COMPACT CAPTURE AND PROTECTED BRIDGES ON THE KORAS–RUSSELL CYLINDER

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Abstract. On an adapted model of the Koras–Russell cylinder, we give an explicit compression of a closed $\mathbb{C}^*$ carrier and a sufficient condition for ambient compact capture: the planar polynomial hull of the scalar image under $q = xy+1$ must avoid 1. The construction yields fixed Runge capture domains, including the connected domain $q^{-1}(\mathbb{C} \setminus [1, \infty))$. Joint capture can be corrected to approximate the identity on a globally holomorphically convex parametrized ball in a fixed Runge Euclidean domain. This gives protected capture of prescribed compact bridges of positive thickness, with no separation of the moving and protected scalar ranges. A complementary convex-ball argument treats certain protected bodies through $q = 1$. An analytic disk explains why the weighted contraction cannot simply be extended to the entire punctured chart. The results are qualitative capture criteria.

1. The model and the main statements

Let

$$Y = \{F = x^2y + 2xzu + z^2 + x + t^3 = 0\} \subset \mathbb{C}^5, \qquad q = xy + 1. \tag{1}$$

The coordinates are always ordered (x, y, z, t, u). Writing $X = \{x^2y_0 + z_0^2 + x + t^3 = 0\} \subset \mathbb{C}^4$, the polynomial map

$$(x, y_0, z_0, t, u) \longmapsto (x, y_0 + u^2, z_0 - xu, t, u) \tag{2}$$

identifies $X \times \mathbb{C}$ with Y. In particular Y is a smooth affine Stein fourfold and is diffeomorphic to $\mathbb{R}^8$; the corresponding real-Euclidean property of X is recalled in [6]. Throughout, automorphisms are holomorphic automorphisms, without an algebraicity or volume-preservation requirement.

For a compact K in a Stein manifold Z, write

$$\widehat{K}_{\mathcal{O}(Z)} = \{p \in Z : |f(p)| \le \sup_K |f| \text{ for every } f \in \mathcal{O}(Z)\}.$$

In the plane, $\widehat{E}_{\mathbb{C}}$ denotes the polynomial hull of a compact E; it is E together with the bounded components of $\mathbb{C} \setminus E$. A Stein open subset $D \subset Z$ is *Runge* if restrictions of $\mathcal{O}(Z)$ are dense in $\mathcal{O}(D)$, equivalently if the ambient hull of every compact in D remains in D.

Definition 1.1. A compact $K \subset Y$ is *capturable into* an open set $W \subset Y$ if $A(K) \subset W$ for some $A \in \mathrm{Aut}(Y)$. A *parametrized closed ball* in a complex N-manifold is a set $P = \psi(\overline{\mathbb{B}})$, where $\mathbb{B} \subset \mathbb{C}^N$ is the unit ball and ψ is a holomorphic embedding on an open neighborhood of $\overline{\mathbb{B}}$.

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Theorem 1.2 (Scalar-hull capture)**.** *Let $K \subset Y$ be compact. If*

$$1 \notin \widehat{q(K)}_{\mathbb{C}}, \tag{3}$$

then K is capturable into every nonempty open set $W \subset Y$. More precisely, fix a smooth proper embedded ray $\Gamma : [0,\infty) \to \mathbb{C}$ with $\Gamma(0) = 1$ and $0 \notin \Gamma([0,\infty))$. Every compact subset of the fixed open set

$$U_\Gamma = q^{-1}(\mathbb{C} \setminus \Gamma([0,\infty))) \tag{4}$$

is capturable into every such W.

