Compact metric graphs with constant mean distance function are loops

Scott Schubr
Luminary Oddities

November 26, 2026

Abstract

Baptista, Kennedy and Mugnolo, in their 2024 paper on mean distance on metric graphs, wrote that examples and intuition suggest that loops should be the only compact metric graphs whose mean distance function is constant; this function assigns to a point the average of its distances to all points of the graph. We prove this statement: a compact connected metric graph, with loops, multiple edges and vertices of degree two allowed, has constant mean distance function if and only if it is isometric to a circle. The proof starts from an exact formula for the one-sided derivatives of the mean distance function: in each direction, the derivative is the total length minus twice the length of the points reached by shortest paths leaving in that direction. Constancy is therefore a continuum form of the distance-balanced property of discrete graphs, imposed at every interior point of every edge. On a short segment, this balance fixes the distribution of the difference of the distances to the two ends. It follows that every interior point of an edge has a unique antipode, the two points cut the graph into halves of equal length, and the antipodal map pairs the maximal arcs between branch points isometrically into two-edge cuts. An ear decomposition then yields an arc that lies in no two-edge cut, so the graph has no branch points. For discrete graphs, constant total distance holds for every vertex-transitive graph; for metric graphs, the circle is the only example.

Keywords. Metric graph; quantum graph; mean distance; average distance; distance-balanced graph; self-median graph; two-edge cut; ear decomposition.
MSC 2020. Primary 05C12; Secondary 51K05, 54E45, 05C40, 81Q35.

1 Introduction

1.1 The question

Let G be a compact connected metric graph: finitely many edges, each identified with a compact interval of positive length, glued at their endpoints, with the shortest-path distance d and the length measure dy; loops and multiple edges are allowed. Write $L = |G|$ for the total length. Baptista, Kennedy and Mugnolo [2, Definition 2.1] introduced the *mean distance function*

$$\rho_G(x) = \frac{1}{|G|} \int_G d(x, y)\, dy, \qquad x \in G, \tag{1}$$

and the mean distance $\rho(G)$, the mean value of ρ_G over G. On the loop (circle) C of length L the function ρ_C is constant, equal to $L/4$ [2, Example 4.3]. In Section 6 of [2], in the paragraph following the proof of Lemma 6.1, the authors write that large classes of examples and some intuition suggest that

"loops should be the only graphs $\mathcal{G}$ with the property that $\rho_G(x)$ is constant"

(independent of x), and they leave the statement without proof. The statement is a stated expectation, not a numbered conjecture or problem; the same sentence appears in the published version [2] (Section 6, p. 20 of 25) and in arXiv version 2 (p. 16). This paper proves it.

1.2 Main result

A *loop* of length L is a metric graph isometric to the circle $\mathbb{R}/L\mathbb{Z}$ with its arclength metric; equivalently, every point has degree two, and after suppressing all vertices but one the graph consists of one vertex and one loop edge. At a point $x \in G$ the *directions* are the germs of edges leaving x; their number is the degree of x (two at interior points of edges, and the number of edge ends at a vertex, a loop counting twice). For a direction η at x let

$$A_\eta(x) = \{y \in G : \text{ some shortest path from } x \text{ to } y \text{ leaves } x \text{ in direction } \eta\}. \tag{2}$$

Theorem 1.1. *Let G be a compact connected metric graph with at least one and finitely many edges, each of positive length, loops, multiple edges and vertices of degree two allowed, and let $L = |G| > 0$. The following are equivalent:*

(i) the mean distance function ρ_G is constant;

(ii) $|A_\eta(x)| = L/2$ for every point $x \in G$ of degree two and both directions η at x;

(iii) G is a loop.

In this case $\rho_G \equiv L/4$, and the identity in (ii) holds at every point of G and in every direction.

Theorem 1.1 proves the expectation of [2, Section 6] for every graph in the setting of [2, Section 2]. Condition (ii) is a balance condition at every interior point of every edge (Corollary 2.4); its discrete counterpart is discussed in Section 5.1. Since $\rho(G)$ is the mean value of the continuous function ρ_G, Theorem 1.1 has the following consequence.

Corollary 1.2. *If G is a compact connected metric graph that is not a loop, then*

$$\min_{x \in G} \rho_G(x) < \rho(G) < \max_{x \in G} \rho_G(x).$$

In particular, the set of points minimizing ρ_G (the median set of G) is all of G exactly when G is a loop.

In the language of distance equilibrium measures on compact metric spaces [8], Theorem 1.1 says that the normalized length measure of a compact metric graph has constant distance potential $x \mapsto \int_G d(x,y)\, dy/L$ exactly when the graph is a loop.

