite assertion of Theorem 1.2. The propositions already handle incident pairs and all hosts with smaller part at most six. It remains to consider disjoint pairs with $\min(m,n) \ge 7$. Use the exact compact variables and polynomial $\mathcal{H}(\varepsilon,\zeta,t,V)$ in Section 9; no asymptotic estimate from that section is a premise here. For this proof write $a = \varepsilon$, $q = \zeta$, $b = 1-2q$ and $F(V) = \mathcal{H}(a,q,t,V)$, holding a, q fixed in derivatives. The symbol q in this section is solely this compact aspect parameter, not the matching polynomial or random-cluster parameter. The physical restrictions are

$$0 < q \le 1/4, \quad 0 < a \le 4/7, \quad ab = 1/m + 1/n \le 2/7, \quad J_0 = 1 - ab + a^2q^2 > 0. \tag{39}$$

These relaxed domains need not all represent integer hosts; proving signs on them suffices. The exact normalization (29) makes the actual deficit have the sign of F.

10.1 Strict interval exclusion and a discriminant switch

The remaining continued-fraction tail is strictly positive, so

$$R > \frac{mn}{x-(m+n-1)}, \qquad f > \frac{1}{t+t^{-1}+aq}, \qquad 0 < V < V_1 := \frac{1+aqt}{at(t^2+1+aqt)}.$$

Every denominator is positive. This remains strict at $t = 1$, where the physical argument is still beyond every finite Laguerre root. Put $K = (1 - ab)t^2 - 2aqt - 1$. Exact substitution gives

$$F(0) = J_0 L > 0, \quad F(V_1) = -\frac{t^3 J_0 K}{a(aqt + t^2 + 1)^2}, \quad \mathrm{disc}_V(F) = KT. \tag{40}$$

Here T is a polynomial in a, q, t. The exact certificate below proves $T > 0$ on (39) for $t \geq 1$.

Since $1 - ab \geq 5/7$ and $K(1) = -a < 0$, K has exactly one positive zero $t_c > 1$ and changes from negative to positive there. Equivalently $t_c = (\sqrt{J_0} - aq)^{-1}$; the denominator is positive because $J_0 - a^2q^2 = 1 - ab > 0$. For $1 \leq t < t_c$, the discriminant is negative and $F(0) > 0$, so F is positive on the entire real line. At t_c, $F(V_1) = 0$ and its discriminant vanishes. The positive constant excludes a constant or linear degeneracy: the leading coefficient is $F(0)/V_1^2 > 0$ and the double root is exactly V_1. Strict physical $V < V_1$ excludes it. Thus actual $F > 0$ through t_c.

For $t > t_c$, $F(0) > 0$ and $F(V_1) < 0$, giving exactly one root in $(0, V_1)$. An additional positive root cannot be physically accessible. We control the derivative only on this accessible branch.

10.2 The physical flow and a positive-root coordinate

The exact equation (28), with $dy/dt = 1 - t^{-2}$, yields for $t > 1$

$$aLt^2 V_t = -L + (t^2 - 1)^2 V + at(t^2 - 1)V^2.$$

Define the polynomial

$$P = aLt^2 F_t + F_V[-L + (t^2 - 1)^2 V + at(t^2 - 1)V^2]. \tag{41}$$

Along the actual smooth exterior ratio, $aLt^2 dF/dt = P$. The derivative F_t here holds V fixed.

Write $F = AV^2 + BV + C$ and $P = p_3V^3 + p_2V^2 + p_1V + p_0$. Put

$$N_1 = 1 + aqt, \quad B_0 = at(t^2 + 1 + aqt), \quad X = \frac{V}{V_1 - V} > 0.$$

The transformed quadratic $B_0^2(1 + X)^2 F(N_1X/[B_0(1 + X)])$ has coefficients

$$g_0 = CB_0^2, \quad g_1 = 2CB_0^2 + BN_1B_0, \quad g_2 = CB_0^2 + BN_1B_0 + AN_1^2 = B_0^2 F(V_1) < 0.$$

