Counting the smallest counterexamples to the Zagreb index inequality at degrees two through five

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Luminary Oddities

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Abstract

Ghalavand (MATCH Commun. Math. Comput. Chem., 2026) showed that a graph whose maximum and minimum degrees differ by at most three can violate the normalized Zagreb inequality $M_1/n \le M_2/m$ only if its degrees run from two to five, and he asked two questions about this range. Question I asks for the largest order through which every connected graph with minimum degree two and maximum degree five satisfies the inequality. Question II asks how many counterexamples a given order admits. Husain has shown that the answer to Question I is 45. We give a short proof of this threshold, by pairing small endpoint degrees with large ones to bound the second Zagreb index, and show that planar counterexamples exist at every order from 46 on. Up to order 46 the integer gap in this bound forces the edge-type table of every violating graph, and we use this to answer Question II at every order up to 46. Every connected counterexample of order 46 is a subdivided loopless multigraph on seven vertices joined by a single bridge to a graph on 21 vertices with one vertex of degree two and twenty of degree three. Burnside's lemma then gives exactly 12,607,804,764 such counterexamples up to isomorphism, and 94,955,199,713,504 violating graphs of order 46 when disconnected graphs are included. Consequently 46 is the smallest order of a connected counterexample among all graphs of degree spread at most three. A Pólya-type coefficient formula counts the violating graphs of every order.

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1 Introduction

For a finite simple graph G, write $n = |V(G)|$, $m = |E(G)|$, and $d(v)$ for the degree of a vertex v. The first and second Zagreb indices are

$$M_1 = \sum_{v \in V(G)} d(v)^2, \qquad M_2 = \sum_{uv \in E(G)} d(u)d(v).$$

The conjecture that every connected graph satisfies the normalized inequality $M_1/n \le M_2/m$ goes back to the AutoGraphiX experiments; see [2, Conjecture 1]. Put $\mathcal{D} = mM_1 - nM_2$. For $m > 0$ the inequality holds exactly when $\mathcal{D} \le 0$, and we call a graph with $\mathcal{D} > 0$ a *violating graph*; a connected violating graph is a counterexample to the conjecture.

Hansen and Vukićević [3] proved the inequality for graphs of maximum degree at most four and gave a connected counterexample on 46 vertices with maximum degree ten [3, p. 168]. Sun

and Chen [6] proved it whenever the maximum and minimum degrees differ by at most three and the minimum degree is not two. Andova, Cohen and Škrekovski [1, §5] constructed connected planar counterexamples with all degrees between two and five, the smallest of their family having 137 vertices. Ghalavand [2, Theorem 1] proved that every violating graph whose maximum and minimum degrees differ by at most three has minimum degree two and maximum degree five, and he constructed connected counterexamples in this range, the smallest in his construction having 218 vertices [2, Proposition 3]. In [2, §3, p. 451] he posed two questions about this exceptional degree range.

Question I (Ghalavand). What is the largest integer N such that every connected graph with minimum degree 2, maximum degree 5 and at most N vertices satisfies $M_1/n \le M_2/m$?

Question II (Ghalavand). How many distinct graphs of a given order with minimum degree 2 and maximum degree 5 are counterexamples?

Husain [4] has answered Question I: the answer is $N = 45$, and counterexamples exist at every order from 46 on; his argument is a case analysis of edge-type counts and is formalized in Lean. We give a short independent proof and add planarity.

Theorem 1.1. *For connected simple graphs with minimum degree two and maximum degree five, the answer to Question I is* $N = 45$. *The orders of planar counterexamples in this class are exactly* $46, 47, 48, \ldots$.

Our main contribution concerns Question II. We count up to isomorphism, which is the natural reading of "distinct graphs". The same pairing argument that proves Theorem 1.1 forces the edge-type table of every violating graph of order at most 46, and this turns the count into a product of two classical enumeration problems.

Theorem 1.2. *Up to isomorphism there are exactly* 12,607,804,764 *connected simple graphs of order* 46 *with minimum degree two, maximum degree five and* $M_1/n > M_2/m$. *Including disconnected graphs, the number is* 94,955,199,713,504.

Theorem 1.2 answers Question II at the first order at which counterexamples exist, for connected graphs and for all graphs. Theorem 4.2 answers it at every order $n \le 46$: there are no connected counterexamples below 46, and Table 1 lists the number of violating graphs, connected or not. Section 5 gives a coefficient formula of Pólya type that counts violating graphs of any prescribed order, connected or not (Theorem 5.1).

