A correction to a correlation-kernel density criterion

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Luminary Oddities

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Abstract

A displayed correlation-kernel characterization of the Laguerre–Pólya class in Dimitrov and Xu, *Wronskians of Fourier and Laplace transforms*, fails even when the Fourier transform has only simple real zeros. We construct an even nonnegative compactly supported smooth probability density whose order-one transform has a pure paired Hadamard product and only simple real zeros, but whose stated correlation kernel has non-dense translates in $L^1(\mathbb{R})$ at the explicit width $1/1000$. The sign change is proved by an analytic bound for the entire infinite product, without numerical evaluation of that product. The underlying distinction is between an outer hyperbolic factor, which receives the real part of an analytic Wronskian, and an inner hyperbolic factor, which receives the mixed Jensen curvature. We prove the corrected density equivalence, including its width-zero simplicity endpoint, under explicit moment and paired-product assumptions. The Jensen and Wiener principles are classical.

Keywords: Fourier transform; translation density; Laguerre–Pólya class; Jensen curvature; correlation kernel.

1 The receiving kernel

We use the Fourier convention

$$F(z) = \widehat{K}(z) = \int_{\mathbb{R}} K(u) e^{-izu}\,\mathrm{d}u, \qquad G_F(z) = F'(z)^2 - F(z)F''(z).$$

For an even real kernel K, put

$$N_K(t) = \int_{\mathbb{R}} (t-2s)^2 K(t-s)K(s)\,\mathrm{d}s, \tag{1}$$

$$C_y(t) = \cosh(yt) N_K(t), \tag{2}$$

$$H_y(t) = \int_{\mathbb{R}} (t-2s)^2 \cosh\big(y(t-2s)\big) K(t-s)K(s)\,\mathrm{d}s. \tag{3}$$

We write $\mathcal{T}(L)$ for the linear span of all real translates of L. Density below refers to the complex Banach space $L^1(\mathbb{R})$; the counterexample also gives failure of density over the real scalars.

Theorem 3.2 of Dimitrov–Xu [1] uses the kernel (2). Its setting includes an even Schwartz kernel, a real entire Fourier transform with zeros in $|\operatorname{Im} z| < \alpha$, and the pure paired Hadamard representation

$$F(z) = F(0) \prod_k \left(1 - \frac{z^2}{z_k^2}\right), \qquad z_k \in \{z : |\operatorname{Im} z| < \alpha\}, \tag{4}$$

with zeros arranged by increasing modulus. The displayed criterion identifies $F \in \mathcal{LP}$ with density of $\mathcal{T}(C_y)$ for every $0 < |y| < \alpha$, and includes a refinement involving simple zeros and $y = 0$. Here $\mathcal{LP}$ denotes the locally uniform limits of real-rooted real polynomials. The counterexample below satisfies these assumptions, is nonnegative, and has *only simple* transform zeros. We address the literal receiving formula and this necessity implication.

The relevant distinction is exact, not asymptotic. Define

$$J_F(x,y) = |F'(x+iy)|^2 - \operatorname{Re}\Big(F(x+iy)\overline{F''(x+iy)}\Big). \tag{5}$$

Jensen's classical curvature identity is

$$\frac{1}{2}\frac{\partial^2}{\partial y^2}|F(x+iy)|^2 = J_F(x,y). \tag{6}$$

The complex Laguerre inequalities and their zero-location significance are treated in [2, 3]; our corrected density theorem uses this classical convexity mechanism together with Wiener's theorem [4].

Proposition 1.1. *Suppose K is real and even and its exponential second moments justify the integrals and differentiations at width $|y|$. Then*

$$\widehat{C}_y(x) = 2\operatorname{Re} G_F(x+iy), \qquad \widehat{H}_y(x) = 2J_F(x,y). \tag{7}$$

Proof. Putting $u = t - s$ and $v = s$ in (1) gives

$$\widehat{N}_K(z) = \iint (u-v)^2 K(u)K(v)e^{-iz(u+v)}\,\mathrm{d}u\,\mathrm{d}v = 2\big(F'(z)^2 - F(z)F''(z)\big).$$

Multiplication by $\cosh(yt)$ averages the shifts $x+iy$ and $x-iy$. Since G_F is real entire, this proves the first identity.

