Concavity of Lattice Counts on Toric Threefolds

Scott Schubr

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Abstract

The fourth root of the ordinary Riemann–Roch count polynomial is concave on the entire closed nef cone of every smooth toric Fano threefold. The exponent $1/4$ is uniformly optimal for these real polynomial extensions. Nine of the eighteen threefolds have concave cube root; the other nine have genuine cube-root failure. We prove a sufficient criterion for cubic full-count polynomials using classical Lorentzian volume theory and verify its hypotheses by exact fan and polynomial certificates. The result includes integral nonbig nef classes. For the split bundles $\mathbf{P}_{\mathbf{P}^2}(\mathcal{O} \oplus \mathcal{O}(a))$, the ordinary polynomial has optimal power $1/3$ at twists $a = 0, 1, 2$ and no positive concave power at any integer twist $a \geq 3$. For geometrically weighted lattice counts, biases $r \geq 1$ admit a nonpolynomial concave logarithmic extension for every twist, whereas every fixed $0 < r < 1$ has integral midpoint counterexamples at large twists. The proof uses sharp coefficient conditions for general quadratic prefix weights and uniform rectangular-moment tail identities. It also gives exponential-quadratic extensions. No positive power works uniformly in the unweighted twist family, or at any fixed bias $r > 1$ over prefixes. Ordinary-polynomial optimality and lattice-only sharpness are kept separate.

1 Results and context

For a smooth projective toric threefold X, write K_X for its canonical class. On its real nef cone consider the ordinary polynomial

$$F_X(D) = \frac{D^3}{6} + \frac{(-K_X)D^2}{4} + \frac{\big((-K_X)^2 + c_2(X)\big)D}{12} + 1. \tag{1}$$

Riemann–Roch and toric nef vanishing identify this with $h^0(X, \mathcal{O}_X(D))$ for integral nef D; equivalently it counts lattice points of the associated polytope [4]. Our proofs below also give boundary count agreement by mixed Ehrhart polynomiality, without using continuity of lattice counts.

Theorem 1.1 (Sharp uniform Fano power)**.** *For every smooth projective toric Fano threefold, F_X is positive and $F_X^{1/4}$ is concave on* $\mathrm{Nef}(X)$*, including its boundary. No exponent greater than* $1/4$ *works uniformly for these ordinary real polynomials. Of the eighteen isomorphism types, precisely the nine marked with degree* 3 *in Table 1 have concave cube root; each of the other nine fails cube-root concavity.*

In particular, integral nef divisor families with integral weighted mean satisfy fourth-root Jensen, and hence logarithmic Jensen. The uniform optimality assertion concerns the ordinary real polynomial, not the best exponent when only lattice values and integral means are tested.

There is a complementary infinite family. Fix $a \geq 0$ and put

$$N_a(k, y) = \frac{k+1}{12}\big[6(y+1)(y+2) + 3a(2y+3)k + a^2k(2k+1)\big], \quad k \in \mathbb{Z}_{\geq 0}, \quad y \geq 0. \tag{2}$$

For integral a, k, y this is the lattice/section count of $\mathbf{P}_{\mathbf{P}^2}(\mathcal{O} \oplus \mathcal{O}(a))$ in nef coordinates. Denote its ordinary real polynomial by $F_a(x, y)$.

Theorem 1.2 (Split bundles and sharp geometric bias)**.** *For integer $a = 0, 1, 2$, $F_a^{1/3}$ is concave on the nef quadrant and $1/3$ is its optimal power. For each integer $a \geq 3$, no positive power of F_a, nor $\log F_a$, is concave there. For $r > 0$ define*

$$Z_{a,r}(k, y) = \sum_{i=0}^{k} r^i \binom{y + ai + 2}{2}.$$

Every $r \geq 1$ and real $a \geq 0$ admit a finite concave $J_{a,r}$ on $\mathbb{R}^2_{\geq 0}$ with $J_{a,r}(k, y) = \log Z_{a,r}(k, y)$ for every integer $k \geq 0$ and real $y \geq 0$. Hence finite and countable first-moment laws satisfy logarithmic Jensen whenever the first-coordinate mean is integral. The logarithmic expectation is finite. Conversely, for each fixed $0 < r < 1$, integer-height midpoint Jensen fails at $k = 1$ for all sufficiently large integer twists a. No positive power works uniformly over twists when $r = 1$, and each fixed $r > 1$ fails every positive power for sufficiently large integer prefix parameters even at $a = 0$.

Thus failure of the preferred ordinary polynomial need not obstruct exact logarithmic lattice Jensen, even with every nondecreasing geometric slice bias. A claim that every fixed twist lacks all positive lattice powers would be stronger than our final assertion and is not made.

The smooth toric Fano classification is classical [2, 3]. The eighteen fans, Riemann–Roch formula, Brion enumeration [5], mixed Ehrhart polynomiality [6], and Lorentzian characterization [1] are inputs, not new results here. Our contribution is a coefficient criterion and its classification-wide count-concavity application, together with the precise distinction between polynomial and nonlocal lattice concavity in the split family. The method arose from companion work on low-dimensional unimodular fiber counts [11]; the present paper treats toric nef cones.

Eur–Larson's Lorentzian result [7, Section 3] uses a signed rising-binomial coefficient transformation, homogenization and factorial normalization in special matroid geometry. It is different from (1). Broad smooth-polytope failures are already known: Ferroni [8, Section 5] constructs smooth nine-dimensional polytopes with $E(1)E(3) > E(2)^2$. Our three-dimensional split-bundle threshold neither preempts that discrete result nor follows from it.

2 A cubic Lorentzian sufficient criterion

We first isolate the coefficient bounds needed for the count polynomial. This is a sufficient consequence of classical Lorentzian theory, not a characterization of all count polynomials.

