Counterexamples to a permanental inequality of Drury for positive semidefinite matrices

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Abstract

Zhang's 2016 survey of permanents records an inequality of Drury: for every positive semidefinite matrix A of order n, the sum over $k \geq 2$ of $|a_{1k}|^2$ times the permanent of A without rows and columns 1 and k is at most $\operatorname{per} A$. Zhang derived it from a stronger conjecture of Drury, disproved by Hutchinson (complex case) and Chalmers (real case), and called it an interesting open problem. We show that it is false. A complex positive semidefinite matrix of order 12 and rank two, built from the two poles and two staggered pentagons on the Bloch sphere, violates it, with both sides computed exactly; direct sums give counterexamples of every order $n \geq 12$. A real positive semidefinite integer matrix of order 21 and rank three also violates it. Over complex positive semidefinite matrices, the best constant c_n with sum at most $c_n \operatorname{per} A$ is unbounded: it is at least a fixed multiple of $\log n$ and at most $(n-1)/2$, and $c_2 = c_3 = 1/2$. The proofs use the identity expressing the permanent of a rank-two Gram matrix through the squared coefficients of a product of binary linear forms; rotation-invariant configurations give both sides in closed form, and a Cauchy–Schwarz estimate in a Gaussian (Fock) norm gives the logarithmic growth. Multistart searches over complex matrices of rank two (and ranks three and four at orders 8–11) find no violation below order 12, and searches over real matrices of rank three find none below order 21.

Keywords. permanent; positive semidefinite matrix; Gram matrix; Drury permanent conjecture; Bloch sphere; Fock space; counterexample.
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1 Introduction

1.1 The question

For a square matrix $A = (a_{ij})_{i,j=1}^n$ let $\operatorname{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^n a_{i\sigma(i)}$. For $T \subseteq [n] = \{1, \ldots, n\}$ write $A[T]$ for the principal submatrix with rows and columns in T, and $A(T) = A[[n] \setminus T]$; the permanent of the empty matrix is 1. All positive semidefinite matrices below are Hermitian, and "positive semidefinite" is abbreviated PSD.

In §7 of his survey [15], Zhang records the following "Drury permanent conjecture 2016", citing a communication of Drury of 30 May 2016. In Zhang's words, A is "an $n \times n$ positive semidefinite matrix", B is "the submatrix of A obtained by deleting the first row and column of A", and B_{jk} is "the submatrix of B obtained by deleting the jth row and kth column of B"; then

$$(a_{11} \operatorname{per} B)^2 + \Big(\sum_{k=2}^{n} |a_{1k}|^2 \operatorname{per} B_{kk} \Big)^2 \leq (\operatorname{per} A)^2. \tag{1}$$

Zhang lists two immediate consequences of (1): Lieb's inequality $a_{11} \operatorname{per} B \le \operatorname{per} A$, and

$$\sum_{k=2}^{n} |a_{1k}|^2 \operatorname{per} B_{kk} \le \operatorname{per} A, \tag{2}$$

and he writes that (2) "itself is an interesting open problem" [15, §7, eqs. (20)–(21)]. In our numbering (1) and (2) are Zhang's (20) and (21). As in [7] and [2, §1], we read B_{kk} $(2 \le k \le n)$ as B without the row and column that carry the index k of A, that is, $B_{kk} = A(\{1, k\})$; Chalmers, numbering the rows of B from 1, writes $B_{k-1,k-1}$. This is the only reading under which every term of (2) has the same degree, two, in each row of a Gram factor of A, as both sides of an inequality between PSD matrices of this kind must (Lemma 2.4(a)).

We write

$$S(A) = \sum_{k=2}^{n} |a_{1k}|^2 \operatorname{per} A(\{1, k\}), \qquad \rho(A) = \frac{S(A)}{\operatorname{per} A}, \tag{3}$$

the ratio being taken for PSD matrices with positive diagonal, for which $\operatorname{per} A \ge \prod_i a_{ii} > 0$ by repeated use of Lieb's inequality [8] (stated in the proof of Proposition 5.6) with $|T| = 1$. A PSD matrix with a zero diagonal entry has a zero row, and then $S(A) = \operatorname{per} A = 0$. Inequality (2) states that $\rho(A) \le 1$. For $n \ge 2$ let

$$c_n = \sup\{\rho(A) : A \text{ a complex PSD } n \times n \text{ matrix with positive diagonal}\}, \tag{4}$$

and let $c_n^{\mathbb{R}}$ be the same supremum over real PSD matrices. The question is whether $c_n \le 1$ (and $c_n^{\mathbb{R}} \le 1$) for every n.

The stronger inequality (1) is false: Hutchinson gave a 4×4 complex correlation matrix violating it [7] (see [2, §1]), and Chalmers gave an 8×8 real PSD matrix of rank three with integer entries [2]. Paksoy [12, Remark 1.1] reported a real 5×5 counterexample to an inequality in which the deleted submatrices have order $n - 1$; Chalmers checked that Paksoy's matrix satisfies (1) [2, §3], and it satisfies (2) as well, with ratio $\rho = 0.43$ (Remark 6.3). The counterexample of Chalmers satisfies (2) with $\rho = 170552384/1204741248 = 0.142$, from the values printed in [2, §2].

1.2 Main results

Throughout, $A = GG^*$ denotes the Gram matrix of the rows $g_1, \dots, g_n$ of a matrix G, so that $a_{ij} = \langle g_i, g_j \rangle = \sum_t g_{it}\overline{g_{jt}}$.

Theorem 1.1 (a complex counterexample of order 12)**.** *Let $\zeta = e^{2\pi i/5}$, $\omega = e^{\pi i/5}$, and let G be the 12×2 matrix with rows*

$$g_1 = (1, 0), \qquad g_{2+j} = (17,\ 12\zeta^j), \qquad g_{7+j} = (1,\ 2\omega\zeta^j) \quad (0 \le j \le 4), \qquad g_{12} = (0, 1).$$

Then $A = GG^*$ is a PSD matrix of rank two with entries in $\mathbb{Z}[\zeta]$, and

$$\operatorname{per} A = 3\,669\,815\,787\,908\,365\,497\,600, \qquad S(A) = 3\,672\,160\,721\,602\,011\,936\,000,$$

so that $\rho(A) = 1931902736533045/1930669080338997 = 1.000638\ldots > 1$. For every $n \ge 12$ the matrix $A \oplus I_{n-12}$ violates (2); thus $c_n > 1$ for all $n \ge 12$.

