Mathematical supplement to Collision-uniform kernel asymptotics for Báez–Duarte zero packets

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Abstract

We specify the norm and tail payments needed to connect an arithmetic approximation with a zero certificate. Anchored Dirichlet polynomials give an exact whole-line, height-adapted Hardy norm as a positive arithmetic series, with a complete Abel tail and an accelerated alternating certificate. We also prove that compact boundary bands alone discard zero-detection information, using an explicit countermodel inside the harmonic-step space. An exact exterior-weight law quantifies the lost coercivity. A prefix-Schur identity shows how subtracting a finite approximation baseline changes the remaining zero signal. These supporting results distinguish the native packet recipient from classical localized Hardy criteria.

1 The recipient and its complete norm

Put $\Omega = \{s : \Re s > 1/2\}$. The Mellin transform identifies $L^2(0,1)$ with $H^2(\Omega)$, with boundary norm

$$\|F\|_{H^2}^2 = \frac{1}{2\pi}\int_{\mathbb{R}} |F(1/2+it)|^2 \,\mathrm{d}t.$$

The harmonic-step subspace S consists, before transformation, of functions constant on $I_n = (1/(n+1), 1/n]$. Its evaluation kernel is

$$K_S(s,w) = \sum_{n\geq 1} n(n+1)\frac{n^{-s}-(n+1)^{-s}}{s}\,\frac{n^{-\bar{w}}-(n+1)^{-\bar{w}}}{\bar{w}}.$$

In particular, $K_S(s,s) \leq 1/(2\Re s - 1)$, since restriction to a closed subspace decreases the norm of evaluation. The native target and generators are

$$E(s) = 1/s, \qquad G_k(s) = (k^{-s}-k^{-1})\zeta(s)/s, \quad k \geq 2, \qquad V_N = \operatorname{span}\{G_2,\dots,G_N\}.$$

Their step values are $E_n = 1$ and $(G_k)_n = -\{n/k\}$. The main paper [1] develops finite zero-packet lower bounds in this exact space and generic finite primal and dual certificates in its norm.

A weighted *whole-line* Hardy norm below is another recipient. A compact boundary-band norm is another one again. No equality of their evaluation kernels or of their distances is assumed. The distinction is necessary: a lower bound for the native distance cannot be compared to an upper computed in a smaller band norm. The classical local Nyman–Beurling criteria [2, 3] use global Hardy norms and reproducing kernels. The arithmetic step kernel and tail arguments in [4, Sections 4–5] and [5, Theorems 3–4] also retain their complete recipients.

The whole-line Poisson identity, scalar gain and localized criterion below also appear in the companion arithmetic-cutoff paper [6]. They provide the whole-line norm identities used in the supporting comparisons below.

2 A height-adapted whole-line arithmetic identity

Choose $0 < a < 1/2$ and $T \in \mathbb{R}$, and write

$$\lambda = \tfrac{1}{2} + a + iT, \qquad q = \tfrac{1}{2} - a + iT, \qquad W(s) = \frac{a}{s - \frac{1}{2} + a - iT}.$$

Let $A(s) = \sum_{d \le N} c_d d^{-s}$ be anchored by the exact condition $A(1) = \sum c_d/d = 0$. Define

$$d_n = \sum_{\substack{d \mid n \\ d \le N}} c_d, \qquad S(x) = \sum_{n \le x} d_n = \sum_{d \le N} c_d \lfloor x/d \rfloor, \qquad B_n(q) = \sum_{k \le n} d_k k^{-q}.$$

The anchor gives $|S(x)| \le C := \sum |c_d|$. Write $\sigma = \Re q = 1/2 - a > 0$ and $w_n = n^{-2a} - (n+1)^{-2a}$.

Theorem 2.1 (Exact arithmetic upper recipient)**.** *For every such anchored polynomial,*

$$\mathcal{M}_{a,T}(A) := \frac{2}{a} \|W(1 - \zeta A)\|_{H^2}^2 = \sum_{n \ge 1} w_n |1 - B_n(q)|^2. \tag{1}$$

Let $E_q = C(1 + |q|/\sigma)$. For every integer $M \ge 1$,

$$\mathcal{M}_{a,T}(A) \le \sum_{n \le M} w_n |1 - B_n(q)|^2 + (M+1)^{-2a} \big(|1 - B_M(q)| + 2E_q M^{-\sigma}\big)^2. \tag{2}$$

If $\zeta(\lambda) = 0$, then $\mathcal{M}_{a,T}(A) \ge 1$.

Proof. Bounded partial sums and Abel summation make $Z(q) = \sum_{n \ge 1} d_n n^{-q} = \zeta(q) A(q)$ convergent in $\Re q > 0$. More explicitly,

$$Z(q) - B_n(q) = -S(n) n^{-q} + q \int_n^\infty S(x) x^{-q-1} \, \mathrm{d}x, \qquad |Z(q) - B_n(q)| \le E_q n^{-\sigma}.$$

The Dirichlet series agrees with ζA first for $\Re q > 1$ and then by analytic continuation.