Proposition 2.1 (Cubic full-count criterion)**.** *Let C be a finitely generated cone, modulo translation lineality, with a Minkowski-additive family of convex polytopes $P_a \subset \mathbb{R}^3$. Let E be its linear span, $V(a) = \mathrm{vol}(P_a)$ its cubic volume polynomial, and choose $K \in C$ with P_K full-dimensional. Put $Q = \frac{1}{2} D_K V$, $B = \operatorname{Hess} Q$, and $F = V + Q + L + 1$, where $L = \ell$ is linear and all terms are invariant under the factored translations. Choose cone generators representing nonpoint polytopes after removing translation directions, and assume L is positive on every generator. For an integer $d \geq 3$, suppose:*

1. *B has at most one positive eigenvalue on E, ℓ belongs to its image, and, for any solution $Bu = \ell$, $\rho = \ell(u) \geq d/(d-1)$;*

2. *for every generator a, writing $H_a = \operatorname{Hess} V(a)$ by cubic polarization, H_a has at most one positive direction on E and*

$$L(a) \le \frac{d-2}{4(d-1)} K^T H_a K.$$

Then $F^{1/d}$ is concave on C. More generally the algebraic conclusion holds if nonnegative coefficients, M-convex support and these quadratic-index hypotheses are imposed directly on the construction below, without a polytope interpretation.

Proof. In generator coordinates $a(y) = \sum_j y_j a_j$, form

$$H_d(y,t) = t^{d-3} V(a(y)) + t^{d-2} Q(a(y)) + t^{d-1} L(a(y)) + t^d.$$

Mixed volumes give nonnegative coefficients of V and Q, and positivity of L gives the remaining ones. The support equals that of $t^{d-3}\operatorname{vol}(P_{a(y)} + tP_K)$: cubic terms agree, quadratic terms differ by a positive scalar because $Q = \frac{1}{2} D_K V$, linear coefficients in both are positive for every nonpoint generator against two full-dimensional P_K factors, and constants are positive. This also covers segments and polygons. Volume-polynomial support is M-convex [1, Theorem 4.1 and Corollary 4.2], meaning that its exponent set satisfies symmetric exchange. A monomial factor translates that set, so preserves M-convexity. If B is singular, ℓ annihilates its kernel because $Bu = \ell$; hence ρ does not depend on the chosen solution.

All order-$(d-2)$ partials are classified by their number k of shape derivatives. For $k = 0$, dividing by $(d-2)!$ gives

$$Q + (d-1)tL + \frac{d(d-1)}{2} t^2 = \frac{1}{2}(a + (d-1)tu)^T B(a + (d-1)tu) + \frac{d-1}{2}[d - (d-1)\rho]t^2.$$

For $k = 1$ in a generator direction a_1, division by $(d-3)!$ gives

$$D_{a_1} V + (d-2)t D_{a_1} Q + \frac{(d-1)(d-2)}{2} t^2 L(a_1) = \frac{1}{2}\left(a + \frac{d-2}{2} tK\right)^T H_{a_1} \left(a + \frac{d-2}{2} tK\right) + t^2 \left[\frac{(d-1)(d-2)}{2} L(a_1) - \frac{(d-2)^2}{8} K^T H_{a_1} K\right].$$

Here $H_{a_1} K = 2Ba_1$. Both scalar remainders are nonpositive by hypothesis, so both quadratics have at most one positive direction; no inverse of H_{a_1} is used. For $k = 2$ (only if $d \ge 4$), division by $(d-3)!$ gives $tD_{a_1} D_{a_2} V + (d-2)t^2 B(a_1, a_2)/2$, a product of t and a nonnegative linear form. For $k = 3$ (only if $d \ge 5$) the derivative is $(d-3)! t^2 D_{a_1} D_{a_2} D_{a_3} V/2$; for $k \ge 4$ it is zero. Repeated indices are allowed. When $d = 3$ just the first two classes occur. A homogeneous nonnegative polynomial with M-convex support is Lorentzian when all derivatives of order two below its degree are zero or quadratics with at most one positive Hessian eigenvalue [1, Theorem 2.25]. The preceding cases verify exactly this test. Homogeneous power concavity [1, Proposition 2.33] then applies, and setting $t = 1$ proves the result. Nonunique generator representations and boundary points are handled by interpolation and continuity directly. □

For a genuine Minkowski volume polynomial the index hypotheses come from Alexandrov–Fenchel, also for boundary generators by continuity.

3 The eighteen Fano nef cones

We describe the finite certificate completely enough to reconstruct it. The file `fan_data.json` supplies the eighteen arrays of primitive normals $n_i \in \mathbb{Z}^3$ and maximal triples. They were extracted from the public Fanography table [10], which links GRDB records. These are secondary input data; their labels are only convenient names. The proof of coverage below validates eighteen pairwise distinct smooth Fano fans and then uses the classical count-eighteen theorem. It does not assume that secondary labels certify completeness. The standalone `verify_fano.py` performs all finite operations with integers and rational numbers, with checks retained under optimized Python execution.

3.1 Bounded smooth fans and classification coverage

For each input array form $P_{\mathbf{1}} = \{z : n_i \cdot z \le 1\}$. Enumerate every nonsingular triple I and its solution $v_I = N_I^{-1}\mathbf{1}$. Retain it when all inequalities hold. Every retained triple has determinant ± 1, exactly three active inequalities, and the full retained list equals the input maximal triples. Moreover, each of the six signed coordinate unit vectors has an explicitly computed nonnegative conic representation in three input normals. Hence the normals positively span $\mathbb{R}^3$, so $P_{\mathbf{1}}$ is bounded. These certificates avoid inferring boundedness from Euler characteristic alone. Its complete normal fan is smooth, and the all-one support class is strictly convex, hence anticanonical ample. Using outward rather than inward fan normals is harmless after a global reflection. The polar is a smooth Fano polytope in the standard lattice.