For the second, set $\phi(x) = \operatorname{Re} F(x+iy)$ and $\psi(x) = \operatorname{Im} F(x+iy)$. Evenness gives

$$\widehat{\cosh(y\,\cdot)K}(x) = \phi(x), \qquad \widehat{\sinh(y\,\cdot)K}(x) = i\psi(x).$$

In particular the second transform is imaginary, not ψ itself. The hyperbolic subtraction formula gives

$$H_y = N_{\cosh(y\,\cdot)K} - N_{\sinh(y\,\cdot)K}.$$

Applying the preceding Wronskian calculation to these two kernels yields

$$\widehat{H}_y = 2\{\phi'^2 - \phi\phi'' + \psi'^2 - \psi\psi''\} = 2J_F.$$

The factor i is essential: its square changes the sign of the second Wronskian. Lemma 3.3 and the ensuing correlation calculation in [1] omit that phase for the odd hyperbolic factor. □

2 A simple-zero counterexample

Let p_j be the jth odd prime, and define

$$c_j = \frac{2^{-j}}{\sqrt{p_j}}, \quad m = 10^{13}, \quad \delta = \frac{\sqrt{2}}{m}, \quad b = 1+\delta, \quad y_* = \frac{1}{1000}. \tag{8}$$

As usual, $\operatorname{sinc} z = \sin z/z$ with $\operatorname{sinc} 0 = 1$.

Theorem 2.1. *The entire function*

$$F(z) = \operatorname{sinc} z \operatorname{sinc}(bz) \prod_{j\geq 1} \operatorname{sinc}(c_j z) \tag{9}$$

is the Fourier transform of an even nonnegative $C_c^\infty(\mathbb{R})$ probability density K. Its support is contained in $(-4,4)$, and K is positive throughout the interior of its support. The function F has order one, belongs to $\mathcal{LP}$, possesses (4)*, and every zero of F is real and simple. Nevertheless*

$$\operatorname{Re} G_F(\pi + iy_*) \leq -\frac{1}{512 \cdot 10^6}. \tag{10}$$

The kernel C_{y_*} has a real Fourier zero, and $\mathcal{T}(C_{y_*})$ is not dense in $L^1(\mathbb{R})$.

2.1 Smoothness, the product and global simplicity

Let $U, V, V_1, V_2, \ldots$ be independent uniform random variables on $[-1,1]$. Since

$$S := \sum_{j\geq 1} c_j \leq \frac{1}{\sqrt{3}} < 1, \qquad \sum_{j\geq 1} c_j^2 \leq \frac{1}{9}, \tag{11}$$

the random series $W = \sum_j c_j V_j$ converges absolutely. Let K be the density of $X = U + bV + W$. The first two uniform laws have a continuous convolution density positive on $(-(1+b), 1+b)$. Convolution with the law of W makes K continuous, even, nonnegative, of mass one, and supported on $[-(1+b+S), 1+b+S] \subset (-4,4)$.

The law of W has support exactly $[-S, S]$. For any interior point, sufficiently large finite partial-sum radii contain that point, their finite convolutions have positive probability in its neighborhoods, and the remaining tail radius tends to zero. The endpoints are in the support by taking finitely many uniforms near their common endpoint and then making the tail radius small. It follows that every interior point of the support of K receives a positive contribution from an open set of W values where the first convolution density is positive. Thus K is positive throughout that interior.

Independence and dominated convergence give (9) on the real axis. The product converges normally on complex compact sets, because $\operatorname{sinc}(c_j z) = 1 + O(c_j^2)$ there and $\sum c_j^2 < \infty$. Compact support identifies the entire extensions. For each integer L, retain any L small sinc factors. On the real axis all omitted factors have modulus at most one, so

$$|F(x)| \leq C_L (1 + |x|)^{-L}.$$

Consequently every $x^r F(x)$ is integrable. Fourier inversion and differentiation show $K \in C_c^\infty(\mathbb{R})$, hence K is Schwartz.