Combined with Ghalavand's theorem quoted above, Theorem 1.1 determines the smallest counterexamples in the whole class of bounded degree spread.

Corollary 1.3. *Among connected graphs whose maximum and minimum degrees differ by at most three, the smallest order of a counterexample to* $M_1/n \le M_2/m$ *is* 46. *The counterexamples of order* 46 *in this class are exactly the* 12,607,804,764 *graphs of Theorem 1.2, and each consists of a subdivided loopless multigraph on seven vertices joined by a single bridge to a graph on* 21 *vertices with one vertex of degree two and twenty of degree three.*

Proof. By [2, Theorem 1], a violating graph with degree spread at most three has minimum degree two and maximum degree five. Theorem 1.1, Theorem 1.2 and Theorem 4.1 below give the rest. □

The route is as follows. Section 2 bounds M_2 below for each degree sequence by pairing small endpoint degrees with large ones, and shows that the bound is attained with a unique edge-type table. A finite integer calculation over 97,136 degree profiles then leaves one profile at order 46 for connected graphs. Section 3 gives planar counterexamples at every order from 46 on. Section 4 turns the forced tables into bridge and disjoint-union decompositions and counts the factors by Burnside's lemma. Section 5 gives the general formula.

2 The pairing bound and its equality table

Write $n_i = \#\{v : d(v) = i\}$ for $2 \le i \le 5$, and call (n_2, n_3, n_4, n_5) the *profile* of G. Give each vertex one half-edge for each incident edge, labelled by the degree of that vertex. The sorted list $\sigma_1 \le \cdots \le \sigma_{2m}$ of these labels contains in_i copies of i. Define

$$B = \sum_{j=1}^{m} \sigma_j \, \sigma_{2m+1-j}, \qquad H = mM_1 - nB.$$

Both depend only on the profile. Pair the first label in the list with the last, the second with the penultimate, and so on, and let T be the table of unordered degree pairs among these m pairs; we call T the *antisorted table*. Its *support graph* has one vertex for each degree present and an edge ij for each positive off-diagonal entry T_{ij}. The *edge-type table* of G records, for each unordered pair $\{i, j\}$, the number of edges joining a vertex of degree i to one of degree j.

Lemma 2.1. *Every graph with a given profile satisfies $M_2 \ge B$, and equality holds only if its edge-type table equals T. If $0 < H < n$, every violating graph with this profile has $M_2 = B$ and edge-type table T. If a connected graph has edge-type table T, then the support graph of T is connected.*

Proof. Regard the two orientations of each edge as distinct, so there are $2m$ oriented edges. For $p = 3, 4, 5$ let τ_p be the number of half-edges with label at least p, and let J_{pq} be the number of oriented edges whose tail has degree at least p and whose head has degree at least q. Each oriented edge has one tail and one head, so inclusion–exclusion gives $J_{pq} \ge [\tau_p + \tau_q - 2m]_+$, where $[x]_+ = \max(0, x)$. Writing each degree as $2 + [d \ge 3] + [d \ge 4] + [d \ge 5]$ and expanding $d(u)d(v)$ over the oriented edges gives

$$2M_2 = 8m + 4 \sum_{p=3}^{5} \tau_p + \sum_{p,q=3}^{5} J_{pq}.$$

In the antisorted pairing, list position j is paired with position $2m + 1 - j$. The tails of degree at least p occupy the last τ_p positions and the heads of degree at least q are paired with the first τ_q positions, so all nine inequalities are equalities simultaneously. This proves $M_2 \ge B$.

If $M_2 = B$, each J_{pq} attains its lower bound, so every J_{pq} of G equals that of the antisorted pairing. Together with the marginal values $J_{2q} = \tau_q$, $J_{p2} = \tau_p$, $J_{22} = 2m$ (where $\tau_2 = 2m$) and $J_{pq} = 0$ when p or q equals 6, these numbers determine every count of oriented edges with tail degree p and head degree q by taking first differences in both indices. Hence the edge-type table equals T.

If $\mathcal{D} > 0$, then $M_2 < mM_1/n = B + H/n$, so $0 \le M_2 - B < H/n$. When $H < n$ the integer $M_2 - B$ is therefore 0.