The inputs are pairwise lattice-inequivalent. Indeed fix a unimodular maximal cone in the first fan. Any equivalence must take its ordered basis to an ordered maximal cone of the other fan. For every target cone and each of its six orderings, that choice determines the unique possible integral unimodular transformation. None maps the first ray set onto the second. Unequal ray cardinalities exclude the remaining pairs. This tests all $\binom{18}{2} = 153$ pairs exhaustively. Watanabe–Watanabe and Batyrev classify exactly eighteen smooth toric Fano threefolds [2, 3]; therefore these eighteen validated inequivalent fans cover that class.

3.2 Complete nef generators

For an array with m rays, quotient support parameters by translations by setting the three bounds in its first unimodular maximal cone to zero. The remaining $r = m - 3$ bounds are integral Picard coordinates. If v_0 solves that first cone's all-one equalities, the anticanonical vector is $K_i = 1 - n_i \cdot v_0$ on the free indices. For neighboring maximal cones I, J, the vertex $N_I^{-1}h_I$ must satisfy the additional J inequality. Primitive-normalizing these rows gives the wall inequalities.

These walls define the full closed nef cone, not just a necessary subcone: the linear functions prescribed on maximal cones agree on shared faces; each wall inequality is their local convexity condition. Along a generic line crossing the complete fan all slope jumps are upward, so the resulting homogeneous piecewise-linear function is globally convex. Nongeneric lines follow by continuity. It is therefore a support function, and its polytope is the convex hull of the linear vertex formulas. Conversely convex support functions satisfy every wall. Thus support functions add throughout this closed cone and

$$P_{h+h'} = P_h + P_{h'}.$$

The vector K satisfies every wall strictly, proving full dimension. If both h and $-h$ satisfy the walls, every slope jump vanishes, so the support function is linear. The gauge then gives $h = 0$, proving pointedness. A zero-dimensional polytope likewise represents the zero class.

In a full-dimensional pointed rational cone of dimension r, an extreme ray is cut out by $r-1$ independent active walls. Enumerate all such subsets, compute their one-dimensional kernels by cofactors, retain feasible orientations, and primitive-normalize. A nonzero cofactor certifies independence. This produces the entire generator list, including the six rays of each nonsimplicial rank-five cone; no assumption of simpliciality is used. Full vertex-feasibility inequalities are also checked at these rays.

3.3 Exact polynomial identities and boundary counts

For every smooth maximal triple I, the vertex is $N_I^{-1}h_I$ and its primitive tangent directions are the columns of $-N_I^{-1}$. Choose an integral objective ξ nonorthogonal to each direction d_{Ij}. Brion's theorem [5] identifies the count polynomial with the constant coefficient in t of

$$\sum_I e^{t\langle \xi, N_I^{-1}h_I\rangle} \prod_{j=1}^{3} (1-e^{t\langle \xi, d_{Ij}\rangle})^{-1}.$$

The Laurent expansion $(1-e^x)^{-1} = -x^{-1}+1/2-x/12+O(x^3)$ and the exponential determine every coefficient of degrees zero through three. Repeating the full reconstruction with two independent generic objectives gives the same polynomial coefficientwise. The output decomposes as $F = V+Q+L+1$, where V is volume, $Q = \frac{1}{2}D_K V$ and $L = \ell$. The identity for Q is verified by differentiating every cubic monomial; the constant is exactly one. This is not interpolation from sample counts.

For integral strict nef parameters the formula is the actual lattice count, since every tangent cone is unimodular. To treat an integral boundary class a, choose an integral strict class v in the same nef cone. All vertex formulas are integral, and Minkowski additivity gives $P_{pa+qv} = pP_a + qP_v$. Mixed Ehrhart polynomiality [6, Theorem 2.1] applies for all nonnegative integers p, q, including zero and lower-dimensional summands. For $p \geq 0, q \geq 1$ the count agrees with $F(pa+qv)$ by the strict formula. Polynomial identity gives agreement at $(p,q) = (1,0)$ as well. There is no appeal to continuity of lattice counts. Direct enumeration at $K, 2K$, every primitive nef ray and K plus every ray supplies 152 additional finite consistency checks.

3.4 The uniform certificate and sharpness

The exact quadratic matrix $B = \operatorname{Hess} Q$ has inertia $(1, r-1, 0)$ in every case. Solve $Bu = \ell$ and put

$$s = \ell^T B^{-1}\ell, \qquad R = \max_j \frac{L(a_j)}{K^T \operatorname{Hess} V(a_j) K},$$

where a_j are the complete primitive nef generators. All numerator and denominator values are positive. The table supplies the exact scalar bounds; the accompanying checker reconstructs the full matrices, cubics, walls and rays rather than trusting this table. Every row has $s \geq 4/3$ and $R \leq 1/6$, exactly the $d = 4$ thresholds of Proposition 2.1. Mixed volumes supply its index hypotheses. Consequently $H_4 = tV + t^2Q + t^3L + t^4$ is Lorentzian and $F^{1/4}$ is concave. Nonnegative generator coefficients imply $F \geq 1$ on the whole closed nef cone. The nine degree-three rows also have $s \geq 3/2$ and $R \leq 1/8$, giving cube-root concavity.