1.3 Relation to earlier work and route of the proof

Baptista, Kennedy and Mugnolo [2, Theorem 3.2] proved that $\rho(G) \le L/3$, with equality only for the interval, and that $\rho(G) \le L/4$ for doubly connected graphs, with equality only for the loop. The second bound comes from the pointwise inequality $\rho_G(x) \le L/4$ for doubly connected G [2, (6.2)], a comparison with the constant function ρ_C of the loop. Mean distance on continuous graphs has also been studied computationally [5].

For a finite connected graph, the total distance $\sum_w d(v,w)$ of a vertex v is called its transmission. The graphs in which it is the same for every vertex are called self-median. A graph is distance-balanced if, for every edge uv, as many vertices are closer to u than to v as the reverse [6]. Balakrishnan, Changat, Peterin, Špacapan, Šparl and Subhamathi [1] observed that a connected graph is distance-balanced if and only if it is self-median; see also [4, Section 1]. Every vertex-transitive graph is self-median. Lemma 2.3 below is the continuum form of the identity behind this equivalence: the one-sided derivative of $L\rho_G$ in a direction equals the length of the points closer behind minus the length of the points closer ahead. In a metric graph this balance is required at every point of every edge, and Theorem 1.1 shows that this singles out the circle.

The proof of (ii)⇒(iii) has three steps.

1. Balance excludes cut points of degree two and loop edges at vertices of degree at least three (Lemma 3.1). So if G is not a loop, its *core* H (vertices of degree at least three, edges the arcs between them) is a loopless bridgeless multigraph of minimum degree at least three.

2. On a short segment $e = [v, w]$ of an edge, balance determines the distribution of $\varphi = d(w,\cdot) - d(v,\cdot)$ on the rest of G: it is uniform with density $1/2$ on $(-\ell, \ell)$, with atoms of mass $(L-2\ell)/2$ at $\pm\ell$ (Lemma 3.2). From this, every interior point x of an edge has a unique *antipode* x^* such that $G \setminus \{x, x^*\}$ consists of two components of length $L/2$. The antipodal map pairs each core edge f isometrically with a core edge $\sigma(f) \neq f$, and $\{f, \sigma(f)\}$ is a two-edge cut of H (Proposition 3.4).

3. The last ear of an ear decomposition of H is a single edge that lies in no two-edge cut (Lemma 4.1), which contradicts step 2.

Section 5 contains examples and remarks: vertex-transitive equilateral graphs, whose mean distance function is constant on the vertex set and not on the edges, and loops with a short pendant edge or a short parallel edge, which show that the scale-invariant oscillation $(\max \rho_G - \min \rho_G)/L$ over non-loops admits no positive lower bound depending only on the combinatorial type.

2 Preliminaries

Definition 2.1. A *compact metric graph* G is a finite connected multigraph with at least one edge, loops allowed, each edge f of which is identified with an interval $[0, m_f]$, $m_f > 0$, the endpoints being glued according to the incidences. The distance $d(x, y)$ is the infimum of the lengths of paths from x to y, and dy is the length measure. For a measurable set $S \subseteq G$ we write $|S|$ for its length, and $L = |G|$. We put

$$F(x) = F_G(x) = \int_G d(x,y)\, dy = L\, \rho_G(x).$$

For background on metric graphs see [3]. Inserting or deleting vertices of degree two changes neither d nor the length measure, so all statements below concern the metric measure space G. Thus $L > 0$. We call G *balanced* if $|A_\eta(x)| = L/2$ for every point $x \in G$ of degree two and both directions η at x, with $A_\eta(x)$ as in (2); Lemma 2.3 shows that $A_\eta(x)$ is a Borel set.

Since $|d(x,y) - d(x',y)| \le d(x,x')$, the function F is Lipschitz with constant L. Shortest paths exist because G is compact and has finitely many edges.

Lemma 2.2. *Let f be an edge with endpoints a, b (possibly $a = b$) and length m, let $f(t)$, $0 \le t \le m$, be the point at arclength t from a, and let $y \in G$ lie outside the open edge f°. Then*

$$d(f(t), y) = \min\{t + d(a,y),\ m - t + d(b,y)\}.$$

In particular, on every edge of G the function $d(p,\cdot)$, for a point p outside the open edge, is the minimum of two affine functions of arclength with slopes $+1$ and -1.

Proof. A path from $f(t)$ to $y \notin f^\circ$ leaves f° through a or through b, after a length of at least t, respectively $m-t$. The two candidate paths along f followed by a shortest path realize the bound. The second statement follows by exchanging the roles of the two points. □

Lemma 2.3. *Let $x \in G$, let η be a direction at x, and let x_δ be the point at distance δ from x along η. Then the one-sided derivative*

$$D_\eta F(x) = \lim_{\delta \downarrow 0} \frac{F(x_\delta) - F(x)}{\delta}$$

exists and equals $L - 2|A_\eta(x)|$.