Theorem 1.2 (a real counterexample of order 21)**.** *Let G be the 21×3 integer matrix whose rows $g_1, \dots, g_{21}$ are, in order,*

$$(0,0,1),\ (-3,0,2),\ (4,-3,1),\ (-3,2,4),\ (3,-5,2),\ (-1,5,1),\ (4,4,1),$$
$$(3,1,4),\ (7,2,1),\ (-4,-2,1),\ (3,-2,4),\ (-3,4,2),\ (-2,-4,2),\ (-3,-2,4),$$
$$(7,-1,2),\ (0,-3,4),\ (3,4,3),\ (0,-4,1),\ (-6,2,1),\ (0,3,4),\ (1,5,1).$$

Then $A = GG^T$ is a real PSD matrix of rank three with integer entries, and

$$\begin{aligned} \operatorname{per} A &= 870\,497\,884\,264\,612\,360\,893\,905\,837\,845\,079\,654\,400, \\ S(A) &= 873\,350\,821\,400\,784\,379\,038\,880\,704\,619\,728\,076\,800, \end{aligned}$$

so that $\rho(A) = 1.00327\ldots > 1$. For every $n \geq 21$ the matrix $A \oplus I_{n-21}$ violates (2)*; thus $c_n^{\mathbb{R}} > 1$ for all $n \geq 21$.*

Theorem 1.3 (size of the best constant)**.** *The constants* (4) *are nondecreasing in n, and*

(i) $c_2 = c_3 = c_2^{\mathbb{R}} = c_3^{\mathbb{R}} = \frac{1}{2}$*;*

(ii) $c_n^{\mathbb{R}} \leq c_n \leq \frac{n-1}{2}$ *for every* $n \geq 2$*;*

(iii) $c_n \geq \dfrac{4}{13\pi} \log \dfrac{n-1}{24}$ *for every* $n \geq 25$*, and* $\liminf\limits_{n\to\infty} \dfrac{c_n}{\log n} \geq \dfrac{4}{27}$.

The lower bounds in (iii) *are realized by complex PSD matrices of rank two.*

Theorems 1.1 and 1.2 answer the question of Zhang [15, §7] in the negative, for complex and for real matrices: (2) fails for complex PSD matrices of every order $n \geq 12$ and for real PSD matrices of every order $n \geq 21$. Theorem 1.3 shows that no fixed multiple of $\operatorname{per} A$ bounds $S(A)$ uniformly in n, and that (2) holds for $n \leq 3$.

A concrete consequence concerns the cycle structure of the permanent. By Lemma 2.4(d) below, $S(A)$ is the total weight $\prod_i a_{i\sigma(i)}$ of the permutations σ in which 1 lies in a transposition. For PSD A the total weight of all permutations is $\operatorname{per} A \geq \prod_i a_{ii} > 0$, and the weight of those fixing 1 is $a_{11} \operatorname{per} A(\{1\}) \geq 0$. The permutations in which 1 lies in a cycle of length at least three therefore carry total weight $\operatorname{per} A - a_{11} \operatorname{per} A(\{1\}) - S(A) \leq (1-\rho(A)) \operatorname{per} A$. By Theorem 1.3(iii) this weight can be smaller than $-\big(\frac{4}{13\pi} \log \frac{n-1}{24} - 1\big) \operatorname{per} A$; it is negative already at order 12 (Theorem 1.1), and for large n its size exceeds any fixed multiple of $\operatorname{per} A$. Equivalently (Remark 6.2), the Loewner inequality $\operatorname{diag} C(B) \preceq C(B) + \operatorname{per}(B)\, \overline{B}^{-1}$ between the permanental cofactor matrix $C(B)$ of a positive definite matrix B and its diagonal fails for some B of order 11.

1.3 Relation to earlier work and route of the proof

Rank-deficient Gram matrices, complex [14, 4, 5] and real [6], drive the known counterexamples to the permanent-on-top conjecture and to the question of Bapat and Sunder; Drury's real counterexample [6] has order 16 and rank three, the setting of our Theorem 1.2, and Drury's 7×7 counterexample of [4] has rank two and entries built from fifth roots of unity [15, §5]. Our analysis rests on the classical identity of Marcus and Newman [9, 10] (see also [11]), in the coordinate form of Lemma 2.2, that writes $\operatorname{per}(GG^*)$ as the Fischer norm of the product of the linear forms $\ell_i(z) = \sum_t g_{it} z_t$; in rank two the right-hand side is a weighted sum of squared coefficients of a binary form. Section 2 proves it, together with a formula over the Bloch sphere (Proposition 2.5) that explains the geometry. Section 3 treats configurations made of the north pole, two staggered regular m-gons on latitude circles and K copies of the south pole: the products of the linear forms of a regular m-gon are binomials, which gives $\operatorname{per} A$ and $S(A)$ in closed form (Proposition 3.2) and proves Theorem 1.1. Section 4 proves Theorem 1.2. Section 5 lets the number of copies of the south pole grow. After rescaling, the Fischer norm turns into the Gaussian norm of the Fock space of entire functions [1], and testing each deleted product against a single monomial gives the lower bound of Proposition 5.2. Concentric regular polygons whose radii grow like the square root of their

index then give a harmonic sum and Theorem 1.3(iii); Lieb's inequality [8] gives (ii). Section 6 records numerical maxima for small orders and the relation to (1) and to the Chollet conjecture.

Every computation that carries part of a proof is exact. The scripts accompanying this article evaluate both sides of (2) for the matrices of Theorems 1.1 and 1.2 twice: by Ryser's formula [13] applied to A in exact arithmetic (in $\mathbb{Z}[\zeta]$, respectively $\mathbb{Z}$), which uses no identity of this paper, and by Lemma 2.2.