For the converse statements, at the origin $F = 1$, $\nabla F = \ell$ and $\operatorname{Hess} F = B$, whence

$$\alpha^{-1} \operatorname{Hess}(F^\alpha)(0) = B - (1-\alpha)\ell\ell^T.$$

Family	r	rays	s	R	least criterion degree
1-17	1	1	121/72	11/96	3
2-36	2	2	71/45	1/8	3
2-35	2	2	113/72	1/8	3
2-33	2	2	14/9	11/90	3
2-34	2	2	14/9	1/8	3
3-31	3	3	83/54	1/8	3
3-29	3	3	307/216	3/22	4
3-30	3	3	383/252	1/8	3
3-27	3	3	3/2	1/8	3
3-28	3	3	3/2	1/8	3
3-26	3	3	13/9	3/22	4
4-12	4	4	131/90	1/7	4
3-25	3	3	131/90	11/84	4
4-11	4	4	298/207	1/7	4
4-10	4	4	10/7	1/7	4
4-9	4	4	394/279	1/7	4
5-2	5	6	4/3	1/6	4
5-3	5	6	4/3	1/6	4

Table 1: Exact certificates on the eighteen nef cones. Family names follow Fanography; coverage uses validated inequivalence and the classical classification, independently of those names.

Its value on $u = B^{-1}\ell$ is $s-(1-\alpha)s^2$. Every degree-four row has $s < 3/2$, so actually fails cube-root concavity. The last two rows have $s = 4/3$, ruling out every $\alpha > 1/4$ uniformly. Although zero is a cone vertex, the positive curvature persists at εK for sufficiently small positive ε; this is an interior point admitting a short feasible two-sided segment. Thus the curvature test is a genuine concavity obstruction.

An intrinsic upper-bound example avoids quotient coordinates altogether. For $X = \mathbf{P}^1 \times S_6$, where S_6 is the degree-six toric del Pezzo surface, restrict to $\mathcal{O}_{\mathbf{P}^1}(x)$ and $y(-K_{S_6})$. The anticanonical hexagon has area three and six boundary lattice points, so Pick's theorem gives

$$F(x,y) = (x+1)(3y^2+3y+1).$$

At zero the gradient is $(1,3)$ and the Hessian is $\left(\begin{smallmatrix} 0 & 3 \\ 3 & 6 \end{smallmatrix}\right)$. The determinant of $(1-\alpha)\nabla F \nabla F^T - \operatorname{Hess} F$ is $3 - 12\alpha$. It is negative for $\alpha > 1/4$ and remains so nearby in the positive quadrant. This proves the optimal uniform power in Theorem 1.1.

4 Split bundles: the ordinary polynomial

Consider, for an integer $a \geq 0$, the polytopes

$$R_a(x,y) = \{(s,u,v) : 0 \leq s \leq x,\ u,v \geq 0,\ u+v \leq y+as\}, \qquad x,y \geq 0.$$

For nonnegative integer x, y their counts are

$$\begin{aligned} N_a(x,y) &= \sum_{i=0}^{x} \binom{y+ai+2}{2} = F_a(x,y), \\ F_a(x,y) &= (x+1)\left[\frac{(y+1)(y+2)}{2} + \frac{a(2y+3)x}{4} + \frac{a^2x(2x+1)}{12}\right]. \end{aligned} \tag{3}$$

This is also the section-count polynomial for $X_a = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(a)) \to \mathbf{P}^2$ in nef coordinates (x, y). We use the quotient convention $\mathbf{P}(E) = \operatorname{Proj} \operatorname{Sym}(E)$, for which $\pi_* \mathcal{O}(x) = \operatorname{Sym}^x E$ [9]. The monomial summands give (3) directly. The nef quadrant follows from global generation and restriction to a fiber and to the quotient section $E \to \mathcal{O}$. For $x, y > 0$ there are six vertices; at each, one of $\pm e_s$ and two of $-e_u, -e_v, (-a, 1, 1)$ are active outward normals. Their determinants have magnitude one. Thus for integral parameters these are smooth three-dimensional lattice polytopes for every twist.

Proposition 4.1 (Split-bundle dichotomy)**.** *The ordinary real polynomial extension F_a has optimal concavity power $1/3$ if $a = 0, 1, 2$. For every integer $a \geq 3$, no positive power of F_a is concave on the nef quadrant; $\log F_a$ is not concave there either. These negative assertions do not decide integral-mean Jensen for the lattice values N_a.*

Proof. Summing i and i^2 in (3) gives the homogeneous parts

$$\begin{aligned} V &= a^2x^3/6 + ax^2y/2 + xy^2/2, \\ Q &= a(a+3)x^2/4 + (a+3)xy/2 + y^2/2, \\ L &= (1 + 3a/4 + a^2/12)x + 3y/2. \end{aligned}$$

The polytopes are Minkowski-additive on their fixed nef cone. For $K = (2, 3 - a)$ direct differentiation gives $Q = \frac{1}{2} D_K V$ and

$$B = \begin{pmatrix} a(a+3)/2 & (a+3)/2 \\ (a+3)/2 & 1 \end{pmatrix}, \qquad \det B = (a+3)(a-3)/4.$$

For $a = 0, 1, 2$, K is interior and the two index conditions hold by mixed volumes. The values needed in Proposition 2.1 are

a	ρ	$L(e_x)/(K^T H_{e_x} K)$	$L(e_y)/(K^T H_{e_y} K)$
0	14/9	1/9	1/8
1	113/72	11/96	1/8
2	71/45	17/150	1/8

They satisfy $\rho \geq 3/2$ and both ratios at most $1/8$, exactly the $d = 3$ bounds. This proves cube-root concavity. Every interior ray has positive cubic leading volume, so any power greater than $1/3$ has positive second derivative sufficiently far along that ray.