Theorem 1.3 (Protected capture)**.** *Let $\Omega \subset Y$ be a Stein Runge domain biholomorphic to $\mathbb{C}^4$, and let $P \subset \Omega$ be an $\mathcal{O}(Y)$-convex parametrized closed ball. Suppose that $K \subset Y$ is compact and*

$$1 \notin q(\widehat{K \cup P})_{\mathbb{C}}. \tag{5}$$

For every $\epsilon > 0$, there is $A \in \mathrm{Aut}(Y)$ such that

$$A(K \cup P) \subset \Omega, \qquad \sup_{p \in P} \mathrm{dist}(A(p), p) < \epsilon. \tag{6}$$

Here dist *is any fixed distance inducing the topology of Y. The same conclusion holds for every compact $K \subset U_\Gamma$ and every such ball $P \subset \Omega \cap U_\Gamma$, with Γ fixed in advance.*

The condition concerns the hull of the *whole union*; separate conditions on K and P are not substituted for it. Their scalar images may overlap, as may the sets themselves. A further compact bridge B can be included in K whenever $1 \notin q(\widehat{K \cup B \cup P})_{\mathbb{C}}$, or whenever the whole union lies in one fixed U_Γ. In particular, the ray $\Gamma(s) = 1 + s$ gives the connected Runge domain

$$U = q^{-1}(\mathbb{C} \setminus [1,\infty)), \qquad V = \{|q| < 1\} \subsetneq U. \tag{7}$$

Section 6 explains prescribed positive thickness and nonvacuous protected balls crossing $|q| = 1$. Section 7 gives a different result for certain balls containing points of $q = 1$, where (5) cannot hold.

The geometric input is an explicit compression of the carrier $\gamma(s) = (s, -1/s, 0, 0, 0)$, combined with weighted contraction and classical Andersén–Lempert approximation. Runge normal-bundle constructions already combine fiber contraction and ambient approximation [4]. Relative transport near a holomorphically convex obstacle is likewise established machinery [3]; the classical shrink-and-slide construction is described in [2, Section 4]. Our statements concern prescribed compact subsets of these concrete domains and one ambient automorphism acting on the whole compact. These are applications of the cited approximation principles.

2. Approximation and convexity tools

A complex manifold has the *holomorphic density property* if the Lie algebra generated by its complete holomorphic vector fields is dense in the Lie algebra of all holomorphic vector fields, for uniform convergence on compact sets. Only this ordinary density property is used. It holds on Y directly by Leuenberger's Main Theorem [6]: in the presentation $x^2 y = a(z,t,u) + xb(z,t,u)$ take

$$a = -z^2 - t^3, \qquad b = -1 - 2zu.$$

The degrees in the distinguished variable z are 2 and 1. In condition (A) of that theorem choose $k = 0$. The common zeroes of a and $\partial_z a$ have $z = t = 0$, where $b = -1$; and $\partial_z a = -2z$ does not vanish identically on the corresponding z-line. Thus all the stated conditions are satisfied.

We use the following standard forms of Andersén–Lempert approximation [5, Theorem 2.2], [2, Theorems 1.3 and 1.5]. On a Stein density-property manifold, an injective holomorphic isotopy

starting with inclusion on an open neighborhood of a compact convex set is approximable there by ambient automorphisms if all moving compact images are holomorphically convex. The neighborhood may be disconnected. For a Stein Runge source and Stein Runge isotopy images, approximation holds on every compact subset of the source. We use smooth isotopies; piecewise smooth paths can be smoothly reparametrized and concatenated.

We also use approximation of the local flow of a time-dependent *globally defined* holomorphic vector field, on any compact where that flow exists through a fixed finite time. This follows from density by time subdivision and approximation of the frozen global fields by Lie combinations of complete fields; see [1, Lemma 2.5 and Section 6]. There is no extra Runge condition on the local flow neighborhood in this global-field formulation. The density property implies transitivity of $\mathrm{Aut}(Y)$, and a Stein density-property N-manifold contains Runge Euclidean domains through each point; see [2, Sections 2 and 4].

Lemma 2.1 (Scalar separation)**.** *Let Z be Stein and $L_1, L_2 \subset Z$ be holomorphically convex compact sets. If $h \in \mathcal{O}(Z)$ maps them into disjoint closed planar disks, then $L_1 \cup L_2$ is holomorphically convex. In particular, adding a point outside a holomorphically convex compact preserves holomorphic convexity.*

Proof. Polynomials on the disjoint disks approximate the two locally constant selectors taking values 0 and 1. Multiplying such selectors, composed with h, by holomorphic separating functions for the individual compacta separates every point outside their union. This proves the first assertion. For a point $p \notin L_1$, choose a global separator h with $|h(p)| > \sup_{L_1} |h|$ and apply the first assertion to the disk containing $h(L_1)$ and a sufficiently small disk about $h(p)$. □