Since $g_0 > 0$, it has exactly one positive root. The transformed cubic $Q(X) = B_0^3(1 + X)^3 P(N_1X/[B_0(1 + X)])$ has coefficients

$$\begin{aligned} Q_0 &= p_0B_0^3, & Q_1 &= 3p_0B_0^3 + p_1N_1B_0^2, \\ Q_2 &= 3p_0B_0^3 + 2p_1N_1B_0^2 + p_2N_1^2B_0, & Q_3 &= p_0B_0^3 + p_1N_1B_0^2 + p_2N_1^2B_0 + p_3N_1^3. \end{aligned}$$

Modulo the quadratic, its linear remainder is $(R_1X + R_0)/g_2^2$, where

$$\begin{aligned} R_1 &= Q_1g_2^2 - Q_2g_1g_2 + Q_3(g_1^2 - g_0g_2), \\ R_0 &= Q_0g_2^2 - Q_2g_0g_2 + Q_3g_0g_1. \end{aligned} \tag{42}$$

These identities follow by replacing X^2 and X^3 using the quadratic; its denominator g_2^2 is strictly positive.

The exact factorizations are

$$\begin{aligned} R_1 &= a^5t^9(aqt + 1)^2(aqt + t^2 + 1)^2 J_0LK\, H_L, \\ R_0 &= -a^6t^{10}(aqt + 1)^2(aqt + t^2 + 1)^3 J_0^2LK\, H_C. \end{aligned} \tag{43}$$

Both H_L, H_C are polynomials. The next subsection proves $H_L < 0$, $H_C > 0$. For $t > t_c$, every other factor is positive; therefore $R_1, R_0 < 0$. At the accessible zero $X > 0$, $Q < 0$ and hence $P < 0$. No sign at an inaccessible upper root has been used or asserted.

10.3 Complete finite sign certificates

All identities (40)–(43) are reproducible from the original A_i or the five displayed E_j, by ordinary polynomial differentiation and division. The supplied `barrier_coefficients.json` contains the complete original polynomials $-H_L, H_C, T$, their clearing factors, and every coefficient below.

For $-H_L, H_C$ substitute

$$a = \frac{4}{7+u}, \quad q = \frac{1}{4(1+v)}, \quad t = 1+s, \qquad u, v, s \geq 0.$$

This covers independently $0 < a \leq 4/7$, $0 < q \leq 1/4$, $t \geq 1$. If a polynomial has degrees d_a, d_q in a, q, multiply by $(7+u)^{d_a}[4(1+v)]^{d_q}$. The resulting polynomials have respectively 341 and 85 nonzero strictly positive integer coefficients, with strictly positive constants.

For T use the sharper physical restriction $ab \leq 2/7$:

$$q = \frac{1}{4(1+v)}, \quad a = \frac{4(1+v)}{(7+u)(1+2v)}, \quad t = 1+s.$$

Then $b = (1+2v)/[2(1+v)]$ and $ab = 2/(7+u)$. Conversely, for any allowed q, ab choose $v = (4q)^{-1} - 1$ and $u = 2/(ab) - 7$; thus every actual host in the new range is covered. The positive clearing factor is $(7+u)^{d_a}[4(1+v)]^{d_q}(1+2v)^{d_a}$. Its expansion has 175 nonzero strictly positive integer coefficients and a strictly positive constant. These complete 601 coefficients prove the claimed strict signs, including $u = v = s = 0$ where applicable.

The verifier reconstructs F, P from the original A_i, forms the generic fractional-linear transforms, takes a Euclidean pseudo-remainder and divides by the explicit prefactors in (43). It checks that the quotients are polynomials and reconstructs all three complete coefficient lists. It does not choose a factor merely by its size, sample roots, fit a polynomial, or use floating-point acceptance. This finite exact calculation is a proof premise, separate from the analytic crossing argument.

10.4 Backward continuation and conclusion

Fix an actual host with $\min(m, n) \geq 7$. Proposition 1.4 gives positive actual F on the terminal interval $\lambda \geq 1$, or $t \geq r$. If a nonpositive value occurred at some $t > t_c$, continuity and that terminal positivity would give a largest zero before a sufficiently late positive time. The zeros in this compact interval form a nonempty closed set. Immediately to the right of the last zero the values are positive, so its derivative must be nonnegative. But (41) and the strict accessible-root sign give $aLt^2 dF/dt = P < 0$, a contradiction. This also excludes an isolated zero or tangency; a sequence of earlier zeros does not evade the largest-zero argument. The direction is backward from a positive terminal interval.