For the negative assertion set $S = a^2 + 9a + 12$. At the origin $F_a = 1$, $\nabla F_a = (S/12, 3/2)$, and its perpendicular vector $v = (3/2, -S/12)$ satisfies

$$v^T \operatorname{Hess} F_a(0) v = \frac{a^4 + 51a^2 - 504}{144} > 0 \qquad (a \geq 3).$$

The numerator is 36 at $a = 3$ and strictly increases for $a > 0$. For every fixed $\alpha > 0$, the gradient-square term in $v^T \operatorname{Hess}(F_a^\alpha)(0) v$ vanishes. The remaining curvature is strictly positive; by continuity it stays positive at a nearby interior point, disproving concavity along a sufficiently short feasible segment. The identical tangent argument applies to $\log F_a$. At $a = 3$, a rational interior witness has center $(1/100, 1/100)$ and endpoints $(13/1000, 1/500)$, $(7/1000, 9/500)$; the product of endpoint values minus the squared central value is $3372033985551/4000000000000000000 > 0$. □

At $a = 3$, $K = (2, 0)$ is the nef and big anticanonical direction, yet B has rank one and $\ell = (4, 3/2)$ does not belong to its image. Thus the range condition is substantive; weak-Fano positivity alone does not imply the ordinary-polynomial conclusion. Smoothness is also weaker than unimodularity of the fixed normal configuration: the maximal minor with rows $-e_u, -e_v, (-a, 1, 1)$ has determinant $-a$. For $a \geq 2$ its magnitude exceeds one, invariant under a unimodular change of lattice basis. The twists $a \geq 3$ are not Fano, so the obstruction is outside Theorem 1.1.

The fractional witness concerns the ordinary polynomial; clearing its denominators changes an inhomogeneous function and does not produce a lattice Jensen violation. We now prove that every twist nevertheless has a suitable concave logarithmic extension through its lattice rows.

5 Weighted arithmetic prefixes and nonlocal Jensen

For a geometric bias $r > 0$ and a positive increasing quadratic or affine polynomial g, put

$$Q_r(k, y) = \sum_{i=0}^{k} r^i g(y + ai), \qquad a, y \geq 0.$$

Theorem 5.1 (Sharp quadratic input criterion)**.** *Let $g(z) = Az^2 + Bz + C$ with $A, C > 0$, $B \geq 0$ and $B^2 \geq 2AC$. For every $r \geq 1$, every row $\log Q_r(k, y)$ is concave, and for $k \geq 1$, $u, v \geq 0$,*

$$Q_r\left(k, \frac{u+v}{2}\right)^2 \geq Q_r(k-1, u) Q_r(k+1, v).$$

There is a finite concave extension of $\log Q_r$ touching every entire integer first-coordinate row. Finite weighted Jensen, and countable first-moment Jensen, follow whenever the first-coordinate mean is integral. The input condition $B^2 \geq 2AC$ is necessary for such a real-row contact extension, already at $k = 0$. For affine $g(z) = Bz + C$, $B \geq 0$, $C > 0$, all the positive conclusions hold for every $r > 0$.

The toric multiplicity polynomial $(z+1)(z+2)/2$ satisfies the criterion. The condition is weaker than real negative roots, which would require $B^2 \geq 4AC$. Necessity here concerns continuous row contact.

5.1 Rows and reduction to a middle strip

Set $w_i = r^i$, $W = \sum_{i=0}^{k} w_i$, $\mu = W^{-1} \sum_i i w_i$, $\nu = W^{-1} \sum_i i^2 w_i$. After dividing Q_r by W, its row coefficients are

$$A, \qquad B' = B + 2aA\mu, \qquad C' = C + aB\mu + a^2 A\nu.$$

For $r \geq 1$ the weights are nondecreasing. Pairing i with $k - i$ gives $\mu \geq k/2$, while $i^2 \leq ki$ gives $\nu \leq k\mu \leq 2\mu^2$. Therefore

$$(B')^2 - 2AC' = (B^2 - 2AC) + 2aAB\mu + 2a^2A^2(2\mu^2 - \nu) \geq 0.$$

The logarithmic second derivative of $Ay^2 + B'y + C'$ has numerator $2AC' - (B')^2 - 2AB'y - 2A^2y^2 \leq 0$. This also holds for $k = 0$, when $\mu = \nu = 0$. For $A = 0$ every row is positive affine, for any positive bias. If g is nonconstant its row derivative is positive, and g'/g is positive and decreasing. Conversely at $k = 0, y = 0$ the curvature numerator is $2AC - B^2$; if positive it stays positive nearby in the interior, precluding a concave extension touching that entire real row.

To prove the mixed neighboring-row inequality, fix $w = (u + v)/2$, $\delta = (u - v)/2$ and consider

$$L(\delta) = \log Q_r(k - 1, w + \delta) + \log Q_r(k + 1, w - \delta), \quad |\delta| \leq w.$$

This is concave by row concavity. Each prefix score is an $r^i g$-weighted average of the decreasing score of g. Adding the two largest arguments lowers that average, so $L'(0) \geq 0$. If $w \geq a$, at $\delta = a$ the left argument list $w+a, \ldots, w+ka$ is contained in the right list $w-a, \ldots, w+ka$; its two extra smaller arguments have larger scores. After the index shift, their geometric weights agree up to one common factor on each list. Thus $L'(a) \leq 0$. The maximum lies in $[0, \min(a, w)]$. It suffices to treat $0 \leq u - v \leq 2a$. For $a = 0$, the maximum is at zero and the conclusion follows directly from log-concavity of g. For $a > 0$ normalize by a and write $u = b + 2t, v = b$, with $b \geq 0$, $0 \leq t \leq 1$; the coefficients of g rescale and still satisfy $B^2 \geq 2AC$; the geometric bias remains fixed.