Euler's product for each sinc gives a paired real-zero product for F. The sum of reciprocal squared positive zeros is a finite constant times $1 + b^2 + \sum c_j^2$. The aggregate paired product therefore converges normally, may be ordered by increasing zero modulus, and has no additional exponential factor. Finite real-rooted polynomial products approximate it locally uniformly, proving $F \in \mathcal{LP}$. Away from the individual zero sets the normally convergent product is nonzero. Compact support gives order at most one; on the positive imaginary axis $\operatorname{sinc}(iy) = \sinh y / y$ and all other factors are at least one. Thus the order is exactly one.

Distinct sinc factors have disjoint nonzero zero sets when their scale ratio is irrational. All the ratios between distinct members of $\{1, b, c_1, c_2, \ldots\}$ are irrational. This is immediate for $b/1$, $c_j/1$ and c_j/c_k, $j \neq k$. For the remaining ratios, the square of b/c_j is a nonzero rational multiple of

$$b^2 = 1 + \frac{2}{m^2} + \frac{2\sqrt{2}}{m},$$

which is irrational. Thus b/c_j cannot be rational. Every individual sinc has simple nonzero zeros, and only finitely many factors have zeros in any bounded set. Every zero of F is therefore simple. These facts also verify the strip assumption for any $\alpha > 0$, in particular $\alpha = 1/2$.

2.2 An analytic sign bound for the infinite product

The mechanism is visible at a double zero. If a real entire function F_0 has $F_0(a+w) = qw^2 + O(w^3)$ with real $q \neq 0$, then

$$G_{F_0}(a+w) = 2q^2w^2 + O(w^3),$$

so $\operatorname{Re} G_{F_0}(a+iy) < 0$ for all sufficiently small nonzero y. We now pay an explicit width and perturbation while making all zeros simple.

Use the reference

$$F_0(z) = \operatorname{sinc}^2 z \prod_j \operatorname{sinc}(c_j z), \qquad G_0 = G_{F_0}.$$

It is the transform of $X_0 = U + V + W$, with $|X_0| < 3$. At $z = \pi$,

$$F_0(z) = q(z-\pi)^2 + O((z-\pi)^3), \qquad q = \pi^{-2} \prod_j \operatorname{sinc}(\pi c_j) > \frac{1}{16}. \tag{12}$$

Indeed $0 \le \pi c_j < 1$. The bounds $\operatorname{sinc} t \ge 1 - t^2/6$ on $[0,1]$, (11), and $\prod(1-a_j) \ge 1 - \sum a_j$ for $a_j \in [0,1]$ give

$$\prod_j \operatorname{sinc}(\pi c_j) \ge 1 - \frac{\pi^2}{54}, \qquad q \ge \frac{1}{\pi^2} - \frac{1}{54} > \frac{1}{10} - \frac{1}{54} > \frac{1}{16}.$$

In particular the reference zero is exactly double.

Differentiation gives

$$G_0(\pi) = G_0'(\pi) = 0, \qquad G_0''(\pi) = 4q^2,$$
$$G_0^{(4)} = 2(F_0''')^2 + F_0'' F_0^{(4)} - 2F_0' F_0^{(5)} - F_0 F_0^{(6)}.$$

The compact probability representation yields $|F_0^{(k)}(\pi + it)| \le 3^k e^{3|t|}$. For $|t| \le y_*$,

$$|G_0^{(4)}(\pi + it)| \le 6 \cdot 3^6 e^{6y_*} < 12 \cdot 3^6.$$

Taylor's integral remainder along the vertical segment, whose odd terms have zero real part, therefore gives

$$\begin{aligned} \operatorname{Re} G_0(\pi + iy_*) &\le -2q^2y_*^2 + \frac{729}{2} y_*^4 \\ &\le -\frac{y_*^2}{128} + \frac{729}{2} y_*^4 \le -\frac{y_*^2}{256}. \end{aligned} \tag{13}$$

Here $e^{6/1000} < 2$ and the final inequality are elementary; the arithmetic is rational apart from the already paid bounds in (12).