2 Gram matrices of linear forms

Definition 2.1. Let $r \ge 1$. For homogeneous polynomials $p = \sum_\alpha p_\alpha z^\alpha$ and $q = \sum_\alpha q_\alpha z^\alpha$ in $z = (z_1, \dots, z_r)$ of the same degree, the *Fischer inner product* is $\langle p, q\rangle_{\mathrm{F}} = \sum_\alpha \alpha!\, p_\alpha \overline{q_\alpha}$, where $\alpha! = \alpha_1! \cdots \alpha_r!$, and $\|p\|_{\mathrm{F}}^2 = \langle p, p\rangle_{\mathrm{F}}$. For $G \in \mathbb{C}^{n\times r}$ with rows g_i put $\ell_i(z) = \sum_{t=1}^r g_{it} z_t$.

Lemma 2.2. *Let $G, H \in \mathbb{C}^{m\times r}$ have rows g_i, h_i, and let ℓ_i, ℓ'_i be the linear forms of the rows of G and H. Then $\operatorname{per}(GH^*) = \langle \prod_{i=1}^m \ell_i, \prod_{j=1}^m \ell'_j\rangle_{\mathrm{F}}$. In particular $\operatorname{per}((GG^*)[T]) = \|\prod_{i\in T} \ell_i\|_{\mathrm{F}}^2$ for $G \in \mathbb{C}^{n\times r}$ and $T \subseteq [n]$.*

Proof. Expanding the products over all maps $f\colon [m] \to [r]$ gives $\prod_i \ell_i = \sum_\alpha c_\alpha z^\alpha$ and $\prod_j \ell'_j = \sum_\alpha d_\alpha z^\alpha$ with $c_\alpha = \sum_{f\in F_\alpha} \prod_i g_{if(i)}$ and $d_\alpha = \sum_{f\in F_\alpha} \prod_j h_{jf(j)}$, where F_α is the set of maps with $|f^{-1}(t)| = \alpha_t$ for every t. On the other hand

$$\operatorname{per}(GH^*) = \sum_{\sigma\in S_m} \prod_{i=1}^m \sum_{t=1}^r g_{it}\overline{h_{\sigma(i)t}} = \sum_{f\colon [m]\to[r]} \sum_{\sigma\in S_m} \prod_{i=1}^m g_{if(i)} \prod_{j=1}^m \overline{h_{jf(\sigma^{-1}(j))}},$$

where the second product was reindexed by $j = \sigma(i)$. For $f \in F_\alpha$ the map $\sigma \mapsto f\circ\sigma^{-1}$ sends S_m onto F_α, and each element of F_α has exactly $\alpha!$ preimages, namely a coset of the group of permutations that preserve every fibre of f. Hence the double sum equals $\sum_\alpha \alpha! \sum_{f,f'\in F_\alpha} \prod_i g_{if(i)}\overline{\prod_j h_{jf'(j)}} = \sum_\alpha \alpha!\, c_\alpha \overline{d_\alpha}$. The second statement is the case $H = G$ applied to the rows indexed by T. □

Corollary 2.3. *If $G \in \mathbb{C}^{m\times 2}$ and*

$$\prod_{i=1}^m (g_{i1}z_1 + g_{i2}z_2) = \sum_{j=0}^m c_j z_1^j z_2^{m-j}, \qquad \text{then} \qquad \operatorname{per}(GG^*) = \sum_{j=0}^m j!\,(m-j)!\,|c_j|^2.$$

Lemma 2.4. *Let A be PSD of order n with positive diagonal.*

(a) If Δ is an invertible diagonal matrix, then $\rho(\Delta A\Delta^*) = \rho(A)$.

(b) If C is PSD of order p with positive diagonal, then $\operatorname{per}(A\oplus C) = \operatorname{per} A \operatorname{per} C$ and $S(A\oplus C) = S(A)\operatorname{per} C$; in particular $\rho(A\oplus I_p) = \rho(A)$, and c_n, $c_n^{\mathbb{R}}$ are nondecreasing in n.

(c) If $A = GG^*$ with $G \in \mathbb{C}^{n\times r}$ and U is an $r\times r$ unitary matrix, then $(GU)(GU)^* = A$.

(d) $S(A) = \sum \prod_{i=1}^n a_{i\sigma(i)}$, the sum over the permutations σ with $\sigma(1) \ne 1$ and $\sigma(\sigma(1)) = 1$.

Proof. (a) Let $\Delta = \operatorname{diag}(d_i)$. Each term of $\operatorname{per}(\Delta A\Delta^*)$ carries the factor $\prod_i |d_i|^2$; each term $|a_{1k}|^2 \operatorname{per} A(\{1,k\})$ of S is multiplied by $|d_1|^2|d_k|^2\prod_{i\ne1,k}|d_i|^2 = \prod_i |d_i|^2$. (b) A permutation with nonzero weight in $\operatorname{per}(A\oplus C)$ preserves the two blocks, which gives the product formula. The first row of $A\oplus C$ vanishes outside the first block, and for k in the first block $(A\oplus C)(\{1,k\}) = A(\{1,k\})\oplus C$. Monotonicity follows by taking $C = I_1$. (c) is immediate. (d) Such a σ is $(1\,k)\tau$ with $k = \sigma(1)$ and τ a permutation of $[n]\setminus\{1,k\}$, and its weight is $a_{1k}a_{k1}\prod_{i\ne1,k} a_{i\tau(i)}$ with $a_{k1} = \overline{a_{1k}}$. □

By Lemma 2.4(a) we may rescale rows of a Gram factor freely; in particular every value of ρ is attained by correlation matrices. By Lemma 2.4(c) and (a), for a complex Gram factor of rank r we may assume $g_1 = e_1$ (or e_r).

The geometry of the rank-two counterexamples is best seen on the Bloch sphere. For a unit vector $g = (g_1, g_2) \in \mathbb{C}^2$ let

$$\beta(g) = \big(2\,\mathrm{Re}(g_1\overline{g_2}),\ 2\,\mathrm{Im}(g_1\overline{g_2}),\ |g_1|^2 - |g_2|^2\big) \in S^2.$$

A direct expansion shows that $|\langle g, h\rangle|^2 = \frac{1}{2}(1+\beta(g)\cdot\beta(h))$ for unit vectors g, h. The point $\beta(g)$ has polar angle θ and azimuth $-\phi$ when g is a unimodular multiple of $(\cos\frac{\theta}{2}, e^{i\phi}\sin\frac{\theta}{2})$; in particular $(1,0)$ and $(0,1)$ give the north and south poles. Let μ denote the rotation-invariant probability measure on S^2.