Proof. Let e be the edge that carries η. Choose $\delta_1 > 0$ smaller than half the length of every edge and smaller than the arclength along e from x to the next vertex in direction η (for a loop e at a vertex x this arclength is m_e). For $0 < t \le \delta_1$ the segment $[x, x_t]$ of e is the unique shortest path from x to x_t: every other path from x to x_t reaches x_t from the side opposite to x, so it traverses all of e except the open segment between x and x_t, and has length at least $m_e - t > t$. Hence the points x_t, $0 < t \le \delta_1$, lie in $A_\eta(x)$.

Fix $y \neq x$. If $y \in A_\eta(x)$, a shortest path from x to y leaving in direction η passes through x_δ for $0 < \delta < \min\{\delta_1, d(x,y)\}$, so $d(x_\delta, y) = d(x,y) - \delta$. If $y \notin A_\eta(x)$, put $c(y) = \delta_1 + d(x_{\delta_1}, y) - d(x,y)$. The segment $[x, x_{\delta_1}]$ followed by a shortest path from x_{δ_1} to y is a path leaving x in direction η, so $c(y) > 0$. For $0 < \delta < \min\{\delta_1, c(y)/2\}$, every path from x_δ to y passes through x or through x_{δ_1}, because y lies outside the open arc between them. Therefore

$$d(x_\delta, y) \ge \min\{\delta + d(x,y),\ \delta_1 - \delta + d(x_{\delta_1}, y)\} = d(x,y) + \min\{\delta,\ c(y) - \delta\} = d(x,y) + \delta,$$

and the triangle inequality gives equality. Thus the continuous functions $y \mapsto (d(x_\delta, y) - d(x,y))/\delta$ converge pointwise on $G \setminus \{x\}$ to a function g equal to -1 on $A_\eta(x) \setminus \{x\}$ and to $+1$ on $G \setminus A_\eta(x)$. In particular $A_\eta(x) \setminus \{x\} = \{g = -1\}$ is a Borel set. The difference quotients are bounded by 1 in absolute value, and dominated convergence gives $D_\eta F(x) = |G \setminus A_\eta(x)| - |A_\eta(x)| = L - 2|A_\eta(x)|$. □

Corollary 2.4. *The function ρ_G is constant if and only if G is balanced, and in that case $|A_\eta(x)| = L/2$ for every point $x \in G$ and every direction η at x. If G is balanced and x is a point of degree two with directions $\eta_\pm$, then $|A_{\eta_+}(x) \cap A_{\eta_-}(x)| = 0$.*

Proof. If F is constant, all one-sided derivatives vanish at every point, and Lemma 2.3 gives $|A_\eta(x)| = L/2$ for every x and η; in particular G is balanced. Conversely, if G is balanced, then by Lemma 2.3 on each edge the Lipschitz function $t \mapsto F(f(t))$ has vanishing one-sided derivatives at every interior point, hence vanishing derivative almost everywhere, and so it is constant on each edge. By continuity and connectedness F is constant. For the second statement, every $y \neq x$ is reached by a shortest path, which leaves x in direction η_+ or η_-; hence $A_{\eta_+}(x) \cup A_{\eta_-}(x) = G$, and $|A_{\eta_+} \cap A_{\eta_-}| = L/2 + L/2 - L = 0$. □

Example 2.5. On the loop C of length L, every point x has $d(x,\cdot)$ ranging over $[0, L/2]$ with two preimages for almost every value, so $F_C(x) = 2\int_0^{L/2} t\,dt = L^2/4$ and $\rho_C \equiv L/4$. This is the implication (iii)⇒(i) of Theorem 1.1, already recorded in [2, Example 4.3].

3 Local structure of balanced graphs

Throughout this section G is balanced. A point x of degree two is a *cut point* if $G \setminus \{x\}$ is disconnected.

Lemma 3.1. *Let G be balanced.*

(a) No point of degree two is a cut point.

(b) If $C \subseteq G$ is a simple closed curve and $u \in C$ is a point such that every point of $C \setminus \{u\}$ has degree two in G, then $G = C$, and G is a loop.

(c) If G is not a loop, then G has no point of degree one and at least one point of degree at least three. Let H be the multigraph whose vertices are the points of degree at least three and whose edges (the core edges*) are the closures of the components of the complement of these points. Then H is connected, has no loops, has no bridges, and has minimum degree at least three.*

Proof. (a) Let x have degree two and suppose $G \setminus \{x\}$ is disconnected. Every component has x in its closure, since G is connected, and x has two branches, so there are exactly two components P and Q, each containing one branch. A shortest path from x to a point of P stays in $P \cup \{x\}$, so $A_\eta(x) = P \cup \{x\}$ for the direction η into P, and balance gives $|P| = |Q| = L/2$. Let $x' \in P$ be the point at a small distance $\delta > 0$ from x on the same edge, with $\delta < L/2$. The open arc between x and x' is the only connection between x and P, so the component of $G \setminus \{x'\}$ containing x is $Q \cup \{x\} \cup (x, x')$, of length $L/2 + \delta$. The other component, the part of P beyond x', is nonempty. Hence x' is a cut point of degree two whose complement has a component of length $L/2 + \delta$. The first part of the argument, applied to the cut point x', shows that both components of $G \setminus \{x'\}$ have length $L/2$, a contradiction.