Positivity through t_c and this continuation prove $F > 0$ for all $t \geq 1$. Proposition 1.4 covers $0 < t \leq 1$. The positive actual denominator transfers the sign to disjoint probability deficits. Proposition 1.5 covers smaller parts two through six, and Corollary 1.3 covers incident pairs. This proves the bipartite assertion of Theorem 1.2 for all $m, n \geq 2$, with independent edges on stars.

Riccati characteristic-root comparison, recurrence improvement and exterior Laguerre bounds are classical; see Segura [12]. The deduction here compares the actual flow with its accessible edge-covariance zero, through the exact finite remainder. The proof uses the classical Riccati comparison principle.

11 Complete graphs at every common fugacity

We now prove the complete-graph assertions of Theorem 1.2 and Corollary 1.3. All forests in this section are unrooted and include isolated vertices. Write $Z_N = \sum_F \lambda^{|F|}$ on K_N. The cases $N = 1, 2$ have no pair of distinct host edges.

11.1 Hermite counts and strict incident correlation

Define

$$H_k(a, \lambda) = k![u^k] \exp(au - \lambda u^2/2), \qquad H_0 = 1.$$

Then $H_{k,a} = kH_{k-1}$ and $H_k = aH_{k-1} - (k-1)\lambda H_{k-2}$. The rooted tree series $U = xe^{\lambda U}$ and unrooted tree series $U - \lambda U^2/2$ give the forest series $\exp(U - \lambda U^2/2)$. The substitution $x = ue^{-\lambda u}$ has logarithmic Jacobian $1 - \lambda u$, so

$$Z_N = (H_N - \lambda H_{N,a})|_{a=1+N\lambda}. \tag{44}$$

In particular $Z_2 = 1 + \lambda$ and $Z_3 = 1 + 3\lambda + 3\lambda^2$, fixing the normalization. Put $\mathcal{W}_N = H_N - \lambda H_{N,a}$ before specialization.

Removing a distinguished vertex leaves rooted branches joined by degree-marked edges and a separate forest. Its series is $\exp(U - \lambda U^2/2) \exp(\lambda z U)$. For $N \geq 2$ its degree polynomial is therefore

$$D_N(z) = H_{N-1}(a_z, \lambda) - (N-1)\lambda H_{N-2}(a_z, \lambda), \qquad a_z = 1 + (N - 1 + z)\lambda.$$

Equivalently it is $\mathcal{W}_{N,a}/N$ at a_z. For $N = 1$ the degree is zero.

For $\lambda > 0$, H_{N-1} is a scaled probabilists' Hermite polynomial and has simple real zeros [13, Chapter18]. If a polynomial q has simple real zeros, the equation $q'/q = 1/\lambda$ has one simple solution in each internal root interval and one beyond its largest root. Indeed $q'/q = \sum_j (a - r_j)^{-1}$ is strictly decreasing, with the appropriate infinite endpoint limits; below the smallest root it is negative. These account for all zeros of $q - \lambda q'$, and none is a pole because q' is nonzero there. Applying this to H_{N-1} proves that D_N has simple real zeros. Its genuine forest coefficients are nonnegative, its constant and leading coefficients positive, so all zeros are strictly negative.

The normalized degree law is consequently a sum of $N - 1$ Bernoulli variables with distinct parameters in $(0, 1)$. Actual incident indicators are exchangeable, and the variance identity (14) gives strict negative covariance when $N \geq 3$. This proves the complete-graph degree assertion; it does not declare the actual incident indicators independent.

Let primes on Z_N denote total physical λ derivatives. There are $N(N-1)/2$ edges, $N(N-1)(N-2)$ ordered adjacent pairs, and $N(N-1)(N-2)(N-3)/4$ ordered disjoint pairs. Degree marking and total edge-count differentiation give

$$\begin{aligned} N_e &= \frac{2\lambda Z_N'}{N(N-1)}, \\ N_{\mathrm{adj}} &= \frac{\lambda^2 h_3}{N(N-1)(N-2)}, \qquad h_3 = \mathcal{W}_{N,aaa}|_{a=1+N\lambda}, \\ N_{\mathrm{dis}} &= \frac{4\lambda^2 (Z_N'' - h_3)}{N(N-1)(N-2)(N-3)}. \end{aligned} \tag{45}$$

Here $h_3 = (N)_3[H_{N-3} - (N-3)\lambda H_{N-4}]$ at the physical argument; a zero-factor term is omitted. The partial derivatives precede specialization. These counts weight each forest once by $\lambda^{|F|}$. It remains to prove $\mathcal{R}_N = N_e^2 - Z_N N_{\mathrm{dis}} > 0$ for $N \geq 4$.