Proposition 2.5. *Let $g_1, \dots, g_n \in \mathbb{C}^2$ be unit vectors, $A = GG^*$, $r_i = \beta(g_i)$ and $f_i(x) = \frac{1}{2}(1+x\cdot r_i)$ for $x \in S^2$. For every $T \subseteq [n]$,*

$$\operatorname{per} A[T] = (|T|+1)! \int_{S^2} \prod_{i\in T} f_i \, d\mu.$$

Consequently $\rho(A)$ depends only on $r_1, \dots, r_n$, is invariant under rotations of S^2, and

$$\rho(A) = \frac{1}{n(n+1)} \sum_{k=2}^{n} f_k(r_1)\, \frac{\int_{S^2} \prod_{i\neq 1,k} f_i \, d\mu}{\int_{S^2} \prod_{i=1}^{n} f_i \, d\mu}. \tag{5}$$

Proof. Let p be a binary form of degree d. Integrating in polar coordinates on $\mathbb{C}^2 = \mathbb{R}^4$, whose unit sphere S^3 has area $2\pi^2$ and carries the normalized measure ν,

$$\frac{1}{\pi^2}\int_{\mathbb{C}^2} |p(z)|^2 e^{-|z|^2}\, dV(z) = \frac{2\pi^2}{\pi^2}\int_0^\infty u^{2d+3} e^{-u^2}\, du \int_{S^3} |p|^2\, d\nu = (d+1)! \int_{S^3} |p|^2\, d\nu.$$

The left side equals $\|p\|_{\mathrm{F}}^2$, because $\pi^{-2}\int z^\alpha \overline{z^{\alpha'}} e^{-|z|^2} dV = \alpha!$ when $\alpha = \alpha'$ and 0 otherwise. Apply this to $p = \prod_{i\in T} \ell_i$ and use Lemma 2.2. For $z \in S^3$ we have $\ell_i(z) = \langle z, \overline{g_i}\rangle$, so $|\ell_i(z)|^2 = \frac{1}{2}(1 + \beta(z)\cdot\beta(\overline{g_i}))$, and $\beta(\overline{g_i}) = Jr_i$ with $J = \mathrm{diag}(1,-1,1)$. The map $\beta\colon S^3 \to S^2$ intertwines the action of $SU(2)$ on S^3 with its action by rotations on S^2, so it pushes ν forward to the unique rotation-invariant probability measure μ. Hence $\operatorname{per} A[T] = (|T|+1)! \int \prod_{i\in T} \frac{1}{2}(1 + x\cdot Jr_i)\, d\mu(x)$, and the substitution $x \mapsto Jx$, which preserves μ, removes J. Formula (5) follows from $|a_{1k}|^2 = f_k(r_1)$ and the cases $T = [n]$ and $T = [n]\setminus\{1,k\}$. □

Formula (5) indicates where violations of (2) can be found. The factor $1/(n(n+1))$ has to be compensated by the ratios of integrals. The function $\prod_i f_i$ vanishes to second order at each antipode $-r_i$, and deleting f_k removes the zero at $-r_k$; the k-th ratio of integrals in (5) is large when $-r_k$ lies where the mass of $\prod_i f_i$ sits, while the weight $f_k(r_1)$ is large when r_k is close to r_1. The configurations below place r_1 at the north pole, regular polygons on latitude circles, and further points at or near the south pole, where the antipodes of the polygons near r_1 lie.

3 Rings on the Bloch sphere: proof of Theorem 1.1

Definition 3.1. Let $m \geq 1$, $K \geq 0$ and $x, y > 0$; put $\zeta_m = e^{2\pi i/m}$ and $\omega_m = e^{\pi i/m}$. Let $A_{m,K}(x,y) = GG^*$, where G has the $n = 2m+1+K$ rows

$$(1,0); \qquad (1,\, x\zeta_m^j)\ \ (0 \leq j < m); \qquad (1,\, y\,\omega_m\zeta_m^j)\ \ (0 \leq j < m); \qquad (0,1)\ \ (K \text{ times}).$$

On the Bloch sphere these rows are the north pole, a regular m-gon on the circle of polar angle $2\arctan x$, a second regular m-gon on the circle of polar angle $2\arctan y$ rotated by π/m against the first (together the vertices of an antiprism), and the south pole with multiplicity K.

Proposition 3.2. *For $A = A_{m,K}(x,y)$,*

$$\begin{aligned}\operatorname{per} A &= (2m+1)!\,K! + (m+1)!\,(m+K)!\,(y^m - x^m)^2 + (2m+K)!\,x^{2m}y^{2m},\\ S(A) &= m\,\big(\Sigma(x,y) + \Sigma(y,x)\big),\end{aligned}$$

where

$$\Sigma(x,y) = \sum_{t=0}^{m-1} x^{2t}\Big((2m-1-t)!\,(t+K)! + y^{2m}(m-1-t)!\,(t+m+K)!\Big).$$

Proof. From $\prod_{j=0}^{m-1}(X - \zeta_m^j Y) = X^m - Y^m$ with $X = z_1$ and $Y = -xz_2$, respectively $Y = -y\omega_m z_2$, and from $\omega_m^m = -1$, the two polygons contribute

$$\prod_j (z_1 + x\zeta_m^j z_2) = z_1^m - \eta\, x^m z_2^m, \qquad \prod_j (z_1 + y\omega_m\zeta_m^j z_2) = z_1^m + \eta\, y^m z_2^m, \qquad \eta = (-1)^m.$$

The product of all n linear forms is therefore $z_1 z_2^K\big(z_1^{2m} + \eta(y^m - x^m)z_1^m z_2^m - x^m y^m z_2^{2m}\big)$, and Corollary 2.3 gives $\operatorname{per} A$.