5.2 Uniform rectangular-moment sectors

Lemma 5.2. *For integers $k \geq 1$, $q \geq 0$, $p \in \{0, 1, 2\}$, and $b \geq 0$, $0 \leq t \leq 1$, the following difference is nonnegative:*

$$M_{p,q} = \sum_{i,j=0}^{k} [(b+t+i)(b+t+j)]^p (2b+2t+i+j)^q - \sum_{i=0}^{k-1} \sum_{j=0}^{k+1} [(b+2t+i)(b+j)]^p (2b+2t+i+j)^q.$$

All upper diagonal tails with $q = 0$ are also nonnegative. There is no upper bound on q.

Proof. Let $D_s^{(p)}$ be the target-minus-source contribution with $q = 0$ on the diagonal $i + j = s$, and let $T_S^{(p)} = \sum_{s=S}^{2k} D_s^{(p)}$. Both products on a diagonal have the same argument sum $A_s = 2b + 2t + s$. Summation by parts shows that nonnegative upper tails imply all q:

$$M_{p,q} = T_0^{(p)} A_0^q + \sum_{s=1}^{2k} T_s^{(p)} (A_s^q - A_{s-1}^q) \geq 0.$$

For $q = 0$ take each $A_s^0 = 1$, also if $A_s = 0$. There is a useful lift in b as well:

$$\partial_b D_s^{(p)} = p A_s D_s^{(p-1)}.$$

If the $p - 1$ upper tails are nonnegative, another summation by parts on any truncated tail makes $\partial_b T_S^{(p)} \geq 0$. Thus it suffices to prove the p tails at $b = 0$ once the previous sector is established.

For $p = 0$ all diagonal cardinalities agree except for one extra target pair at $s = k$. For $p = 1$ direct arithmetic sums give

$$D_s^{(1)} = \begin{cases} (s+1)t^2, & s < k, \\ b^2 + b(k+2t) + (k+1)t^2, & s = k, \\ (2k-s+1)(1-t)^2, & s > k. \end{cases}$$

All are nonnegative. For example on a lower diagonal the pointwise product difference is $t(i-j)+t^2$, and the antisymmetric term sums to zero. The central diagonal has the extra pair; a unit index shift gives the upper-diagonal formula.

It remains to prove the $p = 2$ tails at $b = 0$. For $S \geq k+1$, put $r = 2k - S$, $x = k + t$, $d = 1 - t$. Reversing indices at the upper corners expresses that tail as

$$\sum_{I,J \geq 0,\ I+J \leq r} \left([(x-I)(x-J)]^2 - [(x-d-I)(x+d-J)]^2 \right).$$

Write $n = (r+1)(r+2)/2$. On this triangle,

$$\sum I = \sum J = nr/3, \quad \sum I^2 = \sum J^2 = nr(r+1)/6, \quad \sum IJ = nr(r-1)/12.$$

Expanding and canceling antisymmetric terms gives exactly

$$nd^2\left[2x^2 - \frac{4r}{3}x - \frac{2r}{3} - d^2\right].$$

Here $x = r + 1 + K + t$ for $K = k - r - 1 \geq 0$, and the bracket becomes

$$\frac{2}{3}r^2 + \left[2 + \frac{8}{3}(K+t)\right]r + 1 + 4K + 6t + 2K^2 + 4Kt + t^2 \geq 0.$$

For lower cutoffs $S \leq k$, put $R = S - 1$, so $-1 \leq R \leq k - 1$. The lower-diagonal identity is $D_s^{(2)} = (s+1)t^2[t^2 + s(2t-1)]$ for $s < k$. Its prefix through R equals

$$P_R = \frac{(R+1)(R+2)}{2}t^2\left[t^2 + \frac{2R}{3}(2t-1)\right],$$

including $P_{-1} = 0$. These individual diagonals can be negative, so retaining the compensating upper tails is essential. Let $U(z) = z(z+1)(2z+1)/6$. The full $p = 2$ gap is

$$F(k,t)^2 - L(k,t)U(k+1), \qquad F(k,t) = (k+1)\left[t^2 + kt + \frac{k(2k+1)}{6}\right],$$
$$L(k,t) = k\left[4t^2 + 2(k-1)t + \frac{(k-1)(2k-1)}{6}\right].$$

Subtract P_R to obtain $T_S^{(2)}$. Write its Bernstein expansion as

$$T_S^{(2)} = \sum_{j=0}^{4} \beta_j \binom{4}{j} t^j (1-t)^{4-j}.$$

The coefficients are

$$\beta_0 = k(k+1)(2k^2+2k-1)/6, \qquad \beta_1 = k(k+1)^2(k+2)/6,$$
$$\beta_2 = [k(k+1)(k+2)^2 + R(R+1)(R+2)]/18, \quad \beta_3 = 0,$$
$$\beta_4 = U(k+1) - U(R+1).$$

They are all nonnegative on the stated integer domain. In particular the cubic R factor is zero at $R = -1, 0$ and nonnegative thereafter, and U increases at the relevant nonnegative integers. These formulas follow by expansion of the displayed gap; the checker verifies the entire polynomial identity, not selected parameter values. This proves all tails for $p = 2$ at $b = 0$. The b-lift and final summation by parts prove the lemma for every b and every q. □

5.3 Mixed rows and a concave logarithmic extension

For two arguments U, V, abbreviate $P = UV$, $S = U + V$. Direct expansion gives

$$g(U)g(V) = A^2P^2 + ABPS + ACS^2 + (B^2 - 2AC)P + BCS + C^2.$$

All coefficients are nonnegative under the input criterion. On diagonal $s = i + j$, the geometric factor is r^s in both rectangles. The upper-tail proof of Lemma 5.2 applies by summation by parts

to $r^s(2b+2t+s)^q$, which is nonnegative and increasing for $r \geq 1$. This proves the mixed inequality in the middle strip; the score argument covers all u, v. If $A = 0$, only sectors $p = 0, 1$ occur, and their individual diagonals are nonnegative. Multiplication by any positive r^s preserves that sign, so every $r > 0$ works in the affine case. No series approximation or degree cutoff is required.