11.2 An exact quadratic and its physical flow

For $H = H_N$, the identities

$$H_{aa} = (aH_a - NH)/\lambda, \quad H_\lambda = -H_{aa}/2, \quad (H_a)_\lambda = \frac{aN}{2\lambda^2}H + \left[\frac{N-1}{2\lambda} - \frac{a^2}{2\lambda^2}\right]H_a$$

reduce all physical derivatives $N\partial_a + \partial_\lambda$ in (45) to H, H_a. Thus $\mathcal{R}_N/H^2 = \mathcal{N}(r_c)/[N^2\lambda(N-3)(N-2)(N-1)^2]$, where $r_c = \lambda H_a/H$ and $\mathcal{N}$ is a quadratic. The verifier reconstructs this differential reduction directly from these identities.

Put $e = 1/N$, $U = N\lambda = t^2$, $z = t + t^{-1}$ and

$$f(z) = N^{-1/2}\frac{\mathrm{He}_N'(\sqrt{N}z)}{\mathrm{He}_N(\sqrt{N}z)}, \qquad \omega = (1 - tf)/e.$$

Since $H_N(a, \lambda) = \lambda^{N/2}\,\mathrm{He}_N(a/\sqrt{\lambda})$, we have $r_c = tf$ and $Z_N/H = e\omega$ exactly. The normalized Hermite Jacobi matrix has off-diagonal entries $\sqrt{j/N}$ and norm strictly less than two by its row sums. Hence all zeros and lower-truncation zeros lie strictly inside $(-2, 2)$. The physical $z \geq 2$ is exterior. Partition positivity at both activities t^2/N and $1/(Nt^2)$ also gives $0 < f < \min(t, t^{-1})$, so $\omega > 0$.

The Hermite ODE gives $ef_z = zf - 1 - f^2$, hence exactly

$$2eU^2\omega_U = -U + [eU + (U-1)^2]\omega + e(U-1)\omega^2 =: \mathcal{B}. \tag{46}$$

After substitution into $\mathcal{N}$, set $G = e\mathcal{N}$. Its full polynomial is

$$\begin{aligned} G = {} & 2U^3\omega^2e^2 + 4U^2\omega^2e^2 - 8U^2\omega e^2 + 5U^2\omega e - U^2\omega \\ & + 6U\omega^2e^4 - 14U\omega^2e^3 + 11U\omega^2e^2 - U\omega^2e \\ & - 12U\omega e^3 + 19U\omega e^2 - 13U\omega e + 2U\omega + 6Ue^2 - 5Ue + U \\ & - \omega^2e^2 + \omega^2e + \omega e - \omega. \end{aligned}$$

It has the sign of $\mathcal{R}_N$. Write $G = A\omega^2 + B\omega + C$; then $C = U(1-2e)(1-3e) > 0$. For later crossings define

$$S = 2eU^2G_U + G_\omega\mathcal{B}, \qquad 2eU^2\,dG/dU = S$$

along the physical ratio. The exact probability deficit, not just its sign, is

$$\frac{\mathcal{R}_N}{Z_N^2} = \frac{e^2G}{U(1-e)^2(1-2e)(1-3e)\omega^2}. \tag{47}$$

Indeed $Z_N/H = 1 - r_c = e\omega$ and $\mathcal{N} = G/e$. Every denominator factor is positive.

11.3 A full finite seam and both strict endpoints

For $4 \leq N \leq 31$ use the exact recurrence

$$Z_0 = 1, \qquad Z_n = \sum_{k=1}^{n}\binom{n-1}{k-1}k^{k-2}\lambda^{k-1}Z_{n-k},$$

where the singleton tree count is one. It chooses the component containing a fixed vertex. The verifier checks its identity with (44) and every coefficient of

$$J_N = (N-2)(N-3)(Z_N')^2 - N(N-1)Z_N(Z_N'' - h_3).$$

Each of these 28 complete coefficient vectors is nonnegative with a positive term; all are supplied. Therefore $\mathcal{R}_N = 4\lambda^2 J_N/[N^2(N-1)^2(N-2)(N-3)] > 0$ for every $\lambda > 0$. This is an entire-polynomial check, not an activity grid. For the remaining argument assume $0 < e \le 1/32$.