(b) Let $c = |C|$ and let $x \in C$ be at arclength $t \in (0, c/2)$ from u along C. Every path from a point of C to a point of $G \setminus C$ passes through u, and every path between two points of C that leaves C leaves and returns through u, so it is no shorter than an arc of C. Hence C with the metric d is a circle of length c, and the shortest path from x to u is the arc of length t. Let η be the direction at x toward u along this arc. Then $A_\eta(x)$ contains $G \setminus C$ and meets C in a closed half-circle of length $c/2$. Balance gives $L/2 = |A_\eta(x)| = (L - c) + c/2$, so $c = L$. The set $G \setminus C$ is open of length zero, hence empty, so $G = C$ is a loop.

(c) If p has degree one, the interior points of the edge at p are cut points, contradicting (a). If every point has degree two, then G is a compact connected one-dimensional manifold, hence a circle, and with its arclength metric a loop. Otherwise the set Z of points of degree at least three is finite and nonempty. Each component of $G \setminus Z$ consists of points of degree two, so it is an open arc or a circle. A circle component would be open and closed in G, hence equal to G, which is impossible since $Z \neq \emptyset$. So each component is an open arc of finite length whose ends converge to points of Z, and H is a well-defined multigraph. A core edge whose two ends lie at the same vertex u is a closed curve as in (b), which would make G a loop; so H has no loops. A bridge of H would consist of cut points of degree two, contradicting (a). Connectedness of H follows from that of G, and the degree condition holds by construction. □

For the rest of this section we assume moreover that G is not a loop, so the core H of Lemma 3.1(c) is defined.

Lemma 3.2. *Let f be a core edge of length m, let $e = [v, w] \subseteq f^\circ$ be a closed segment of length $\ell \leq m/2$, and let x be its midpoint. Put $\Gamma = G \setminus e^\circ$ and $\varphi(y) = d(w, y) - d(v, y)$ for $y \in \Gamma$. Then:*

(a) $|\{\varphi \le \tau\}| = (L - \ell + \tau)/2$ *and* $|\{\varphi = \tau\}| = 0$ *for* $-\ell < \tau < \ell$, *and* $|\{\varphi = \ell\}| = |\{\varphi = -\ell\}| = (L - 2\ell)/2$; *here the sets are subsets of* Γ.

(b) The set $\Omega = \{y \in \Gamma : |\varphi(y)| < \ell\}$ *is an open arc of length* ℓ *consisting of points of degree two. Its closure has two distinct endpoints* p_+, p_-, *and* φ *is affine of slope* 2 *along* Ω, *from* $\varphi(p_-) = -\ell$ *to* $\varphi(p_+) = \ell$.

(c) $\Gamma \setminus \Omega$ *has exactly two components,* $X = \{\varphi = \ell\} \ni v, p_+$ *and* $Y = \{\varphi = -\ell\} \ni w, p_-$, *each of length* $(L - 2\ell)/2$.

(d) Let x^* *be the midpoint of* Ω. *Then* $G \setminus \{x, x^*\}$ *has exactly two components, each of length* $L/2$.

Proof. By Lemma 3.1(c), f is not a loop, so a path from v to w other than the segment e runs through both ends of f and has length at least $m - \ell \ge \ell$. Hence $d(v, w) = \ell$, the segment e is a shortest path, and $|\varphi| \le \ell$ on Γ. The set Γ is connected. Indeed, if $\Gamma = P \sqcup Q$ with P, Q disjoint, closed and nonempty, then each of P and Q contains one of v, w, since a part containing neither would be open and closed in the connected space G. Then $G \setminus \{x\} = (P \cup [v, x)) \sqcup (Q \cup (x, w])$ would be disconnected, contradicting Lemma 3.1(a).