Lemma 2.2 (Planar pullbacks)**.** *For every planar Runge domain $D \subset \mathbb{C}$, $q^{-1}(D)$ is Stein and Runge in Y.*

Proof. The graph $\{(p, w) \in Y \times D : q(p) = w\}$ is a closed complex submanifold of the Stein manifold $Y \times D$, and is biholomorphic to $q^{-1}(D)$. For every compact $E \subset q^{-1}(D)$,

$$q(\widehat{E}_{\mathcal{O}(Y)}) \subset \widehat{q(E)}_{\mathbb{C}} \subset D.$$

The first inclusion follows by testing polynomials composed with q; the second is the planar Runge hull criterion. The ambient hull is compact because Y is Stein, and remains in the pullback, proving Runge inclusion. □

3. An explicit compressible carrier

For $0 \le d < 1$ set $r(d) = \sqrt{1 - d^{12}} > 0$ and

$$j_d(s) = \big(r(d)s, -r(d)/s, d^9, -d^6, -d^3/2\big), \qquad s \in \mathbb{C}^*. \tag{8}$$

Then $xy = -(1 - d^{12})$, $2zu = -d^{12}$ and $z^2 + t^3 = 0$, so $F \circ j_d = 0$. Each j_d is a holomorphic embedding and $j_0 = \gamma$. On every compact annulus in $\mathbb{C}^*$, as $d \to 1$ these embeddings converge uniformly to

$$p_* = (0, 0, 1, -1, -1/2) \in Y. \tag{9}$$

Only parameters strictly below 1 will be used as isotopy endpoints.

Proposition 3.1. *Every compact subset of $\gamma(\mathbb{C}^*)$ is capturable into every nonempty open subset of Y.*

Proof. We realize (8) by a global tangent velocity. Put $b = y - u^2$ and define on $\mathbb{C}^5$

$$R = (1 - 2xb)(\partial_x - u\partial_z) + 4b^2\partial_y. \tag{10}$$

Since $F_x - uF_z = 1 + 2xb$ and $F_y = x^2$, one has

$$R(F) = (1 - 2xb)(1 + 2xb) + 4x^2b^2 = 1.$$

In particular Y is smooth. On $|d| < 1$, take the holomorphic branch of r with $r(0) = 1$, and put

$$\begin{aligned} h(d) &= r'(d)/r(d) = -6d^{11}/(1 - d^{12}), \\ W_d &= h(d)(x\partial_x + y\partial_y) + 9d^8\partial_z - 6d^5\partial_t - \tfrac{3}{2}d^2\partial_u, \\ X_d &= W_d - W_d(F)R. \end{aligned}$$

Then $X_d(F) = 0$, so X_d is globally holomorphic and tangent to Y. Along $j_d(\mathbb{C}^*)$, $W_d = \partial_d j_d$ and differentiation of $F \circ j_d = 0$ gives $W_d(F) = 0$. Hence $X_d(j_d(s)) = \partial_d j_d(s)$.

Fix a compact annulus $A \subset \mathbb{C}^*$ and $d_0 < 1$. The time-dependent flow starting on $\gamma(A)$ follows $j_d(A)$ for $0 \le d \le d_0$. Compactness of this trace and holomorphic ODE existence give a neighborhood of $\gamma(A)$ on which the flow exists through d_0. The global-field approximation from Section 2 approximates this finite flow by automorphisms on $\gamma(A)$.