We also require actual strict initial and terminal intervals, for each fixed N. At small activity,

$$\mathcal{R}_N = 2\lambda^4 + O(\lambda^5); \qquad \mathcal{R}_N = 4N^{2N-8}\lambda^{2N-3} + O(\lambda^{2N-4}) \quad (\lambda \to \infty). \tag{48}$$

Here are finite symbolic derivations of these cancellations. With falling factorials $(k)_j$, define

$$c(k,h) = \sum_{j=0}^{h} \frac{(k)_{2j}(-1/2)^j}{j!} \frac{(k-2j)_{h-j}}{(h-j)!} N^{h-j}.$$

This is $[\lambda^h]H_k(1+N\lambda,\lambda)$. For $0 \le h \le 4$ set $z_h = c(N,h) - Nc(N-1,h-1)$, interpreting a negative index as zero; for $0 \le h \le 2$ set $d_h = (N)_3[c(N-3,h) - (N-3)c(N-4,h-1)]$. Equation (45) then gives the λ^{h+1} coefficient of N_e as $2(h+1)z_{h+1}/[N(N-1)]$, and its disjoint λ^{h+2} coefficient as

$$\frac{4[(h+2)(h+1)z_{h+2} - d_h]}{N(N-1)(N-2)(N-3)}.$$

Multiplication gives the Rayleigh coefficients at powers two, three and four as $0, 0, 2$, identically in N.

For the upper endpoint put

$$b(j,h) = \sum_{i=0}^{h} \frac{(N-j)_{2i}(-1/2)^i}{i!} \frac{(N-j-2i)_{h-i}}{(h-i)!} N^{-i}.$$

Then $[\lambda^{N-j-h}]H_{N-j}(1+N\lambda,\lambda) = N^{N-j-h}b(j,h)$. The terms $h \le 2$ give

$$\begin{aligned} Z_N &= N^{N-2}\lambda^{N-1} + \frac{(N-1)(N+6)}{2} N^{N-4}\lambda^{N-2} + O(\lambda^{N-3}), \\ h_3 &= 3(N-2)(N-1)N^{N-3}\lambda^{N-3} \\ &\quad + \frac{(N-3)(N-2)(N-1)(3N+20)}{2} N^{N-5}\lambda^{N-4} + O(\lambda^{N-5}). \end{aligned}$$

Inserting these into the marked counts cancels λ^{2N-2} and leaves the second formula in (48). The verifier checks both universal endpoint identities symbolically. In particular $\mathcal{R}_N/Z_N^2 \sim 4/(N^4\lambda) > 0$: zero spanning-tree covariance alone would not prove a strict terminal interval.

11.4 The low interval and a forward crossing

For $0 < U < 1$, $A = e\overline{A} > 0$. An exact certificate is the substitution $e = 1/(32+w)$, $U = v/(1+v)$, $v, w \ge 0$: the cleared numerator has strictly positive coefficients and its denominator is $(1+v)^3(32+w)^4 > 0$. Put $\mathcal{D} = \mathrm{disc}_\omega(G)/(e-1)^2$. For $U = 1/(1+v)$,

$$(1+v)^4\mathcal{D} = v^4 + 18ev^3 + (105e^2+6e)v^2 + (192e^3+74e^2-4e)v + e^2(192e-15).$$

The first three coefficients are positive and the constant negative. Either sign or zero for the fourth leaves exactly one sign change. Descartes' rule and opposite endpoint signs give one positive root counted with multiplicity, hence simple. There is therefore a unique $U_b \in (0,1)$ with $\mathcal{D} > 0$ before it, zero there and negative afterwards; $\mathcal{D}_U(U_b) < 0$. Both real roots before birth are positive: near

zero they approach zero and $1/e$, and $A, C > 0$ prevent crossing zero or infinity on an interior compact interval.