Finally, a path in G between vertices of degrees $i \ne j$ projects, after deleting consecutive repetitions, to a walk from i to j in the support graph of the edge-type table. □

Lemma 2.2. *For every order $n \le 46$, every profile with $n_2, n_5 \ge 1$ and $H > 0$ satisfies $H < n$. For $n \le 45$ the support graph of the antisorted table of each such profile is disconnected. At $n = 46$ exactly one such profile has a connected support graph, namely $(n_2, n_3, n_4, n_5) = (18, 21, 0, 7)$, with $H = 2$ and nonzero entries $T_{23} = 1$, $T_{25} = 35$, $T_{33} = 31$.*

Proof. A vertex of degree five needs $n \ge 6$. For each $6 \le n \le 46$, take all $n_2 = 1, \ldots, n - 1$, $n_3 = 0, \ldots, n - n_2 - 1$, $n_4 = 0, \ldots, n - n_2 - n_3 - 1$, and $n_5 = n - n_2 - n_3 - n_4$, and keep the profile when $n_3 + n_5$ is even. These are exactly the profiles with both extreme degrees and an

even degree sum. For each, form the sorted list and its antisorted pairing, compute B and H, and test the support graph of T for connectivity. The complete integer calculation gives the following counts, where the last column counts profiles with $H > 0$ and connected support.

Orders	Profiles	$H > 0$	$H \geq n$	Connected support
6–15	1,160	4	0	0
16–25	7,560	12	0	0
26–35	24,210	19	0	0
36–45	56,110	37	0	0
46	8,096	5	0	1

The program `threshold.py` reproduces the table and evaluates B both by the pairing and by the nine threshold counts of Lemma 2.1. The program `independent_threshold.py` recomputes it from the explicit sorted list and also confirms, by enumerating all perfect matchings of every multiset of at most ten labels from $\{2, 3, 4, 5\}$, that the antisorted pairing is the unique minimizing edge-type table. A third program, `tree_check.py`, uses a different necessary condition: it reserves one edge for each link of a spanning tree on the degree classes present, pairs the remaining half-edges antisortedly, and minimizes over the trees. It leaves the same single profile, with bound 635. □

3 Planar counterexamples at every order from 46 on

For odd $h \in \{5, 7\}$, take a cycle on vertices $0, \ldots, h-1$ and give the edge $\{i, i+1\}$ (indices mod h) multiplicity three when i is odd and two otherwise. This loopless multigraph has degree four at 0 and degree five elsewhere. Subdivide every edge once, using a fresh vertex for every parallel copy; the result is simple. For even $s \geq 6$, take the prism formed by two cycles of length $s/2$ with corresponding vertices joined by a matching, and subdivide one edge of the outer cycle by a new vertex w. Finally join 0 to w by a path of length two through one more new vertex. Denote the resulting connected simple graph by $G(h, s)$. A family of the same shape appears in [4].

Vertex 0 now has degree five, w has degree three, and the new middle vertex has degree two. The graph $G(h, s)$ is planar: its blocks are the blocks of the subdivided multigraph (a cycle with parallel subdivided paths), the blocks of the subdivided prism, and the two edges of the joining path, and a graph is planar exactly when all its blocks are planar.

Proof of Theorem 1.1. The graph $G(h, s)$ has h vertices of degree five, $(5h+1)/2$ of degree two and $s+1$ of degree three. Its edges of types 25, 23, 33 number $5h$, 1, $3s/2+1$, respectively. Summing degrees, squared degrees and edge products gives, for every integer $t \geq 0$,

Graph	n	m	M_1	M_2	$\mathcal{D}$
$G(7, 20+2t)$	$46+2t$	$67+3t$	$436+18t$	$635+27t$	$2+2t$
$G(5, 28+2t)$	$47+2t$	$69+3t$	$438+18t$	$643+27t$	$1+t$

Both defects are positive, and the two progressions cover every order from 46 on. Let $n \leq 45$ and let G be a violating graph with minimum degree two and maximum degree five. By Lemma 2.2 its profile has $0 < H < n$, so by Lemma 2.1 its edge-type table is the antisorted table, whose support graph is disconnected by Lemma 2.2. By the last statement of Lemma 2.1, G is disconnected. □

The program `independent_threshold.py` also builds $G(7, 20+2t)$ and $G(5, 28+2t)$ for $0 \leq t \leq 11$, checks simplicity, connectivity, degrees and the table entries, and confirms planarity by an explicit rotation system satisfying Euler's formula.

4 Classification and enumeration through order 46

We use three counts for an ordinary degree $\kappa \in \{3, 5\}$ and k ordinary vertices. Let α_k be the number of unlabelled κ-regular graphs on k vertices, ρ_k the number of unlabelled graphs on $k+1$ vertices with one vertex of degree $\kappa - 1$ and k of degree κ, and γ_k the number of connected ones among the latter. We take loopless multigraphs for $\kappa = 5$ and simple graphs for $\kappa = 3$, and write $\alpha_k^{(5)}, \alpha_k^{(3)}$, and so on.