The first row is $(1,0)$, so $|a_{1k}|^2$ is the squared modulus of the first entry of row k: it equals 1 on both polygons and 0 on the copies of $(0,1)$. Let row k be $(1, x\zeta_m^{j_0})$ on the first polygon. With $Y_0 = -x\zeta_m^{j_0} z_2$, which satisfies $Y_0^m = (-x)^m z_2^m$,

$$\prod_{j\neq j_0} (z_1 + x\zeta_m^j z_2) = \frac{z_1^m - Y_0^m}{z_1 - Y_0} = \sum_{t=0}^{m-1} z_1^{m-1-t} {Y_0}^t,$$

so the product of the linear forms of $A(\{1,k\})$ is $z_2^K\big(\sum_t z_1^{m-1-t}Y_0^t\big)\big(z_1^m + \eta\, y^m z_2^m\big)$. Its monomials $z_1^{2m-1-t}z_2^{t+K}$ and $z_1^{m-1-t}z_2^{t+m+K}$ $(0 \le t < m)$ are pairwise distinct, since the exponents of z_1 lie in $[m, 2m-1]$ and in $[0, m-1]$ respectively, and their coefficients have moduli x^t and $x^t y^m$. Corollary 2.3 gives $\operatorname{per} A(\{1,k\}) = \Sigma(x,y)$. The same computation with the roles of the polygons exchanged, using that $z_1^m - \eta\, x^m z_2^m$ has coefficients of moduli 1 and x^m, gives $\Sigma(y,x)$ for rows of the second polygon. Summing over the $2m$ rows with $|a_{1k}| = 1$ gives $S(A)$. □

Remark 3.3. If the second polygon is rotated by an angle ϕ instead of π/m, that is, its rows are $(1, ye^{i\phi}\zeta_m^j)$, the proof of Proposition 3.2 applies verbatim: the coefficients of the products for the deleted rows keep their moduli, while the middle coefficient of the full product has modulus $|x^m + y^m e^{im\phi}| \ge |y^m - x^m|$, with equality exactly when $e^{im\phi} = -1$. The staggered position therefore minimizes $\operatorname{per} A$ and maximizes ρ within this family. Maximizing the closed form over x, y gives $\max\rho(A_{5,1}) = 1.0010581$, at polar angles 70.650° and 125.768° (order 12), $\max\rho(A_{6,0}) = 1.0254174$ at 72.365° and 133.783° (order 13), and $\max\rho(A_{6,1}) = 1.0736917$ (order 14).

Proof of Theorem 1.1. Take $m = 5$, $K = 1$, $x = 12/17$ and $y = 2$ in Proposition 3.2. Exact rational arithmetic gives

$$\operatorname{per} A_{5,1}(\tfrac{12}{17}, 2) = \frac{3669815787908365497600}{17^{10}}, \qquad S(A_{5,1}(\tfrac{12}{17}, 2)) = \frac{3672160721602011936000}{17^{10}}.$$

The matrix G of Theorem 1.1 is obtained from the Gram factor of $A_{5,1}(\frac{12}{17}, 2)$ by multiplying the five rows of the first pentagon by 17, which multiplies both quantities by 17^{10} (Lemma 2.4(a)). This

gives the stated values and $\rho(A) = 1931902736533045/1930669080338997 > 1$. The entries $\langle g_i, g_j\rangle$ lie in $\mathbb{Z}[\zeta]$ because $\omega = -\zeta^3$ and $\overline{\zeta} = \zeta^4$, and A has rank two because g_1, g_{12} are independent. The statement for $n \geq 12$ follows from Lemma 2.4(b). The script `v1_n12_cyclotomic.py` recomputes per A and $S(A)$ by Ryser's formula applied to the 12×12 matrix A in exact arithmetic in $\mathbb{Z}[\zeta]$, by Corollary 2.3 in $\mathbb{Z}[\zeta]$, and by Proposition 3.2; the three computations agree. □

Remark 3.4. A counterexample with Gaussian-integer entries is obtained by rounding the two pentagons. Let G' have rows $g_1 = (1, 0)$, then $g_2, \ldots, g_{11} = (24, b)$ for the ten values of b below (in any order, since ρ is unchanged by permutations of $g_2, \ldots, g_n$), and $g_{12} = (0, 1)$, with

$$b \in \{17,\ 5 \pm 16i,\ -14 \pm 10i\}\ \cup\ \{38 \pm 28i,\ -14 \pm 45i,\ -47\}.$$

Then $A' = G'G'^*$ has entries in $\mathbb{Z}[i]$, and Ryser's formula in exact Gaussian-integer arithmetic gives

$$\begin{aligned} \operatorname{per} A' &= 6\,238\,154\,934\,938\,963\,866\,658\,023\,401\,168\,568\,320, \\ S(A') &= 6\,243\,311\,031\,933\,208\,358\,165\,068\,968\,760\,442\,880, \end{aligned}$$

so that $\rho(A') = 1.000826\ldots$ (script `v2_n12_gaussian_integer.py`).

4 A real counterexample: proof of Theorem 1.2

Proof of Theorem 1.2. The first row of G is e_3, so a_{1k} is the third entry of g_k; all these entries are positive. Further, G^TG has rows $(260, -3, 18)$, $(-3, 208, 5)$, $(18, 5, 134)$ and is nonsingular, so $\operatorname{rank} A = 3$. Lemma 2.2 with $r = 3$ reduces $\operatorname{per} A$ and the twenty permanents $\operatorname{per} A(\{1, k\})$ to Fischer norms of products of ternary linear forms with integer coefficients, which the script `v3_n21_real.py` evaluates in exact integer arithmetic. The script also evaluates the same twenty-one permanents by Ryser's formula applied to the integer matrices A and $A(\{1, k\})$. Both computations give the values stated in Theorem 1.2, and $S(A) - \operatorname{per} A = 2\,852\,937\,136\,172\,018\,144\,974\,866\,774\,648\,422\,400 > 0$. Lemma 2.4(b) gives the statement for $n \geq 21$. □

Remark 4.1. The matrix G is a rounding of a local maximizer of ρ over real Gram factors of rank three and order 21, at which $\rho = 1.01885$. Since ρ is unchanged when a row of G is replaced by its negative, the rows are lines through the origin. Up to this sign, the maximizer consists of e_3, six vectors at angles between 38° and 41° from e_3 with azimuths close to the vertices of a regular hexagon, and fourteen vectors at angles between 64° and 89° from e_3. In the real projective plane the set of lines farthest from e_3 is a full circle rather than a point, and the fourteen vectors spread around it; they play the role of the south pole in Section 3. At order 22 the maximizer found is symmetric: e_3, three regular hexagons at angles 38.35°, 65.16° and 75.34° from e_3, and three equatorial lines at mutual angles 60°, with $\rho = 1.04752$.