(a) Let $x_s \in e$ be the point at distance $s \in (0, \ell)$ from v, and put $\tau = 2s - \ell$. For $y \in \Gamma$, Lemma 2.2 gives $d(x_s, y) = \min\{s + d(v, y),\ \ell - s + d(w, y)\}$, and a path leaving x_s toward w has length at least $\ell - s + d(w, y)$. Hence a shortest path from x_s to y leaves toward w exactly when $\varphi(y) \le \tau$, and toward v exactly when $\varphi(y) \ge \tau$. For $y \in e$, the shortest path is the subsegment of e, which leaves toward w exactly when y lies beyond x_s. So, for the directions η_w and η_v at x_s toward w and v,

$$|A_{\eta_w}(x_s)| = (\ell - s) + |\{\varphi \le \tau\}|, \qquad |A_{\eta_v}(x_s)| = s + |\{\varphi \ge \tau\}|.$$

Balance makes both equal to $L/2$. Since $|\Gamma| = L - \ell$, adding the two identities gives $|\{\varphi = \tau\}| = 0$, and the first gives $|\{\varphi \le \tau\}| = L/2 - \ell + s = (L - \ell + \tau)/2$. Letting $\tau \downarrow -\ell$ in the first formula and $\tau \uparrow \ell$ in $|\{\varphi \ge \tau\}| = (L - \ell - \tau)/2$ gives the two atoms, by continuity of measure along the decreasing families $\{\varphi \le \tau\}$ and $\{\varphi \ge \tau\}$.

(b) By Lemma 2.2, φ is piecewise affine on every edge of Γ, with slopes in $\{-2, 0, 2\}$. Subdivide so that φ is affine on every edge. A piece of slope 0 inside Ω would give a value $\tau \in (-\ell, \ell)$ with $|\{\varphi = \tau\}| > 0$, which (a) excludes, so $|\varphi'| = 2$ almost everywhere on Ω. For $\tau \in (-\ell, \ell)$ outside the finite set of values of φ at vertices, let $n(\tau)$ be the number of points of Ω where $\varphi = \tau$. Change of variables on each affine piece gives, for bounded Borel h,

$$\int_\Omega h(\varphi(y))\, dy = \frac{1}{2} \int_{-\ell}^{\ell} h(\tau)\, n(\tau)\, d\tau.$$

By (a), $|\{y \in \Omega : \varphi(y) \le \tau\}| = |\{\varphi \le \tau\}| - |\{\varphi = -\ell\}| = (\ell + \tau)/2$ for $-\ell < \tau < \ell$, so the left side equals $\frac{1}{2} \int_{-\ell}^{\ell} h(\tau)\, d\tau$. Hence $n(\tau) = 1$ for almost every $\tau \in (-\ell, \ell)$.

Ω contains no point p of degree at least three. At such a point, φ is affine on an initial segment of each of the at least three branches, with slope ± 2 by the previous paragraph. Two branches would then have slopes of the same sign, and $n \ge 2$ on an interval of values next to $\varphi(p)$. Also $v, w \notin \Omega$, since $\varphi(v) = \ell$ and $\varphi(w) = -\ell$. Thus Ω is an open subset of Γ consisting of points of degree two, and each of its components is an open arc or a simple closed curve. A closed-curve component would be open and closed in the connected set Γ, which is impossible because $v \notin \Omega$.

Let J be a component of Ω. Along J, φ is piecewise affine with slopes ± 2. A change of sign of the slope would make $n \ge 2$ on an interval of values, so φ is strictly monotone along J. Since

$\varphi(J) \subseteq (-\ell, \ell)$ and $|\varphi'| = 2$, the arc J has length at most ℓ, so each of its ends converges to a point of Γ. If an end of J converged to a point $q \in \Omega$, then $q \in J$, since $J \cup \{q\}$ is connected, and the limit of φ at that end, $\varphi(q)$, would be attained in the interior of J, against strict monotonicity. Hence the ends of J converge to points $p_\pm \notin \Omega$, where $|\varphi| = \ell$. By strict monotonicity the two limits differ, so they are $\pm\ell$, $p_+ \neq p_-$, $\varphi(J) = (-\ell, \ell)$, and J has length ℓ because $|\varphi'| = 2$. Finally, $|\Omega| = |\Gamma| - |\{\varphi = \ell\}| - |\{\varphi = -\ell\}| = \ell$ by (a), so $\Omega \setminus J$ is an open set of length zero, hence empty.

(c) $\Gamma \setminus \Omega$ is the disjoint union of the closed sets $X = \{\varphi = \ell\}$ and $Y = \{\varphi = -\ell\}$, with $v, p_+ \in X$ and $w, p_- \in Y$. Each component K of the finite metric graph $\Gamma \setminus \Omega$ contains p_+ or p_-: otherwise K would be open and closed in Γ, because near points of K the sets Γ and $\Gamma \setminus \Omega$ coincide. Hence X and Y are connected, and their lengths are given by (a).