Theorem 4.1. *Every connected violating graph of order* 46 *with minimum degree two and maximum degree five has a unique edge joining a vertex of degree two to a vertex of degree three, and that edge is a bridge. Deleting it and suppressing the remaining degree-two vertices on its side gives a bijection, on isomorphism classes, onto ordered pairs* (M, K)*, where* M *is a connected loopless multigraph on seven vertices with degrees* $4, 5, 5, 5, 5, 5, 5$ *and* K *is a connected simple graph on* 21 *vertices with degrees* $2, 3, \dots, 3$*. Consequently their number is* $\gamma_6^{(5)}\,\gamma_{20}^{(3)} = 822 \cdot 15{,}337{,}962 = 12{,}607{,}804{,}764$*.*

Proof. By Lemmas 2.1 and 2.2, G has profile $(18, 21, 0, 7)$ and edge-type table $T_{23} = 1$, $T_{25} = 35$, $T_{33} = 31$. Thus exactly one edge xw joins the degree-three class to its complement, with $d(x) = 2$ and $d(w) = 3$, and every other edge at a degree-three vertex stays inside that class. Since G is connected, xw is a bridge and every degree-three vertex is joined to w inside the class. Deleting xw therefore leaves the component K on the 21 degree-three vertices, in which w is the unique vertex of degree two, and a second component containing x, which is now a leaf attached to a degree-five vertex v_0. Remove x. Each of the other 17 degree-two vertices has two neighbours of degree five, which are distinct because G is simple. Suppressing these 17 vertices yields a connected loopless multigraph M on the seven degree-five vertices in which v_0 has degree four and the others degree five.

Conversely, given (M, K), subdivide every edge of M once with a fresh vertex for each parallel copy, and join the degree-four vertex of M to the degree-two vertex of K by a path of length two. The result is simple and connected, has the table above, and has $\mathcal{D} = 67 \cdot 436 - 46 \cdot 635 = 2$. The two constructions are mutually inverse. An isomorphism between two such graphs maps the unique 23 edge to the 23 edge, so it induces isomorphisms of the multigraphs and of the graphs K; the distinguished vertices are the unique vertices of their degrees, so conversely isomorphisms of M and K extend. Hence the classes correspond bijectively.

It remains to compute γ_k. In a graph counted by ρ_k, the component containing the exceptional vertex is connected with j ordinary vertices for some $j < k$, and the remaining $k - j$ vertices form an arbitrary κ-regular graph. Therefore $\alpha_0 = 1$, $\gamma_0 = 0$ and

$$\gamma_k = \rho_k - \sum_{j=0}^{k-1} \gamma_j \, \alpha_{k-j}. \tag{1}$$

The numbers α_k and ρ_k are orbit counts of the symmetric group on the k ordinary vertices acting on labelled (multi)graphs, so by Burnside's lemma each is the average, over permutations π, of the number of labelled (multi)graphs fixed by π; the exceptional vertex is a fixed point. A permutation with c_ℓ cycles of length ℓ occurs $k!/\prod_\ell \ell^{c_\ell} c_\ell!$ times. A fixed (multi)graph has constant multiplicity on each orbit of π on unordered vertex pairs. Between two different cycles of lengths ℓ and ℓ' there are $g = \gcd(\ell, \ell')$ pair orbits, and each contributes ℓ'/g to the degree of every vertex of the first cycle and ℓ/g to every vertex of the second. Within an ℓ-cycle there are $\lfloor (\ell - 1)/2 \rfloor$ orbits contributing two to each vertex, and for even ℓ one further orbit contributing one. Distributing total multiplicity j over g interchangeable orbits can be done in $\binom{g}{j}$ ways for simple graphs and

$\binom{g+j-1}{j}$ ways for multigraphs. Counting the multiplicity assignments that meet every prescribed degree gives each fixed count. The resulting values are

κ	k	α_k	ρ_k	γ_k
5	6	54	847	822
3	20	516,344	15,451,913	15,337,962

and $822 \cdot 15{,}337{,}962 = 12{,}607{,}804{,}764$. □

The values were computed by two separately written programs, `count_cores.py` and `independent_count.py`, which agree on every entry. The second also compares the Burnside counts with exhaustive enumeration of all labelled (multi)graphs up to isomorphism for $\kappa = 3$, $k \le 8$ and $\kappa = 5$, $k \le 6$; in particular $\rho_6^{(5)} = 847$ and $\gamma_6^{(5)} = 822$ are confirmed by direct enumeration. The regular counts $\alpha_k^{(3)}$ used below, for $k \le 38$, agree with the numbers of cubic graphs in [5].