For $1 \le \beta \le b$, let F_β be the transform of $X_\beta = U + \beta V + W$, and $G_\beta = G_{F_\beta}$. At $z = \pi + iy_*$ we have $|z| < 4$, $|X_\beta| < 4$ and $e^{4y_*} < 2$. Differentiating the compact probability integral in β gives

$$|\partial_\beta F_\beta| \le 8, \quad |\partial_\beta F_\beta'| \le 34, \quad |\partial_\beta F_\beta''| \le 144. \tag{14}$$

For derivative order k, a bound for its differentiated integrand is

$$(k4^{k-1} + 4 \cdot 4^k)e^{4y_*},$$

with the first term absent for $k = 0$; this proves (14). Also $|F_\beta| \le 2$, $|F'_\beta| \le 8$ and $|F''_\beta| \le 32$. Hence

$$|\partial_\beta G_\beta| \le 2 \cdot 8 \cdot 34 + 8 \cdot 32 + 2 \cdot 144 = 1088 < 1200.$$

Since $F_b = F$, $F_1 = F_0$, and $\delta < 2/10^{13}$,

$$|G_F(z) - G_0(z)| \le 1200\delta < \frac{2400}{10^{13}} < \frac{y_*^2}{512}.$$

Combining this with (13) proves (10).

2.3 A bounded annihilator

At $x = 0$ the sign is positive. Write $M(y) = F(iy) > 0$, and let $\mathrm{Var}_y(X)$ denote variance under exponential tilting of the density K by $e^{yu}/M(y)$. Direct differentiation gives

$$G_F(iy) = M(y)^2 \,\mathrm{Var}_y(X) > 0.$$

The tilted density is nondegenerate. By Proposition 1.1, the real continuous function $x \mapsto \widehat{C}_{y_*}(x)$ is positive at zero and negative at π. It has a zero $x_* \in (0, \pi)$. The nonzero bounded functional

$$\ell(f) = \int_{\mathbb{R}} f(t) e^{-ix_* t} \,\mathrm{d}t$$

annihilates every translate of C_{y_*}. Thus the translates are not dense in $L^1(\mathbb{R})$. Since C_{y_*} is even, the real cosine functional also annihilates its real translates. This completes the proof of Theorem 2.1.

3 The corrected density equivalence

For completeness we state the receiving criterion with (3). The moment assumptions below make every Fourier operation explicit. They hold for all compact kernels in Section 2 and for the Riemann theta kernel.

Theorem 3.1. *Let K be a nonzero even nonnegative $L^1(\mathbb{R})$ density and let $\alpha > 0$. Assume that for every $0 \le Y < \alpha$,*

$$\int_{\mathbb{R}} (1 + t^2) e^{Y|t|} K(t) \,\mathrm{d}t < \infty. \tag{15}$$

Assume $F = \widehat{K}$ extends to a real entire function, all its zeros lie in $|\operatorname{Im} z| < \alpha$, and it has the pure paired representation (4) with $\sum_k |z_k|^{-2} < \infty$. Then the following are equivalent:

1. *$F \in \mathcal{LP}$;*
2. *$\mathcal{T}(H_y)$ is dense in $L^1(\mathbb{R})$ for every $0 < |y| < \alpha$;*
3. *$J_F(x, y) > 0$ for every $x \in \mathbb{R}$ and $0 < |y| < \alpha$.*

Moreover, density for every $|y| < \alpha$, including $y = 0$, is equivalent to $F \in \mathcal{LP}$ and simplicity of all its real zeros.