(d) Let Ω_+ be the half-open arc from p_+ to x^*, excluding x^*, and Ω_- the half-open arc from x^* to p_-, excluding x^*, each of length $\ell/2$. Then $G \setminus \{x, x^*\}$ is the disjoint union of

$$X \cup [v, x) \cup \Omega_+ \qquad \text{and} \qquad Y \cup (x, w] \cup \Omega_-,$$

which are connected by (c) and meet each other only through the removed points x, x^*. Each has length $(L - 2\ell)/2 + \ell/2 + \ell/2 = L/2$. □

Definition 3.3. Let $x \in G$ have degree two. An *antipode* of x is a point $z \neq x$ of degree two such that $G \setminus \{x, z\}$ has exactly two components, each of length $L/2$.

Proposition 3.4. *Let G be balanced and not a loop, with core H.*

(a) Every point x of degree two has exactly one antipode x^*, and $(x^*)^* = x$.

(b) For every core edge f there is a core edge $\sigma(f) \neq f$, with $m_{\sigma(f)} = m_f$, such that $x \mapsto x^*$ maps f° isometrically onto $\sigma(f)^\circ$, and $\sigma(\sigma(f)) = f$.

(c) $\{f, \sigma(f)\}$ is a two-edge cut of H, that is, $H - f - \sigma(f)$ is disconnected.

Proof. (a) Existence: x lies in the interior of a core edge f; apply Lemma 3.2(d) to a segment $e \subseteq f^\circ$ centred at x, of length $\ell \leq m_f/2$ small enough that $e \subseteq f^\circ$. The point x^* given there has degree two.

For uniqueness, first let z be any antipode of x, with components A, B of $G \setminus \{x, z\}$. Each of x and z has one branch into A and one into B. Indeed, if z were not in the closure of B, then B would be open and closed in $G \setminus \{x\}$, and x would be a cut point, against Lemma 3.1(a). So z lies in the closures of both A and B. Each of its two branches is connected and avoids x, so it lies in A or in B, and therefore one branch goes to each. Since z has degree two, the same argument with x and z exchanged applies to x.

Now suppose $z_1 \neq z_2$ are antipodes of x, with $G \setminus \{x, z_1\} = A_1 \sqcup B_1$ and, say, $z_2 \in A_1$. The connected set B_1 avoids x and z_2, so it lies in one component K of $G \setminus \{x, z_2\}$. Since z_1 is joined to B_1 by its branch into B_1, and $z_1 \notin \{x, z_2\}$, K also contains z_1 and an initial segment of the branch of z_1 into A_1 that avoids z_2. Then $|K| > |B_1| = L/2$, a contradiction. The definition of antipode is symmetric in x and z, so $(x^*)^* = x$.

(b) Let $x \in f^\circ$, and let A, B be the components of $G \setminus \{x, x^*\}$. For small $\delta > 0$, let x' be the point at distance δ from x on its branch into A, and let z' be the point at distance δ from x^* on its branch into B. Then

$$A' = (A \setminus (x, x']) \cup [x^*, z'), \qquad B' = (B \setminus (x^*, z']) \cup [x, x')$$

are connected (each removed arc hangs from the rest of its component only at its far end, and each added arc is attached at the point x^* or x adjacent to that component). They are disjoint, they

cover $G \setminus \{x', z'\}$, and they meet only through x', z'. Each has length $L/2 - \delta + \delta = L/2$. Hence z' is an antipode of x', and by (a), $(x')^* = z'$. Exchanging the roles of A and B treats points on the other side of x. Consequently the map $x \mapsto x^*$ is continuous on f° and preserves distances between nearby points.

The image of f° is a connected set of points of degree two. Since the points of degree two of G form the disjoint union of the open core edges, this image lies in a single open core edge g°. A map from the interval $f^\circ \cong (0, m_f)$ to $g^\circ \cong (0, m_g)$ that is injective (by (a)) and locally distance-preserving is an isometry onto an open subinterval. Applying the same to g, the image of g° lies in one open core edge and contains the image of the image of f°, which is f° itself; so it lies in f°. Hence the map sends f° onto g°, and $m_f = m_g$. If $g = f$, the map would be an isometric involution of an open interval onto itself: the identity or the reflection, both of which have fixed points, whereas $x^* \neq x$. So $g \neq f$. Put $\sigma(f) = g$; then $\sigma(g) = f$ by (a).

(c) Write $g = \sigma(f)$, and identify a subgraph of H with the closed subset of G formed by its vertices and edges. Let $x \in f^\circ$. By the first step of the uniqueness argument in (a), the two branches of x go into different components of $G \setminus \{x, x^*\}$. Since $x^* \in g^\circ$ and $g \neq f$, these branches lead along f, without meeting x^*, to the two ends of f, which therefore lie in different components of $G \setminus \{x, x^*\}$. As $G \setminus (f^\circ \cup g^\circ) \subseteq G \setminus \{x, x^*\}$, the two ends of f lie in different components of $H - f - g$. □

4 An edge in no two-edge cut, and the proof of Theorem 1.1

In this section H is a finite connected multigraph without loops. An edge of a connected graph is a *bridge* if its removal disconnects the graph; equivalently, it lies on no cycle. A pair $\{f, g\}$ of distinct edges of H is a *two-edge cut* if $H - f - g$ is disconnected. The following lemma is a consequence of Robbins' ear decomposition [7]; we include the short proof.