For $q > 0$ define $M_k(q) = \sup_{y \geq 0} Q_r(k, y)e^{-qy}$ and $m_k(q) = \log M_k(q)$. Suprema are finite and attained. The mixed inequality with independent second coordinates gives $M_{k-1}(q)M_{k+1}(q) \leq M_k(q)^2$. Let $\phi_q(x)$ be the concave piecewise-linear interpolation of this logarithmic sequence and set

$$J(x, y) = \inf_{q>0} \{\phi_q(x) + qy\}.$$

An infimum of concave functions is concave. Also $\phi_q \geq \log g(0)$, so $J \geq \log g(0)$, and choosing one q proves finiteness above. For nonconstant g, at each integer row choose the supporting slope $q = (\log Q_r(k, y))' > 0$. Row concavity, including the right derivative at zero, gives exact contact $J(k, y) = \log Q_r(k, y)$; the reverse inequality follows directly from the supremum definition. $g = C$ constant is handled by putting $G_k(r) = \sum_{i=0}^{k} r^i$. The identity $G_k^2 - G_{k-1}G_{k+1} = r^k$ gives concavity of its logarithmic interpolation ϕ. Explicitly $J(x, y) = \log C + \phi(x)$ works; equivalently take $q \downarrow 0$ in the transform. This handles the constant polynomial factor without assuming a supporting slope above zero.

Finite Jensen follows from exact contact whenever the first-coordinate mean is integral. For countable first-moment laws, group each remaining tail at its conditional mean, apply finite concavity to $J - \log g(0) \geq 0$, drop its nonnegative tail term and pass to the limit. This proves both the inequality and logarithmic integrability.

5.4 A sharp classification of geometric bias

For the toric multiplicities define

$$Z_{a,r}(k, y) = \sum_{i=0}^{k} r^i \binom{y + ai + 2}{2}, \qquad r > 0.$$

Theorem 5.1 proves the positive part of Theorem 1.2 for every $r \geq 1$ and every real step $a \geq 0$. For a fixed $0 < r < 1$, use $k = 1$ and integral heights $0, 1, 2$. Direct expansion gives

$$Z_{a,r}(1, 0)Z_{a,r}(1, 2) - Z_{a,r}(1, 1)^2 = \frac{r(1-r)a^2 - 5r(1+r)a - 6(1+r)^2}{2}. \tag{4}$$

Its leading coefficient is positive, so it is positive for all sufficiently large integer a. This is an actual finite weighted-count Jensen failure with integral midpoint divisor parameters. Thus $r \geq 1$ is exactly the universal positive-bias class on the domain $r > 0$. For example $a = 20, r = 1/4$ gives values $235/4, 265/4, 75$ and gap $275/16$. These are the actual section multiplicities with the indicated slice weights, not a counterexample to the unweighted theorem.

The logarithm cannot be uniformly replaced by a positive power. For each fixed $r > 1$, already $a = 0, y = 0$ gives $Z_{0,r}(k, 0) = G_k(r)$ and, for every $\alpha > 0$,

$$\frac{G_{k-1}(r)^\alpha + G_{k+1}(r)^\alpha}{2G_k(r)^\alpha} \longrightarrow \frac{r^{-\alpha} + r^\alpha}{2} > 1.$$

Thus even that fixed bias eventually fails power midpoint Jensen in the integer prefix coordinate. Every fixed proper quadratic also fails for $0 < r < 1$ and sufficiently large integer steps: write $Q_r(1, y) = A_1y^2 + B_1y + C_1$, where $A_1 = A(1 + r)$, $B_1 = B(1 + r) + 2rAa$, $C_1 = C(1 + r) +$

$rBa + rAa^2$. The integer-height gap is $2A_1C_1 - (A_1 + B_1)^2$, with leading coefficient $2r(1-r)A^2$ in a. Thus the affine exception is substantive. The $r = 1$ toric family has the separate all-twist power obstruction given below. These elementary growth observations are limitations, not new geometric-series principles.

Corollary 5.3 (Exponential quadratic weights)**.** *Let $h(t) = e^{\lambda t}g(t)$, where g is a quadratic satisfying Theorem 5.1, or a positive affine polynomial, and $\lambda \geq 0$. Then $\sum_{i=0}^{k} h(y + ai)$ has the same row-contact concave logarithmic extension and finite/countable Jensen conclusions for every $a \geq 0$. No negative λ can be admitted uniformly over all such proper quadratics and all real steps. Affine factors work for every real rate.*

Proof. Put $r = e^{\lambda a} \geq 1$. The count is $e^{\lambda y}Q_r(k, y)$; add the affine function λy to its logarithmic extension. For $\lambda < 0$, choose $a = \log 4/(-\lambda)$, $g(t) = (t + a/10)^2$ and $k = 1$. Then $r = 1/4$ and the polynomial factors of the row at $y = 0, a/10, a/5$ are $5a^2/16, 2a^2/5, 41a^2/80$. Their endpoint-product gap is $a^4/6400 > 0$; multiplication by $e^{\lambda a/5}$ preserves its sign. The affine case follows for all real rates from its all-bias result; adding the integrable linear term λY preserves the countable conclusion. Thus the negative-rate converse is a uniform statement over the proper-quadratic class. □

A unit-step integral subcase is classical even for $0 < r < 1$. With integral roots and heights, introduce anchors (x_0, x_1, x_2) and integer states v, w_j subject to $x_1 \leq v \leq x_0$ and $v \leq w_j \leq x_2 + c_j - 1$. These fixed difference constraints define an L-convex set with finite fibers. The weight $r^{x_2 - v}$ is logmodular and invariant under common translation. Speyer's weighted marginal theorem [12, Lemma 5.3] gives L-log-concavity; at $(0, -k, y)$ the marginal is $r^y Q_r(k, y)$, whose affine logarithmic factor can be removed. Thus integral-mean logarithmic Jensen in that unit-step integral setting is prior work for every $r > 0$. Scaling the lattice model to real steps or roots introduces a congruence restriction on the summation state; the present extension treats real parameters and classifies all twists.