5 Growth of the best constant

We first isolate the mechanism in its simplest form.

Example 5.1 (a square and a cluster)**.** Let G have the rows $(1, 0)$; $(13, 2i^j)$ for $0 \leq j \leq 3$; and $K = 150$ copies of $(0, 1)$; the order is $n = 155$. The four middle rows are the vertices of a square near the north pole, at polar angle $2 \arctan \frac{2}{13} \approx 17.5°$. Rescale them to $(1, xi^j)$ with $x = \frac{2}{13}$ and put $s = x^2$. The product of all linear forms is $z_1 z_2^K (z_1^4 - x^4 z_2^4)$, and the computation in

the proof of Proposition 3.2, now with a single polygon, gives $\operatorname{per} A = 5!\,K! + (4+K)!\,s^4$ and $\operatorname{per} A(\{1,k\}) = \sum_{t=0}^{3} s^t (3-t)!\,(t+K)!$ for each vertex k. Hence

$$\rho(A) = \frac{4\sum_{t=0}^{3} s^t\,(3-t)!\,(t+K)!}{5!\,K! + (4+K)!\,s^4}, \qquad s = \frac{4}{169},$$

which equals $1.0019117\ldots$ in exact arithmetic (script `v4_identities_family_growth.py`). If $K \to \infty$ with $s = c/K$, the right side tends to $4(6+2c+c^2+c^3)/(120+c^4)$, whose maximum over $c>0$ is 1.011799, at $c = 3.654$.

For large K the K copies of $(0,1)$ act as a Gaussian weight. If $Q = \sum_a q_a z_1^a z_2^{D-a}$ is a binary form, then $\|z_2^K Q\|_{\mathrm{F}}^2 = \sum_a a!\,(D-a+K)!\,|q_a|^2$, and $(D-a+K)!/K! \sim K^{D-a}$ as $K \to \infty$; so $\|z_2^K Q\|_{\mathrm{F}}^2/K!$ tends to the norm $\sum_a a!\,|f_a|^2$ of the dehomogenized polynomial $\sum_a f_a w^a = Q(w,\sqrt{K})$ in the Fock space of entire functions [1], whenever the coefficients of $Q(w,\sqrt{K})$ are held fixed. The next proposition is an exact finite-K form of a lower bound in that space.

Proposition 5.2. *Let $w_1,\ldots,w_D \in \mathbb{C}\setminus\{0\}$ and $K \ge 1$. Let A be the Gram matrix of the $n = 1+D+K$ rows*

$$(1,0); \qquad (1,-w_j/\sqrt{K}) \;\; (1\le j\le D); \qquad (0,1) \;\; (K \text{ times}).$$

Write $F(w) = \prod_{j=1}^{D}(1-w/w_j) = \sum_{a=0}^{D} f_a w^a$ and $\mathcal{N}(F) = \sum_{a=0}^{D}(a+1)!\,|f_a|^2$. Then

$$\rho(A) \;\ge\; \frac{\sum_{j=1}^{D} |w_j|^{-2}}{\mathcal{N}(F) + D/K}.$$

Proof. Write $\Pi = \prod_j |w_j|^2$ and $N = K+D-1$. The rows $(1,-w_j/\sqrt{K})$ have $|a_{1k}|^2 = 1$, and the rows $(0,1)$ have $a_{1k} = 0$. For a row k of the first kind, the product of the linear forms of $A(\{1,k\})$ is $P_k = z_2^K \prod_{j\ne k}(z_1 - w_j z_2/\sqrt{K})$, a form of degree N whose coefficient of z_2^N is $\prod_{j\ne k}(-w_j/\sqrt{K})$. By the Cauchy–Schwarz inequality for $\langle\cdot,\cdot\rangle_{\mathrm{F}}$ and $\langle P_k, z_2^N\rangle_{\mathrm{F}} = N!\prod_{j\ne k}(-w_j/\sqrt{K})$, $\|z_2^N\|_{\mathrm{F}}^2 = N!$,

$$\operatorname{per} A(\{1,k\}) = \|P_k\|_{\mathrm{F}}^2 \;\ge\; \frac{|\langle P_k, z_2^N\rangle_{\mathrm{F}}|^2}{\|z_2^N\|_{\mathrm{F}}^2} = N!\,K^{-(D-1)}\,\Pi\,|w_k|^{-2}.$$

Summing, $S(A) \ge N!\,K^{-(D-1)}\,\Pi\sum_k |w_k|^{-2}$.

For the denominator, $\prod_j (z_1 - w_j z_2/\sqrt{K}) = \sum_{a=0}^{D} h_a z_1^a z_2^{D-a}$ with $h_a = e_{D-a}(-w/\sqrt{K})$, where e_q denotes the elementary symmetric polynomial of degree q in D variables. The identities

$$e_{D-a}(w_1,\ldots,w_D) = w_1\cdots w_D\; e_a(w_1^{-1},\ldots,w_D^{-1}), \qquad f_a = (-1)^a e_a(w_1^{-1},\ldots,w_D^{-1})$$

give $|h_a|^2 = K^{-(D-a)}\,\Pi\,|f_a|^2$. The full product is $z_1 z_2^K \sum_a h_a z_1^a z_2^{D-a}$, so Corollary 2.3 gives

$$\operatorname{per} A = \sum_{a=0}^{D}(a+1)!\,(D-a+K)!\,|h_a|^2 = N!\,K^{-(D-1)}\,\Pi\sum_{a=0}^{D}\tau_a\,(a+1)!\,|f_a|^2,$$

where $\tau_a = K^{a-1}(D+K-a)!/(D+K-1)!$. Here $\tau_0 = 1 + D/K$, and for $1 \le a \le D$ we have $\tau_a = K^{a-1}/\prod_{q=1}^{a-1}(D+K-q) \le 1$, since each factor $D+K-q$ exceeds K. As $f_0 = 1$, the last sum is at most $D/K + \mathcal{N}(F)$, and dividing the two estimates proves the claim. □