The same method counts all violating graphs through order 46, connected or not.

Theorem 4.2. *Let $n \le 46$. Up to isomorphism, the number of simple graphs of order n with minimum degree two, maximum degree five and $M_1/n > M_2/m$ is the entry of Table 1; all orders not listed give zero. None of these graphs is connected when $n \le 45$.*

Proof. By Lemma 2.2, every profile with $H > 0$ at these orders has $H < n$, so by Lemma 2.1 every violating graph has the antisorted table T of its profile, and conversely every graph with that table violates the inequality. Apart from the profile $(18, 21, 0, 7)$ at order 46, direct inspection of these tables (carried out by `order_table.py`) shows that $n_4 \le 1$, that the only nonzero entries are T_{24}, T_{25} and T_{33}, that $T_{24} + T_{25} = 2n_2 = 4n_4 + 5n_5$, and that $2T_{33} = 3n_3$. Thus every degree-two vertex has two distinct neighbours of degree four or five, those vertices have only degree-two neighbours, and the degree-three vertices induce a cubic graph. As in the proof of Theorem 4.1, suppressing the degree-two vertices gives a bijection on isomorphism classes onto pairs consisting of a loopless multigraph with n_4 vertices of degree four and n_5 of degree five, and a cubic simple graph on n_3 vertices. The contribution of the profile is therefore $\alpha_{n_5}^{(5)}\alpha_{n_3}^{(3)}$ when $n_4 = 0$ and $\rho_{n_5}^{(5)}\alpha_{n_3}^{(3)}$ when $n_4 = 1$. The profile $(18, 21, 0, 7)$ contributes $\rho_6^{(5)}\rho_{20}^{(3)}$ by the argument of Theorem 4.1 without the connectedness requirement. Summing gives the table, and the second statement is Theorem 1.1. □

Proof of Theorem 1.2. The connected count is Theorem 4.1, and the total is the row $n = 46$ of Table 1. □

5 A counting formula for every order

Orbit counting gives a formula for every order. Let $\mathbf{a} = (a_2, a_3, a_4, a_5)$ be a profile, allowing zero entries, and on a fixed set of a_i vertices of colour i prescribe degree i. Put $L = \sum_i i a_i$ and $F = \sum_i i^2 a_i$. For a tuple of integer partitions $\lambda_i \vdash a_i$, choose a permutation π preserving the colours whose cycle type on colour i is λ_i, and let $z_\lambda = \prod_\ell \ell^{c_\ell} c_\ell!$ when λ has c_ℓ parts equal to ℓ, with $z_\varnothing = 1$. Give each vertex v a variable x_v, write $c(v)$ for its colour, and for an orbit O of π on unordered pairs of distinct vertices put

$$W_O = \prod_{\{u,v\} \in O} x_u x_v \, y^{c(u)c(v)}.$$

Table 1: Violating graphs of order $n \le 46$ with minimum degree two and maximum degree five, up to isomorphism, connected or not. Orders not listed give zero. Only order 46 has connected ones, 12,607,804,764 of them.

n	graphs	n	graphs
11	1	31	118,574,726
13	2	32	212,303
15	6	33	2,103,210,814
17	21	34	2,585,226
18	5	35	40,635,098,872
19	94	36	36,880,138
20	10	37	847,882,467,826
21	540	38	593,362,153
22	30	39	18,987,306,410,690
23	4,207	40	10,519,756,801
24	105	41	454,035,339,995,444
25	42,164	42	203,207,540,800
26	470	43	11,544,374,187,125,831
27	516,452	44	4,239,801,705,359
28	2,700	45	310,965,313,606,949,982
29	7,374,248	46	94,955,199,713,504
30	21,035		

Set $P_{\mathbf{0}}(y) = 1$ and otherwise

$$P_{\mathbf{a}}(y) = \sum_{\lambda_i \vdash a_i \ (2 \le i \le 5)} \frac{1}{\prod_{i=2}^{5} z_{\lambda_i}} \left[\prod_v x_v^{c(v)} \right] \prod_O (1 + W_O).$$