Lemma 4.1. *Let H be a finite connected multigraph without loops and without bridges, in which every vertex has degree at least three. Then H has an edge e such that $H - e$ is connected and has no bridges. In particular, e belongs to no two-edge cut.*

Proof. Call a subgraph of H *good* if it is connected and has no bridges. Since H has an edge and no bridges, it contains a cycle K_0, which is good. Suppose a good subgraph $K_{i-1} \neq H$ has been constructed. As H is connected and every vertex has an edge, some edge uv of H lies outside K_{i-1} and has $u \in V(K_{i-1})$. If $v \in V(K_{i-1})$, let P_i be the edge uv and put $w = v$. Otherwise uv is not a bridge of H, so $H - uv$ contains a path from v to u; follow it from v to its first vertex $w \in V(K_{i-1})$, and let P_i be the edge uv followed by this path. In both cases P_i is a path from u to w, or a cycle through u if $w = u$. Its edges lie outside K_{i-1}, and its vertices other than u and w lie outside $V(K_{i-1})$. Put $K_i = K_{i-1} \cup P_i$. It is connected, and it has no bridges. Indeed, every edge of K_{i-1} lies on a cycle of K_{i-1}. Every edge of P_i lies on P_i itself if $w = u$, and otherwise on the cycle formed by P_i and a path from w to u in K_{i-1}; when P_i is the single edge uv, this uses $u \neq v$, as H has no loops. Each step adds at least one edge, so the construction ends with $K_r = H$ for some r.

Every vertex of the cycle K_0 has degree two in K_0, and every vertex of H has degree at least three, so $K_0 \neq H$ and $r \geq 1$. Suppose P_r has a vertex $z \notin V(K_{r-1})$. Every edge of H at z lies in $K_r = H$ but not in K_{r-1}, hence in P_r, which has exactly two edges at z. Then z has degree two in H, which is impossible. Hence P_r is a single edge $e = uv$ with $u, v \in V(K_{r-1})$, and $H - e = K_{r-1}$ is connected and has no bridges. If $\{e, g\}$ were a two-edge cut, then g would be a bridge of $H - e$. □

Proof of Theorem 1.1. Corollary 2.4 gives (i)⇔(ii), and Example 2.5 gives (iii)⇒(i) together with the value $L/4$. It remains to show (ii)⇒(iii). Let G be balanced and suppose G is not a loop. By

Lemma 3.1(c), the core H is a finite connected multigraph without loops and without bridges, in which every vertex has degree at least three. Lemma 4.1 gives a core edge e that belongs to no two-edge cut of H. By Proposition 3.4(b) and (c), the core edge $\sigma(e)$ differs from e, and $\{e, \sigma(e)\}$ is a two-edge cut of H. This contradiction shows that G is a loop. Finally, when the conditions hold, ρ_G is constant, and Corollary 2.4 gives $|A_\eta(x)| = L/2$ at every point and in every direction. □

Proof of Corollary 1.2. By Theorem 1.1, ρ_G is a non-constant continuous function on G with mean value $\rho(G)$. The sets $\{\rho_G < \rho(G)\}$ and $\{\rho_G > \rho(G)\}$ are open. If one of them were empty, $\rho_G - \rho(G)$ would have one sign and integral zero, so it would vanish almost everywhere, hence everywhere by continuity, contradicting Theorem 1.1. So both sets are nonempty, and the two strict inequalities follow. If G is a loop, ρ_G is constant and every point is a minimizer. □

5 Examples and remarks

5.1 The discrete counterpart

For a connected graph with vertex set V and an edge uv, let W_{uv} be the set of vertices closer to u than to v. Then $\sum_w d(v,w) - \sum_w d(u,w) = |W_{uv}| - |W_{vu}|$, because the distances to u and v differ by exactly one outside the set of vertices equidistant from u and v. This identity underlies the equivalence of the distance-balanced and self-median properties [1, 4, 6]. Lemma 2.3 is its continuum analogue, with an edge replaced by a direction at a point. The two settings behave very differently, as the following example shows.