Lemma 5.3. *For every polynomial F, $\mathcal{N}(F) = \frac{1}{\pi}\int_{\mathbb{C}} |wF(w)|^2 e^{-|w|^2}\,dA(w)$.*

Proof. In polar coordinates $\frac{1}{\pi}\int_{\mathbb{C}} w^a \overline{w}^b e^{-|w|^2} dA(w)$ vanishes for $a \neq b$ and equals $\int_0^\infty s^a e^{-s} ds = a!$ for $a = b$. Apply this to $wF(w) = \sum_a f_a w^{a+1}$. □

Lemma 5.4. *For $p > 1$ and $t \geq 0$, $\sum_{q=1}^{\infty} \log(1 + (t/q)^p) \leq \dfrac{\pi t}{\sin(\pi/p)}$.*

Proof. The summand is a decreasing function of $q > 0$, so the sum is at most $\int_0^\infty \log(1+(t/u)^p)\, du = t\int_0^\infty \log(1 + v^{-p})\, dv$. Integration by parts, whose boundary terms vanish because $p > 1$, gives $\int_0^\infty \log(1 + v^{-p})\, dv = p\int_0^\infty \frac{dv}{1+v^p} = \frac{\pi}{\sin(\pi/p)}$. □

Theorem 5.5. *Let $m \geq 3$, $R \geq 1$ and $K \geq 1$ be integers, let $\kappa_m = \dfrac{2\pi}{m \sin(2\pi/m)}$ and $\gamma > \kappa_m$. Apply Proposition 5.2 to the $D = mR$ points $w_{q,j} = \sqrt{\gamma m q}\, e^{2\pi i j/m}$ $(1 \leq q \leq R,\ 0 \leq j < m)$. The resulting rank-two PSD matrix A of order $n = 1 + mR + K$ satisfies*

$$\rho(A) \ \geq \ \frac{H_R/\gamma}{(1 - \kappa_m/\gamma)^{-2} + mR/K}, \qquad H_R = \sum_{q=1}^{R} \frac{1}{q}.$$

Proof. The points form R concentric regular m-gons, and $\prod_j (1 - w/w_{q,j}) = 1 - w^m/(\gamma m q)^{m/2}$. Hence $F(w) = \prod_{q=1}^{R} (1 - w^m/(\gamma m q)^{m/2})$ and, with $s = |w|^2$,

$$\log|F(w)|^2 \leq 2\sum_{q=1}^{\infty} \log\Bigl(1 + \Bigl(\frac{s/(\gamma m)}{q}\Bigr)^{m/2}\Bigr) \leq \frac{2\pi s}{\gamma m \sin(2\pi/m)} = \frac{\kappa_m}{\gamma}\, s$$

by Lemma 5.4 with $p = m/2 > 1$. Lemma 5.3 and polar coordinates give

$$\mathcal{N}(F) \leq \int_0^\infty s\, e^{-(1-\kappa_m/\gamma)s}\, ds = (1 - \kappa_m/\gamma)^{-2}.$$

Finally $\sum_{q,j} |w_{q,j}|^{-2} = \sum_q m/(\gamma m q) = H_R/\gamma$, and Proposition 5.2 applies. □

Proposition 5.6. *Let A be PSD of order $n \geq 2$ with positive diagonal. Then $S(A) \leq \frac{n-1}{2} \operatorname{per} A$. Moreover $\operatorname{per} A - 2S(A) = \det A$ if $n = 2$, and $\operatorname{per} A - 2S(A) = \det A + 2a_{11}|a_{23}|^2$ if $n = 3$.*

Proof. Lieb's inequality [8] states that $\operatorname{per} A \geq \operatorname{per} A[T] \operatorname{per} A([n]\setminus T)$ for PSD A and every $T \subseteq [n]$. With $T = \{1, k\}$ it gives $(a_{11}a_{kk} + |a_{1k}|^2) \operatorname{per} A(\{1,k\}) \leq \operatorname{per} A$. Since $|a_{1k}|^2 \leq a_{11}a_{kk}$, each term of $S(A)$ is at most $\frac{|a_{1k}|^2}{a_{11}a_{kk}+|a_{1k}|^2} \operatorname{per} A \leq \frac{1}{2} \operatorname{per} A$, and there are $n - 1$ terms. For $n = 2$, $S(A) = |a_{12}|^2$ and $\operatorname{per} A = a_{11}a_{22} + |a_{12}|^2$. For $n = 3$, $S(A) = a_{33}|a_{12}|^2 + a_{22}|a_{13}|^2$, and expanding

$$\begin{aligned}\operatorname{per} A &= a_{11}a_{22}a_{33} + a_{11}|a_{23}|^2 + a_{22}|a_{13}|^2 + a_{33}|a_{12}|^2 + 2\operatorname{Re}(a_{12}a_{23}a_{31}),\\ \det A &= a_{11}a_{22}a_{33} - a_{11}|a_{23}|^2 - a_{22}|a_{13}|^2 - a_{33}|a_{12}|^2 + 2\operatorname{Re}(a_{12}a_{23}a_{31})\end{aligned}$$

gives the identity. □

Proof of Theorem 1.3. Monotonicity is Lemma 2.4(b), and $c_n^{\mathbb{R}} \leq c_n$ by definition.

(i) By Proposition 5.6, $\rho(A) \leq \frac{1}{2}$ for $n \leq 3$, because $\det A \geq 0$. Equality holds for the real matrices $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & c & c \\ c & 1 & 0 \\ c & 0 & 1 \end{smallmatrix}\right)$ with $c = 2^{-1/2}$, which are singular with $a_{23} = 0$.

(ii) is the first statement of Proposition 5.6.