Given nonempty open W, choose $B \in \mathrm{Aut}(Y)$ with $B(p_*) \in W$. First choose $d_0 < 1$ sufficiently close to 1 that $j_{d_0}(A) \subset B^{-1}(W)$, and then approximate the flow closely enough that $C(\gamma(A)) \subset B^{-1}(W)$. The map BC captures $\gamma(A)$. Every compact of the carrier is contained in such an annulus. □

4. Cut domains and scalar-hull capture

On $\{x \neq 0\} \subset Y$, eliminating y identifies this chart with $\mathbb{C}^* \times \mathbb{C}^3$. For $a \in \mathbb{C}^*$ define

$$\Phi_a(x, y, z, t, u) = (x, a^6y + (a^6 - 1)/x, a^3z, a^2t, a^3u). \tag{11}$$

Direct calculation on the chart gives

$$F \circ \Phi_a = a^6F, \qquad q \circ \Phi_a = a^6q, \qquad \Phi_a^{-1} = \Phi_{1/a}. \tag{12}$$

Proof of Theorem 1.2. Fix first a ray Γ as in the statement and set $D = \mathbb{C} \setminus \Gamma([0, \infty))$. The complement of a smooth proper embedded planar ray is a connected Runge domain. It contains 0 and excludes 1. Set $\lambda(s) = 1/\Gamma(s)$. This is a smooth nonvanishing path with $\lambda(0) = 1$ and $\lambda(s) \to 0$ as $s \to \infty$. Lift it along the interval to a smooth sixth root $a(s)$ with $a(0) = 1$. No single-valued sixth root on all of $\mathbb{C}^*$ is required.

The source $q^{-1}(D)$ excludes $x = 0$, since that divisor has $q = 1$. Moreover $1 \notin \lambda(s)D$, because $1/\lambda(s) = \Gamma(s) \notin D$. Thus the target also excludes $x = 0$, and (12) proves the exact equality

$$\Phi_{a(s)}(q^{-1}(D)) = q^{-1}(\lambda(s)D). \tag{13}$$

Both sides are Stein Runge open sets by Lemma 2.2. For any finite S, (11) therefore gives an injective holomorphic isotopy, starting with inclusion, whose moving images are Stein Runge. Andersén–Lempert applies on every compact of the source. If the source is regarded componentwise, the same conclusion follows from its compact hull and the compact-image form; no connectivity assumption on a neighborhood is needed.

For a prescribed compact $K \subset q^{-1}(D)$, the function x is bounded away from zero on K. Since $|a(s)| \to 0$,

$$\Phi_{a(s)}|_K \longrightarrow \gamma \circ x|_K \quad \text{uniformly.} \tag{14}$$

Choose an annulus A containing $x(K)$, and first choose a carrier-capturing automorphism B with $B(\gamma(A)) \subset W$. There is an open neighborhood N of $\gamma(A)$ with $B(N) \subset W$. Choose a finite S for which $\Phi_{a(S)}(K) \subset N$, and approximate this endpoint closely enough by $C \in \mathrm{Aut}(Y)$ that $C(K) \subset N$. Then $BC(K) \subset W$. The limiting collapsed map is never used as an approximation endpoint.

Finally suppose (3) holds and $K \neq \varnothing$. The point 1 lies in the unbounded component of $\mathbb{C} \setminus q(K)$. Choose in that open component a smooth simple path from 1 to the exterior of a disk containing $q(K) \cup \{0\}$, avoiding the additional point 0, and continue to infinity outside that disk. Removing loops and smoothing gives a proper embedded ray Γ disjoint from $q(K) \cup \{0\}$. Thus $K \subset U_\Gamma$, and the preceding argument applies. Empty K is immediate. □

Proposition 4.1. *The domain U in (7) is connected, Stein and Runge, strictly contains V, and deformation retracts onto $\gamma(\mathbb{C}^*)$. In particular $H_1(U;\mathbb{Z}) = \mathbb{Z}$.*

Proof. Steinness and Runge inclusion follow from Lemma 2.2. For real $0 < a \leq 1$, (13) becomes

$$\Phi_a(U) = q^{-1}(\mathbb{C} \setminus [a^6, \infty)) \subset U.$$

Extend this homotopy continuously to $a = 0$ by $\gamma \circ x$. It fixes the carrier pointwise, so gives the asserted deformation retraction and connectivity. To see strict inclusion, choose $(x, y, z, u) = (1, -1 + 2i, 0, 0)$ and $t^3 = -2i$. This is a point of Y with $q = 2i$, hence belongs to $U \setminus V$. □

5. Correcting joint capture to protect a ball

We give the target correction explicitly to keep its hypotheses separate from any source-side union-convexity assumption.