Remark 5.1. Let Λ be a finite connected vertex-transitive graph with at least one edge, and let G be the equilateral metric graph over Λ with unit edges. Automorphisms of Λ act as isometries of G preserving the length measure, so F_G is constant on the vertex set. By Theorem 1.1, F_G is not constant on G unless Λ is a cycle. For the complete graph K_4 with unit edges ($L = 6$), a direct computation with Lemma 2.2 gives

$$F_G(\text{vertex}) = 3 \cdot \tfrac{1}{2} + 3 \cdot \tfrac{5}{4} = \tfrac{21}{4}, \qquad F_G(\text{midpoint of an edge}) = \tfrac{1}{4} + 4 \cdot 1 + \tfrac{7}{4} = 6.$$

Thus ρ_G equals $7/8$ at the vertices and 1 at the edge midpoints. The exact computations in the accompanying script give the pairs (vertex value, midpoint value) $(19/2, 23/2)$ for K_5, $(21/2, 45/4)$ for $K_{3,3}$, $(18, 37/2)$ for the cube, $(24, 103/4)$ for the Petersen graph and $(11, 12)$ for the triangular prism.

Remark 5.2. Rescaling all edge lengths by $\lambda > 0$ multiplies ρ_G and L by λ, so the oscillation per unit length

$$\operatorname{osc}(G) = \frac{\max_G \rho_G - \min_G \rho_G}{L}$$

is scale invariant. Over non-loops it admits no positive lower bound in terms of the number of edges and the Betti number alone. Two families with $0 < \varepsilon \le 1$ show this.

First, let G_ε be a loop of length 1 with a pendant edge of length ε attached at a point u, so $L = 1+\varepsilon$. By Lemma 2.2, a point x of the loop at distance $r \in [0, 1/2]$ from u has $F(x) = 1/4 + \varepsilon r + \varepsilon^2/2$, and a point of the pendant edge at distance $s \in [0, \varepsilon]$ from u has $F = 1/4 + s + s^2/2 + (\varepsilon - s)^2/2$, which is increasing in s. Hence $\min F = F(u) = 1/4 + \varepsilon^2/2$, the maximum $1/4 + \varepsilon + \varepsilon^2/2$ is attained at the free end, and $\operatorname{osc}(G_\varepsilon) = \varepsilon/(1+\varepsilon)^2$.

Second, a bridgeless example: let Θ_ε be the theta graph with two vertices u, z joined by three edges of lengths $1, \varepsilon, \varepsilon$, so $L = 1 + 2\varepsilon$. Let C be the circle formed by the edges of lengths 1 and ε, and let e_3 be the remaining edge. A path between points of C that traverses e_3 can use the parallel edge

of C instead, so d restricted to C is the circle metric of length $1+\varepsilon$, and $\int_C d(x,y)\,dy = (1+\varepsilon)^2/4$ for $x \in C$. For $y \in e_3$, $d(x,y) \le d(x,u) + d(u,y) \le (1+\varepsilon)/2 + \varepsilon$. Hence, for $x \in C$,

$$\frac{(1+\varepsilon)^2}{4} \le F_{\Theta_\varepsilon}(x) \le \frac{(1+\varepsilon)^2}{4} + \frac{\varepsilon(1+3\varepsilon)}{2}.$$

The isometry of Θ_ε exchanging the two edges of length ε extends this to all $x \in \Theta_\varepsilon$, so $\operatorname{osc}(\Theta_\varepsilon) \le \varepsilon(1+3\varepsilon)/(2(1+2\varepsilon)^2)$.

Neither graph is a loop, so both mean distance functions are non-constant by Theorem 1.1, yet both oscillations per unit length tend to 0 with ε. Any quantitative form of Theorem 1.1 must therefore involve ratios of edge lengths. The script confirms both statements exactly at $\varepsilon = 1/2, 1/10, 1/100, 1/1000$.

Remark 5.3. The proofs above are self-contained. Two accompanying scripts check several of the statements independently. The first works in exact rational arithmetic on random multigraphs with loops and multiple edges. It checks the derivative formula of Lemma 2.3 at vertices (133 exact comparisons), and the segment identity behind Lemma 3.2(a), $\frac{d}{ds}F(x_s) = \tau + (L-\ell) - 2|\{\varphi < \tau\}|$ (280 exact comparisons). It also checks Example 2.5 on loops with several vertices, and the values and formulas in Remarks 5.1 and 5.2. The second checks Lemma 4.1 in two independent ways: by running the ear construction of its proof, and by listing all two-edge cuts. It does so on all 28,429 connected loopless bridgeless multigraphs of minimum degree at least three with 2 to 5 vertices and bounded edge multiplicities (615 of them have a two-edge cut), and on 3000 random such graphs built as chains of blocks (1839 of them have a two-edge cut).

Statements and Declarations

Funding. This research was self-funded.

Competing interests. The author owns Luminary Oddities and has no other relevant competing interests.

Data and code availability. The verification scripts cited in the text are included with this article.

AI assistance. Anthropic Claude assisted with derivations, proof development, computations, source checks and independent internal review. Scott Schubr directed the research and is responsible for the content.

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