(iii) Take $m = 12$ in Theorem 5.5. Then $\sin(2\pi/12) = \frac{1}{2}$ and $\kappa_{12} = \pi/3$. With $\gamma = \pi$ and $K = 12R$ we get $1 - \kappa_{12}/\gamma = \frac{2}{3}$ and $\rho(A) \geq (H_R/\pi)/(\frac{9}{4} + 1) = \frac{4}{13\pi} H_R$ at order $1 + 24R$. For

$n \geq 25$ let $R = \lfloor (n-1)/24 \rfloor \geq 1$. Monotonicity and $H_R \geq \log(R+1) > \log \frac{n-1}{24}$ give the first bound. For the second, fix $m \geq 3$ and an integer $\lambda \geq 1$, and take $\gamma = 3\kappa_m$ and $K = \lambda m R$ in Theorem 5.5. This gives $\rho(A) \geq \frac{4}{27\kappa_m} H_R \, (1 + \frac{4}{9\lambda})^{-1}$ at order $n_R = 1 + mR(1+\lambda)$. Since $H_R \geq \log R$, $\log n_{R+1} = \log R + O(1)$ as $R \to \infty$, and $c_n \geq c_{n_R}$ for $n_R \leq n < n_{R+1}$, we obtain $\liminf_n c_n / \log n \geq \frac{4}{27\kappa_m}(1 + \frac{4}{9\lambda})^{-1}$. Letting $\lambda \to \infty$ and $m \to \infty$, so that $\kappa_m \to 1$, gives $4/27$. The matrices of Theorem 5.5 have rank two. □

Remark 5.7. For small n the true values exceed the bound of Theorem 5.5 considerably. For instance, the configuration used in the proof of (iii) has $\rho = 0.1614,\ 0.2415,\ 0.2948,\ 0.3347$ for $R = 1, 2, 3, 4$, against the bound $\frac{4}{13\pi} H_R = 0.0979,\ 0.1469,\ 0.1796,\ 0.2040$. At comparable orders, optimized members of the family $A_{m,K}(x, y)$ of Section 3, optimized over x and y, has $\rho = 1.331$ for $(m, K) = (5, 20)$ (order 31), 1.630 for $(10, 20)$ (order 41), 2.016 for $(50, 25)$ (order 126) and 2.272 for $(200, 100)$ (order 501). In the limit $K \to \infty$, a single regular m-gon gives $\sup_{c>0} m \sum_{t=0}^{m-1} (m-1-t)!\, c^t / ((m+1)! + c^m)$, which equals 1.0118, 1.5642 and 2.5211 for $m = 4$, 10 and 160.

6 Further remarks

Remark 6.1 (small orders). Table 1 lists the largest values of ρ found by multistart quasi-Newton maximization (BFGS, with gradients computed from Lemma 2.2). For complex rank two the variables are the Bloch-sphere coordinates of $r_2, \ldots, r_n$, with 30 random starts per order and 60 and 120 starts at orders 10 and 11; at every order the best value was reached by many starts. For complex matrices of rank three (orders 10, 11; 20 starts) and of rank four (orders 8, 9; 6 starts) the searches returned the rank-two values. For real matrices of rank three the table uses 4 to 16 starts per order. At order 20 a random start over real matrices of rank four returned 0.9903, and perturbing the rank-three maximizer into rank four and maximizing again returned it unchanged, 0.99082. In these searches the first violation of (2) occurs at order 12 for complex matrices, where the best value belongs to the configuration $A_{5,1}$ of Remark 3.3, and at order 21 for real matrices. Real matrices of rank two give at most 0.6011 at the even orders $4 \leq n \leq 30$.

n	3	4	5	6	7	8	9	10	11	12
complex rank 2	0.5000	0.5430	0.6270	0.6811	0.7463	0.8023	0.8605	0.9100	0.9542	1.0011

n	19	20	21	22	23	24
real rank 3	0.9616	0.9908	1.0188	1.0475	1.0757	1.1031

Table 1: Largest values of $\rho(A)$ found numerically. The value at $n = 3$ is exact by Theorem 1.3(i).

Remark 6.2 (a Loewner form). Write a PSD matrix with positive definite $B = A(\{1\})$ as $A = \left(\begin{smallmatrix} a & b^* \\ b & B \end{smallmatrix}\right)$, and let $C(B)$ be the permanental cofactor matrix, $C(B)_{lk} = \operatorname{per} B(l \,|\, k)$, where $B(l \,|\, k)$ is B without row l and column k. Expanding per A along its first row and column gives $\operatorname{per} A = a \operatorname{per} B + y^* C(B) y$ and $S(A) = y^* \operatorname{diag}(C(B))\, y$ with $y = \overline{b}$, while $A \succeq 0$ exactly when $a \geq b^* B^{-1} b = y^* \overline{B}^{-1} y$. Hence (2) holds for all A with $A(\{1\}) = B$ if and only if

$$\operatorname{diag} C(B) \;\preceq\; C(B) + \operatorname{per}(B)\, \overline{B}^{-1}.$$

For the matrix A of Theorem 1.1 and small $\varepsilon > 0$, the matrix $A + \varepsilon I$ still violates (2), and $B = A(\{1\}) + \varepsilon I_{11}$ is positive definite; so this Loewner inequality fails for some positive definite B

of order 11. The weaker inequality $C(B) \succeq 0$ holds, since $C(B)$ is the Gram matrix of the products $\prod_{i \neq k} \ell_i$ in the Fischer inner product; this is equivalent to Lieb's inequality $a_{11} \operatorname{per} A(\{1\}) \leq \operatorname{per} A$.

Remark 6.3 (related inequalities). Since the left side of (1) is at least $S(A)^2$, every PSD matrix with $\rho(A) > 1$, in particular those of Theorems 1.1 and 1.2, violates Drury's inequality (1), and Theorem 1.3 shows that the left side of (1) can exceed $(\operatorname{per} A)^2$ by a factor at least of order $(\log n)^2$. The counterexample of Chalmers to (1) satisfies (2), with $\rho = 0.142$, as noted in §1.1. Paksoy's 5×5 correlation matrix [12, eq. (1.3)], with off-diagonal entries $\pm\frac{3}{5}$ in the first row and column and $\pm\frac{3}{20}$ elsewhere, has $\rho = 0.43$ (script `v5_literature_matrices.py`). Zhang observed that (1) implies Chollet's conjecture $\operatorname{per}(A \circ B) \leq \operatorname{per} A \operatorname{per} B$ [3, 15]; the counterexamples of Hutchinson and Chalmers to (1) already close that route [2, §4]. Chollet's inequality compares two matrices, whereas the counterexamples here concern the single-matrix inequality (2).

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