Lemma 5.1 (Closed-ball correction)**.** *Let Z be a Stein density-property N-manifold, $N \geq 2$, and let $\Omega \subset Z$ be a Stein Runge domain biholomorphic to $\mathbb{C}^N$. Let $P \subset \Omega$ be an $\mathcal{O}(Z)$-convex parametrized closed ball. If $K \subset Z$ is compact and $K \cup P$ is capturable into Ω, then for every $\epsilon > 0$ it is capturable there by an automorphism ϵ-close to* id *on P.*

Proof. Choose $A \in \mathrm{Aut}(Z)$ with $A(K \cup P) \subset \Omega$, a biholomorphism $\chi : \Omega \to \mathbb{C}^N$, and a parametrization ψ of P. On a neighborhood of $\overline{\mathbb{B}}$, put $f = \chi\psi$ and $g = \chi A\psi$. Both $f(\overline{\mathbb{B}})$ and $g(\overline{\mathbb{B}})$ are polynomially convex: ambient convexity of P and $A(P)$ implies their intrinsic $\mathcal{O}(\Omega)$-convexity, which becomes polynomial convexity under χ.

For $0 < t \leq 1$, the set $f(t\overline{\mathbb{B}})$ is polynomially convex. Indeed its hull lies in $f(\overline{\mathbb{B}})$. On that larger compact the inverse coordinate functions f^{-1} are holomorphic on a neighborhood, so Oka–Weil approximates them by entire functions. A complex linear combination of source coordinates strictly separates any source point outside $t\overline{\mathbb{B}}$ from that ball. Approximation retains this separation and excludes the rest of the larger compact from the smaller hull. The same argument applies to g.

The normalized rescalings

$$f_t(z) = f(0) + \frac{f(tz) - f(0)}{t} \quad (t > 0), \qquad f_0(z) = f(0) + df_0(z)$$

are smooth holomorphic embedding isotopies on a common neighborhood of $\overline{\mathbb{B}}$, with polynomially convex closed-ball images. Define g_t likewise. On a neighborhood of $g(\overline{\mathbb{B}})$, first follow $g_t g^{-1}$ from $t = 1$ to $t = 0$, then connect the invertible affine maps g_0 and f_0, still composed with g^{-1}, and finally follow $f_t g^{-1}$ from $t = 0$ to $t = 1$. The affine connection exists because $\mathrm{GL}_N(\mathbb{C})$ is connected. Every moving compact image is polynomially convex. After smooth concatenation, compact-image Andersén–Lempert approximates the terminal map fg^{-1} on $g(\overline{\mathbb{B}})$ by an automorphism H of $\mathbb{C}^N$.

Consequently $B = \chi^{-1} H \chi \in \mathrm{Aut}(\Omega)$ approximates A^{-1} as closely as desired on $A(P)$. Choose it so BA is $\epsilon/2$-close to identity on P. Every automorphism H of $\mathbb{C}^N$ is smoothly isotopic to identity: use normalized rescaling for its origin-fixing part, and connect its invertible linear part and translation to identity. Conjugating gives a path $B_t \in \mathrm{Aut}(\Omega)$ from id to B. Viewed as

maps $\Omega \to Z$, all its images are the same Stein Runge domain Ω. Ambient Andersén–Lempert therefore approximates B on $A(K \cup P)$ by $C \in \mathrm{Aut}(Z)$.

The selected compact $B(A(K \cup P))$ has a positive open-containment margin in Ω. Choose the last approximation small enough both to preserve this containment and to make CA ϵ-close to identity on P. Then CA has the required properties. No uniform lower bound on this margin as $\epsilon \to 0$ is needed. □

Proof of Theorem 1.3. Theorem 1.2 captures the entire compact $K \cup P$ into Ω under (5), or under its fixed-cut hypothesis. Apply Lemma 5.1 with $Z = Y$. □

6. Prescribed bridges and overlapping scalar images

Theorems 1.2 and 1.3 apply to arbitrary compact subsets of their stated domains, not just totally real cores. Thus a compact bridge $B \subset U$ may have nonempty complex interior and any fixed positive thickness compatible with compact containment in U. Once K, B, P are selected, they remain unchanged for every protection tolerance. The construction does not first choose a capture map and then take a sufficiently small neighborhood in its inverse image.