With marking variables $\boldsymbol{\xi} = (\xi_2, \xi_3, \xi_4, \xi_5)$ and $\boldsymbol{\xi}^{\mathbf{a}} = \prod_i \xi_i^{a_i}$, define the formal power series

$$\mathcal{A}(\boldsymbol{\xi}, y) = \sum_{\mathbf{a} \ge 0} P_{\mathbf{a}}(y)\, \boldsymbol{\xi}^{\mathbf{a}}, \qquad \mathcal{C}(\boldsymbol{\xi}, y) = \sum_{r \ge 1} \frac{\mu(r)}{r} \log \mathcal{A}(\boldsymbol{\xi}^r, y^r),$$

where μ is the Möbius function and $\boldsymbol{\xi}^r = (\xi_2^r, \dots, \xi_5^r)$.

Theorem 5.1. *For $n \ge 1$, the number of order-n violating graphs with minimum degree two and maximum degree five, up to isomorphism, is*

$$\sum_{\substack{a_2+a_3+a_4+a_5=n,\ a_2a_5>0 \\ L \text{ even}}} \ \sum_{q=0}^{\lfloor (LF-1)/(2n) \rfloor} [\boldsymbol{\xi}^{\mathbf{a}} y^q]\, \mathcal{S}(\boldsymbol{\xi}, y),$$

with $\mathcal{S} = \mathcal{A}$ for all simple graphs and $\mathcal{S} = \mathcal{C}$ for connected simple graphs.

Proof. The factors $1 + W_O$ select whole pair orbits of a simple graph fixed by π. Extracting the coefficient of $\prod_v x_v^{c(v)}$ enforces every degree, and the exponent of y is then M_2. In a graph with all degrees in $\{2, 3, 4, 5\}$ the colours are the degrees, so isomorphism classes with profile $\mathbf{a}$ are the orbits of $\prod_i S_{a_i}$, and Burnside's lemma gives $P_{\mathbf{a}}(y) = \sum_q (\text{number of classes with } M_2 = q)\, y^q$. A graph is a multiset of connected components and both the profile and M_2 add over components, so $\log \mathcal{A}(\boldsymbol{\xi}, y) = \sum_{r \ge 1} \mathcal{C}_0(\boldsymbol{\xi}^r, y^r)/r$, where $\mathcal{C}_0$ is the generating function of connected graphs; Möbius inversion gives $\mathcal{C}_0 = \mathcal{C}$. Finally $2m = L$ and $M_1 = F$, so $\mathcal{D} > 0$ is $2nq < LF$, which is the range of q. □

The program `order_counts.py` evaluates Theorem 5.1 for a prescribed order. Run at each order up to 20, it reproduces the corresponding rows of Table 1, for instance the single violating graph $K_{2,5} \cup K_4$ of order 11, a disjoint union of the kind noted by Ghalavand [2, §2]. The program `formula_bruteforce.py` enumerates every graph with all degrees in $\{2, 3, 4, 5\}$ on at most seven vertices and confirms that every coefficient $[\boldsymbol{\xi}^{\mathbf{a}} y^q]$ of $\mathcal{A}$ and $\mathcal{C}$ produced by `order_counts.py` equals the number of isomorphism classes of that profile and second Zagreb index.

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References

[1] V. Andova, N. Cohen and R. Škrekovski, Graph classes (dis)satisfying the Zagreb indices inequality, *MATCH Commun. Math. Comput. Chem.* **65** (2011), no. 3, 647–658.

[2] A. Ghalavand, On a conjecture about comparing the first and second Zagreb indices of graphs, *MATCH Commun. Math. Comput. Chem.* **96** (2026), no. 2, 441–452. doi:10.46793/match.96-2.04725.

[3] P. Hansen and D. Vukićević, Comparing the Zagreb indices, *Croat. Chem. Acta* **80** (2007), no. 2, 165–168. `https://hrcak.srce.hr/en/12846`.

[4] A. Husain, The sharp order threshold for the Zagreb-index inequality with minimum degree two and maximum degree five, manuscript with Lean 4 formalization, November 2026. `https://github.com/adnanrh/experiment-004-zagreb-indices`.

[5] OEIS Foundation, The On-Line Encyclopedia of Integer Sequences, sequence A005638 (number of cubic graphs on $2n$ vertices), `https://oeis.org/A005638`, accessed 26 November 2026.

[6] L. Sun and T. Chen, Comparing the Zagreb indices for graphs with small difference between the maximum and minimum degrees, *Discrete Appl. Math.* **157** (2009), no. 7, 1650–1654. doi:10.1016/j.dam.2008.10.007.