Before taking any remainder, the exact raw resultant is

$$\mathrm{Res}_\omega(G, S) = -4U^6e^4(e-1)^4(2e-1)(3e-1)\overline{A}\,\mathcal{J},$$

where

$$\begin{aligned}\mathcal{J} = {} & 32U^3e^2 - 16U^3e + 2U^3 - 48U^2e^2 + 36U^2e - 6U^2 \\ & + 18Ue^2 - 24Ue + 6U + 3e - 2.\end{aligned}$$

For $U = v/(1+v)$,

$$(1+v)^3\mathcal{J} = (2e^2 - e)v^3 - (12e^2 + 3e)v^2 + (18e^2 - 15e)v + 3e - 2 < 0.$$

Thus the raw resultant never vanishes in $0 < U \le 1$. At the double root, $G_\omega = 0$ and

$$S = 2eU_b^2G_U = -\frac{eU_b^2(e-1)^2\mathcal{D}_U(U_b)}{2A} > 0.$$

This uses the stationary value $-(e-1)^2\mathcal{D}/(4A)$ of the quadratic. Continuity and nonvanishing resultant propagate $S > 0$ along both real-root branches on $(0, U_b]$. The small-activity endpoint gives actual $G > 0$ initially. A first subsequent zero would have derivative at most zero, contrary to $S > 0$. After U_b the upward quadratic has negative discriminant and is everywhere positive. This includes the only possible double contact and proves the entire low interval.

11.5 The initial critical interval

The positive Hermite continued fraction and its exterior denominators give the strict three-level bound

$$f > \cfrac{1}{z - \cfrac{1-e}{z - \cfrac{1-2e}{z}}}, \qquad 0 < \omega < \omega_3 = \frac{1 + eU^2 + 3eU}{e(U+1)(U^2 + 3eU + 1)}.$$

The omitted tail is positive for $N \ge 32$. Set $U_0 = (1-4e)^{-1}$. Exact substitution gives

$$G(\omega_3) = \frac{U^4(e-1)^2(2e-1)(3e-1)[(4e-1)U+1]}{e(U+1)^2(U^2+3eU+1)^2}, \qquad G_\omega(\omega_3)|_{U_0} = -\frac{e(e-1)}{4e-1} < 0.$$

The coefficients of $\mathcal{D}(1+h)$, in descending powers four through zero, are

$$(1-4e)^2, \quad 6e(1-4e)(1-8e), \quad 9e(2-23e+64e^2), \quad e(4-134e+576e^2), \quad e^2(192e-15).$$

The first four are positive and the last negative for $e \le 1/32$, as can also be checked by clearing the positive denominator after $e = 1/(32+w)$. Hence $\mathcal{D}$ increases strictly on $U \ge 1$ and has one simple zero $U_* > 1$. The identity $\mathcal{D}(U_0) = e^2/(4e-1)^2 > 0$ gives $U_* < U_0$. On $[1, U_0]$,

$$A/e \ge 16e - (U-1) - 14Ue^2 \ge \frac{e(12-78e)}{1-4e} > 0.$$

Below U_* negative discriminant gives positivity everywhere. On (U_*, U_0), $G(\omega_3) > 0$ and $A > 0$. Its derivative at ω_3 cannot vanish, since a stationary value of an upward quadratic with nonnegative discriminant is nonpositive. The negative derivative at U_0 therefore fixes its sign throughout this interval. Thus ω_3 lies below the smaller root; actual $\omega < \omega_3$ is strictly on the positive side. At U_* its bound value is positive; at U_0 the bound is the smaller root and strictness excludes contact. This proves $1 \le U \le U_0$.

11.6 The high interval and a backward crossing

For $U > U_0$, $G(\omega_3) < 0$. Put

$$M = 1 + eU^2 + 3eU, \quad B_0 = e(U+1)(U^2+3eU+1), \quad X = \omega/(\omega_3 - \omega) > 0.$$

Let

$$g = B_0^2(1+X)^2 G(MX/[B_0(1+X)]), \qquad Q = B_0^3(1+X)^3 S(MX/[B_0(1+X)]).$$

Write their coefficients as g_0, g_1, g_2 and Q_0, Q_1, Q_2, Q_3. Here $g_0 = B_0^2 C > 0$ and $g_2 = B_0^2 G(\omega_3) < 0$, giving exactly one positive root. Their linear remainder is $(R_1 X + R_0)/g_2^2$ with the same explicit expressions (42). The exact polynomial factorizations are

$$\begin{aligned} R_1 = & -2U^5 e^5 (U+1)^2 (e-1)^4 (2e-1)(3e-1) \\ & \cdot (U^2+3Ue+1)^2 (U^2 e+3Ue+1)^2 H_L, \\ R_0 = & -2U^6 e^6 (U+1)^3 (e-1)^3 (2e-1)^2 (3e-1)^2 \\ & \cdot (U^2+3Ue+1)^3 (U^2 e+3Ue+1)^2 H_C. \end{aligned}$$