For an explicit existence statement, fix a ball $P \subset \Omega \cap U$ and a point in a thick compact $K_0 \subset V$. Since U is connected, join that point to a point of $\mathrm{Int}\, P$ by a compact smooth path in U. A finite chain of small closed coordinate balls around the path can be chosen with compact union $B \subset U$. This union is connected, has positive thickness, and meets both K_0 and P. Fix it before applying Theorem 1.3 to $K_0 \cup B$. If the endpoint in P lies outside $\overline{V}$, the bridge also leaves the old tube V.

Such protected balls exist in every fixed Ω and can genuinely cross the old unit-circle boundary. The function $q|_\Omega$ is nonconstant, since otherwise the identity theorem would make q constant on Y. In $\Omega \cong \mathbb{C}^4$, restrict it to a complex line on which it is nonconstant. Little Picard gives an attained nonreal unit-circle value $c \neq 1$. A sufficiently small closed coordinate ball P centered at a preimage of c, together with a slightly larger concentric open ball, lies in $\Omega \cap U$. It is globally convex because coordinate balls are convex in $\Omega \cong \mathbb{C}^4$ and Ω is Runge in Y. The nonconstant holomorphic function q is locally open, so $q(\mathrm{Int}\, P)$ contains values on both sides of $|w| = 1$. This P is fixed before K and ϵ.

Overlap does not depend on allowing $K \cap P \neq \varnothing$. For $c \in q(\mathrm{Int}\, P)$ the fiber $q = c$ contains an unbounded family

$$x \neq 0, \quad y = (c-1)/x, \quad z = u = 0, \quad t^3 = -cx.$$

Choose a point of this family outside P and a small thick closed coordinate ball K around it, disjoint from P and inside U. Then $q(\mathrm{Int}\, K)$ and $q(\mathrm{Int}\, P)$ both contain a neighborhood of c. Theorem 1.3 applies despite this scalar-image overlap. Bent cuts in Theorem 1.2 also allow configurations meeting the former positive ray above 1, provided the scalar hull of their whole union still avoids 1.

7. A complementary result through the critical level

The scalar-hull criterion cannot apply to a body containing a point with $q = 1$. The following adaptation of the classical shrink-and-slide argument [2, Section 4], followed by closed-ball correction, gives another protected class.

Theorem 7.1 (Joint-convex ball shuttle)**.** *Let $\Omega \subset Y$ be Stein Runge and biholomorphic to $\mathbb{C}^4$, let $K \subset V$ be compact, and put $L = \widehat{K}_{\mathcal{O}(Y)}$. Let $P \subset \Omega$ be a globally holomorphically convex parametrized closed ball. If $L \cap P = \varnothing$ and $L \cup P$ is globally holomorphically convex, then* (6) *holds for every $\epsilon > 0$. No restriction on $q(P)$ is imposed.*

Proof. The compact L lies in V by Lemma 2.2. The complement $Y \setminus L$ is connected. To see this, use the real-Euclidean topology $Y \cong \mathbb{R}^8$ and a large real ball containing L. There is at most one component escaping every compact, since the exterior of that ball is connected. Any other component G would be relatively compact, with boundary contained in L. For every $f \in \mathcal{O}(Y)$, the maximum principle gives $|f| \le \sup_L |f|$ on G, putting G inside $\widehat{L}_{\mathcal{O}(Y)} = L$, a contradiction. As an open connected manifold, $Y \setminus L$ is path connected.

Write $P = \psi(\overline{\mathbb{B}})$ and $p = \psi(0)$. Choose a smooth path from p to a point of $V \setminus L$, staying outside L. At each time, L plus the moving singleton is globally convex by Lemma 2.1. A holomorphic point isotopy near the path, together with identity near L, can therefore be approximated by ambient automorphisms. Here one may construct the point isotopy using (10): if T_s is the constant ambient field equal to the velocity at the path point, then $T_s - T_s(F)R$ is tangent to Y and has that velocity there. Its local flow exists on a common small initial neighborhood through the finite path interval. Shrink that neighborhood so all moving images remain disjoint from a neighborhood of L. Compact-image approximation gives $B \in \mathrm{Aut}(Y)$ with $B(p) \in V$ and $B(L) \subset V$.