The summation-by-parts mechanism is also classical: partial-diagonal positivity followed by an Abel transfer appears in Johnson–Goldschmidt [13]. Their array conditions differ from the rectangular upper-tail identities proved here; we use the elementary mechanism.

5.5 No uniform positive power in the unweighted family

For completeness the unweighted bias $r = 1$ also has no positive power uniformly over twists, using actual integer parameters. On the fixed row $k = 1$ the three counts at $y = 0, 1, 2$ are

$$\frac{a^2 + 3a + 4}{2}, \qquad \frac{a^2 + 5a + 12}{2}, \qquad \frac{a^2 + 7a + 24}{2}.$$

Fix $\alpha > 0$ and write $C = a^2/2$. Taylor expansion at $a^{-1} = 0$ with uniform remainder over these three values gives

$$N_a(1, y)^\alpha = C^\alpha \left[1 + \frac{\alpha b_y}{a} + \frac{\alpha c_y + \alpha(\alpha - 1)b_y^2/2}{a^2} + O_\alpha(a^{-3})\right],$$

where $b_y = 2y + 3$ and $c_y = 2y^2 + 6y + 4$. The second differences of c_y and b_y^2 are 4 and 8, so

$$N_a(1, 0)^\alpha + N_a(1, 2)^\alpha - 2N_a(1, 1)^\alpha = C^\alpha \left[\frac{4\alpha^2}{a^2} + O_\alpha(a^{-3})\right] > 0$$

for all sufficiently large integer twists depending on α. Hence no fixed positive exponent works uniformly over this family. This does not rule out a positive exponent depending on a fixed twist. For example $a = 13$ already refutes fourth-root Jensen with counts $106, 123, 142$. Exact integer fourth powers certify

$$3208680^4 < 106 \cdot 10^{24}, \quad 3452010^4 < 142 \cdot 10^{24}, \quad 3330246^4 > 123 \cdot 10^{24}.$$

Since $3208680 + 3452010 - 2 \cdot 3330246 = 198 > 0$, $106^{1/4} + 142^{1/4} > 2 \cdot 123^{1/4}$. All midpoint parameters are integral, and the three polytopes are full-dimensional. Logarithmic Jensen remains consistent: $106 \cdot 142 = 15052 < 15129 = 123^2$.

6 Scope and reproducibility

The Fano theorem concerns a fixed toric threefold's full nef cone and its ordinary count polynomial, with sharpness uniform over the eighteen types. The split-family theorem supplies a different concave logarithmic extension, with a sharp universal geometric-bias classification, even when the ordinary polynomial has no positive concave power. The quadratic input criterion includes complex-root polynomials; affine factors admit all positive biases.

The input file and the two standard-library checkers specify all finite data and identities. They use exact arithmetic and explicit checks retained with Python optimization. Finite count replays are consistency checks; the analytic Minkowski, Lorentzian, boundary and transform arguments above supply the all-parameter conclusions.

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References

[1] P. Brändén and J. Huh, *Lorentzian polynomials*, Ann. Math. **192** (2020), 821–891. Author manuscript.

[2] K. Watanabe and M. Watanabe, *The classification of Fano 3-folds with torus embeddings*, Tokyo J. Math. **5** (1982), 37–48. doi:10.3836/tjm/1270215033.

[3] V. V. Batyrev, *Toroidal Fano 3-folds*, Math. USSR-Izv. **19** (1982), 13–25. doi:10.1070/IM1982v019n01ABEH001404.

[4] D. A. Cox, J. B. Little and H. K. Schenck, *Toric Varieties*, Graduate Studies in Mathematics 124, American Mathematical Society, 2011.

[5] M. Brion, *Points entiers dans les polyèdres convexes*, Ann. Sci. École Norm. Sup. **21** (1988), 653–663. doi:10.24033/asens.1572.

[6] C. Haase, M. Juhnke-Kubitzke, R. Sanyal and T. Theobald, *Mixed Ehrhart polynomials*, Electron. J. Combin. **22** (2015), Paper 1.47. Author manuscript.

[7] C. Eur and M. Larson, *K-theoretic positivity for matroids*, Algebr. Geom. **13** (2026), 241–265. doi:10.14231/AG-2026-008.

[8] L. Ferroni, *Unimodality shenanigans in Ehrhart theory*, arXiv:2609.10513v1 (2026). arxiv.org/abs/2609.10513.

[9] The Stacks Project Authors, *The Stacks Project*, Section 27.21, Example 27.21.3. Tag 01OA.

[10] *Fanography: Toric Fano threefolds*, public eighteen-family fan table and linked Graded Ring Database records, accessed October 2026. fanography.info/toric/.

[11] S. Schubr, *Power Concavity of Low-Dimensional Unimodular Fiber Counts*, public edition 2 (2026). Published companion.

[12] D. E. Speyer, *L-log-concavity and a proof of the conjecture of Lam, Postnikov and Pylyavskyy*, arXiv:2601.05007v2 (2026). arxiv.org/abs/2601.05007.

[13] O. Johnson and C. Goldschmidt, *Preservation of log-concavity on summation*, ESAIM Probab. Stat. **10** (2006), 206–215. doi:10.1051/ps:2006008.