Division by these specified prefactors gives polynomials; the verifier checks the identities exactly. Substitute

$$e = \frac{1}{32+u}, \qquad U = \frac{32+u}{28+u} + s, \qquad u, s \geq 0.$$

For a polynomial of degrees d_e, d_U, clear the positive factor $(32+u)^{d_e}(28+u)^{d_U}$. The complete resulting polynomial for H_L has 117 strictly positive integer coefficients and positive constant; that for $-H_C$ has 34, also with positive constant. All original polynomials and coefficients are supplied. Thus $H_L > 0$, $H_C < 0$ on the required domain. The first prefactor is negative and the second positive, so $R_1, R_0 < 0$. At an actual $G = 0$, $X > 0$ gives $Q < 0$ and hence $S < 0$; the remainder denominator is nonzero because $U > U_0$.

The large-activity endpoint gives actual $G > 0$ on a terminal interval. If it vanished earlier in $U > U_0$, choose its largest zero before that positive interval. Immediately to the right its values are positive and the derivative must be nonnegative, contradicting $S < 0$ and the positive flow scale. This excludes crossings and tangencies. Closedness on the intervening compact interval handles accumulated earlier zeros.

The complete finite vectors, low polynomials, endpoint identities and 151 high coefficients are in `complete_graph_coefficients.json`; `complete_graph_verify.py` reconstructs them from the original Hermite counts and checks their exact identities.

The low, initial and high intervals exhaust every $U > 0$ for $N \geq 32$; the full finite seam covers $4 \leq N \leq 31$. Incident pairs were settled for every $N \geq 3$. This proves the complete-graph assertion of Theorem 1.2. All parameter thresholds above are proof partitions, not exclusions from its statement.

12 Verification, limits and further questions

The accompanying Python verifier requires Python 3 and SymPy 1.14.0. Its star helper compares all multivariate coefficients from the original weighted forest sum with the matching–determinant identity on three explicit heterogeneous weighted hosts, including the optional component-mark identity on their vertex deletions. These finite comparisons corroborate normalization; the arbitrary-host stability proof is analytic in Section 2. It verifies the differential reduction, every exact coefficient

in Lemma 7.1 and the discriminant barrier, the two-column transfer identities, the complete Good-polynomial identities for widths four through six, and small-host direct edge-subset enumeration. It also checks the exact Riccati correction and residual identities, the regular compactification, its actual probability normalization, every entry of the weighted high-fugacity degree table, the complete global barrier remainders and all 601 bipartite barrier coefficients. The complete-graph helper additionally reconstructs the original Hermite count reduction, every finite-host vector for $4 \le N \le 31$, the raw low resultant and discriminant, initial critical cover, complete 151-coefficient high barrier, and both universal endpoint identities. It also checks direct complete-graph edge-subset counts through $N = 6$. Those analytic remainder estimates are proved in the text, not certified by finite enumeration. Its coefficient checks are proof premises; its direct finite enumerations are independent corroboration and are not extrapolated to infinite families. A normal and an optimized Python run perform the same explicit acceptance checks.

Theorem 1.2 covers both complete and complete-bipartite hosts at every common positive fugacity with the stated strictness exceptions; no size or aspect threshold remains. Positivity of the partition function or real-rootedness of a one-vertex degree polynomial alone would not settle the disjoint orbit; the exact accessible-root barrier supplies the additional argument. Nor does a fixed-component forest theorem apply without accounting for the component-count mixture. Theorem 1.1 separately supplies arbitrary-weight incident dependence and full star negative association on every finite host. The star marginal theorem concerns incident edges on every finite host.

Use of AI tools. OpenAI ChatGPT/Codex assisted with literature discovery, proof exploration, drafting and exact verification code. Separate AI-assisted internal checks were performed.

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