Proof. First, (15) gives $H_y \in L^1$ at every indicated width: after putting $u = t - s$, the absolute integrand is bounded by a constant times

$$(1+u^2)(1+s^2)e^{|y|(|u|+|s|)}K(u)K(s).$$

It also justifies Proposition 1.1. Since K is a nonzero nonnegative density, the integral of H_y is strictly positive: the set $u = s$ has two-dimensional measure zero, and the remaining integrand is positive wherever both kernel factors are positive. Thus $\widehat{H}_y(0) > 0$.

For a real-rooted polynomial p, at any fixed real x the function $A_p(y) = |p(x+iy)|^2$ is a positive constant times a product of factors $(x-r)^2 + y^2$. Its coefficients as a polynomial in y^2 are nonnegative. Hence $A_p'' \geq 0$ and $yA_p' \geq 0$. Locally uniform polynomial approximation passes both inequalities to LP functions, including their derivatives.

Suppose (1) holds. The Riemann–Lebesgue lemma excludes a constant nonzero F; the pure product then contains a real zero r. Write $F(z) = (z-r)F_1(z)$, with $F_1 \in \mathcal{LP}$. For $y \neq 0$, $A_1(y) := |F_1(x+iy)|^2$ is strictly positive, and

$$\frac{\partial^2}{\partial y^2}|F(x+iy)|^2 = 2A_1(y) + 4yA_1'(y) + \big((x-r)^2 + y^2\big)A_1''(y) > 0.$$

The last two terms are nonnegative. Equation (6) proves (3). The usual closure of LP under division by a real linear zero factor also follows directly from its real canonical product.

Wiener's L^1 Tauberian theorem says that translates of an L^1 kernel are dense exactly when its Fourier transform has no real zeros. Thus (3) implies (2) by (7).

Conversely assume (2). For each nonzero indicated y, Wiener gives a nowhere-zero $\widehat{H}_y$. It is continuous and real because H_y is real and even. Its positive value at zero fixes the sign on the connected real axis, so $J_F(x,y) > 0$ everywhere. By continuity $J_F(x,0) \geq 0$. For each real x, the even function $A_x(y) = |F(x+iy)|^2$ is consequently convex on $(-\alpha,\alpha)$ and nondecreasing for $y \geq 0$. If it vanished at $0 < y_0 < \alpha$, it would vanish on $[0,y_0]$, forcing the entire function F to vanish identically. This contradicts $F(0) = \int K > 0$. Reflection handles negative y. All zeros in the given strip are real; by the pure paired product they are all the zeros, and the product of real-rooted partial polynomials places F in LP. This proves (1).

At $y = 0$, $\widehat{H}_0 = 2(F'^2 - FF'')$. For a nonconstant LP function with the stated pure product, the logarithmic derivative gives $F'^2 - FF'' > 0$ away from its real zeros: it is F^2 times the strictly positive sum of reciprocal squared distances to those zeros. At a simple zero it equals $F'^2 > 0$, while at a multiple zero it vanishes. A final application of Wiener gives exactly the endpoint assertion. □

Corollary 3.2. *For the explicit kernel of Theorem 2.1, $\mathcal{T}(H_y)$ is dense in $L^1(\mathbb{R})$ for every real y, whereas $\mathcal{T}(C_{1/1000})$ is not dense.*

Proof. The compact support supplies all exponential moments and the zeros are real and simple. Apply Theorem 3.1 with arbitrarily large finite α. □

4 Scope and exact arithmetic

The counterexample concerns a general correlation criterion and its simple-zero refinement. It neither supplies a negative Jensen curvature for the actual Riemann theta transform nor determines its zero distribution. The corrected theorem isolates the appropriate receiving object; establishing its universal sign for actual theta remains a different mathematical problem. Nonnegativity of H_y as a kernel only makes its Fourier transform positive definite; it does not make that transform pointwise nonnegative.

The supplementary standard-library program `verify_counterexample.py` checks the four formal Wronskian derivatives and the rational inequalities used in the explicit sign margin. It uses exact fractions and explicit exception gates, and does not evaluate or truncate the infinite product. Those finite controls support normalization and arithmetic; the analytic arguments above establish smoothness, global simplicity, infinite-product convergence and non-density.

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