For $0 < r \le 1$, put $P_r = \psi(r\overline{\mathbb{B}})$. The union $L \cup P_r$ is globally convex. Its hull is first confined to $L \cup P$. For a point of $P \setminus P_r$, a linear combination of ψ^{-1} coordinates strictly separates it from P_r. Prescribe that function near P and zero near L. Oka–Weil on the disjoint convex union $L \cup P$ approximates it globally and retains strict separation. This excludes every remaining candidate hull point.

The piecewise isotopy equal to identity near L and to $\psi(z) \mapsto \psi(rz)$ near P is jointly injective on suitably small neighborhoods for r in any fixed interval $[r_0, 1]$. Its moving compact images $L \cup P_r$ are convex. Choose $r_0 > 0$ after B so $B(P_{r_0}) \subset V$, and approximate this shrinking isotopy by $S \in \mathrm{Aut}(Y)$ closely enough that $BS(L \cup P) \subset V$. Theorem 1.2 captures this entire compact from V into Ω. Thus $K \cup P$ is jointly capturable there. Finally Lemma 5.1 protects the original P to any requested accuracy. Auxiliary maps may move P substantially; the protection assertion concerns their final corrected composition. □

Corollary 7.2. *Fix $0 < \delta < 1$ and $0 < \rho < 1 - \delta$. Let $P \subset \Omega$ be as in Theorem 7.1 and suppose*

$$q(P) \subset \{w : |w - 1| \le \delta\}.$$

Then every prescribed compact $K \subset \{|q| < \rho\}$ admits arbitrarily accurate P-protected capture into Ω. The same fixed P may contain a point of $q = 1$, including a point of $x = 0$.

Proof. The hull $L = \widehat{K}_{\mathcal{O}(Y)}$ remains in $\{|q| < \rho\}$. The disks $\{|w| \le \rho\}$ and $\{|w - 1| \le \delta\}$ are disjoint. Lemma 2.1 proves $L \cup P$ convex and $L \cap P = \varnothing$. Apply Theorem 7.1. □

If Ω contains a designated point of $q = 1$, sufficiently small Ω-coordinate balls around it supply the hypothesis of Corollary 7.2; basins through designated points provide examples. The ball and ρ in this corollary are fixed before K and ϵ.

8. The boundary of the method

The full chart $\{x \ne 0\} \cong \mathbb{C}^* \times \mathbb{C}^3$ is not Runge in Y. For $|\zeta| \le 1/4$, consider the analytic disk

$$d(\zeta) = (\zeta, 1, \sqrt{1 - \zeta - \zeta^2}, -1, 0). \tag{15}$$

The branch with value 1 at zero exists since $|\zeta + \zeta^2| \le 5/16 < 1$. Substitution gives $F \circ d = 0$ and $q \circ d = 1 + \zeta$. Its boundary lies in $\{x \ne 0\}$, while its center, with $x = 0$, lies in the ambient holomorphic hull of the boundary by the maximum principle. This contradicts Runge

hull confinement for the full chart. The coordinate $a^6y + (a^6 - 1)/x$ also has a genuine pole along this disk at $x = 0$ whenever $a^6 \neq 1$.

Thus arbitrary filling of the scalar cut cannot be justified by the same Runge contraction. This is a limitation of the argument, not a noncapture theorem: a Runge target containing an image of the disk boundary must also contain the image of its hull. Likewise (3) is sufficient rather than necessary; single points with $q = 1$ are capturable by transitivity.

The bridge and critical-level results have different hypotheses. A bridge meeting both L and P cannot automatically be inserted in Theorem 7.1, because its disjointness hypothesis may then fail. Conversely, Theorem 1.3 applies to a whole prescribed thick bridge whenever its stated scalar-hull or fixed-domain condition holds. Neither theorem proves arbitrary protected capture across the critical level, capture of a cofinal exhaustion of Y, or biholomorphic identification of Y with $\mathbb{C}^4$. All approximations and open-containment margins may depend on the compact and tolerance.

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