The inverse Laplace transform on $y \ge 0$ is

$$W(s)(1 - \zeta(s) A(s)) = \mathcal{L}\left[a e^{qy} \big(1 - B_{\lfloor e^y \rfloor}(q)\big) \right](s).$$

This identity follows first for $\Re s > 1$ by integrating each step at $y = \log n$. The boundedness of $B_n(q)$ implies that the displayed physical function, multiplied by $e^{-y/2}$, is square integrable. Thus its transform is genuinely in $H^2(\Omega)$. This also proves the required Hardy membership directly, without assuming multiplication by s is a bounded Hardy operator. Plancherel and integration on each interval $[\log n, \log(n+1))$ give (1). For $n > M$, $|B_n - B_M| \le |B_n - Z| + |Z - B_M| \le 2E_q M^{-\sigma}$. Sum the tail weights to obtain (2).

The Hardy reproducing kernel at λ is $k_\lambda(s) = 1/(s + \bar{\lambda} - 1)$, with squared norm $1/(2a)$. Here $W = a k_\lambda$ and $W(\lambda) = 1/2$. At a zero λ, bounded evaluation gives $\|W(1 - \zeta A)\|^2 \ge (1/4)/(1/(2a)) = a/2$. This is exactly the last assertion. □

The height dependence in E_q and the full physical tail in (2) are part of the upper bound. This norm is not an upper bound for the native distance in S.

2.1 The exact scalar gain

Fix an anchored A_0, put $F_0 = \zeta A_0$, and set $b = F_0(\lambda)$ and $Q = (2/a)\|WF_0\|^2$. The same arithmetic formula gives $Q = \sum w_n |B_n(q)|^2$. Also

$$\sum_{n\geq 1} w_n B_n(q) = F_0(\lambda) = b. \tag{3}$$

Indeed the finite sum is $B_M(\lambda) - (M+1)^{-2a} B_M(q)$; its second term tends to zero, and its first converges by bounded partial sums, since $\lambda = q + 2a$. Consequently, for $Q > 0$,

$$\mathcal{M}_{a,T}(cA_0) = 1 - 2\Re(cb) + |c|^2 Q, \qquad \inf_{c\in\mathbb{C}} \mathcal{M}_{a,T}(cA_0) = 1 - \frac{|b|^2}{Q}, \qquad c_{\mathrm{opt}} = \bar{b}/Q. \tag{4}$$

Thus the improvement is paid by an actual arithmetic mean over its full variance. At a hypothetical zero the mean vanishes. The projection criterion is classical; the identity supplies a fully paid arithmetic realization of its norm.

2.2 A complete alternating acceleration

Take $A_0(s) = 1 - 2^{1-s}$, so $d_n = (-1)^{n+1}$ and $|S(x)| \leq 1$. For $\Re z = \sigma > 0$ define $f_z(x) = x^{-z}$, $\Delta f(x) = f(x) - f(x+1)$, and the rising factorial $(z)_j$. The alternating tail $R_n(z) = \sum_{k>n} (-1)^{k+1} k^{-z}$ has, for every integer $K \geq 1$, the enclosure

$$\begin{aligned} R_n(z) &= (-1)^n \sum_{j=0}^{K-1} \frac{\Delta^j f_z(n+1)}{2^{j+1}} + \mathcal{R}, \\ |\mathcal{R}| &\leq \frac{|(z)_K|}{2^K} \left[(n+1)^{-\sigma-K} + \frac{(n+1)^{1-\sigma-K}}{\sigma + K - 1} \right]. \end{aligned} \tag{5}$$

To prove this, for a convergent alternating sum beginning at x use the identity $\mathcal{S}(f) = f(x)/2 + \mathcal{S}(\Delta f)/2$ repeatedly. The K-fold difference is the integral of the Kth derivative over $[0,1]^K$, with magnitude at most $|(z)_K| x^{-\sigma-K}$. An absolute sum and integral comparison bound the last alternating remainder. The denominator $\sigma + K - 1$ is positive. This proves the entire remainder in (5), including its sign convention.

For $n \geq m$, a convenient uniform tail magnitude is

$$L_m(z) = \sum_{j=0}^{K-1} \frac{|(z)_j| m^{-\sigma-j}}{2^{j+1}} + \frac{|(z)_K|}{2^K} \left[m^{-\sigma-K} + \frac{m^{1-\sigma-K}}{\sigma + K - 1} \right]. \tag{6}$$

The same bound applies whenever $m \leq n + 1$. Enclose the whole alternating sum Z by $B_M + R_M$ using (5). For any fixed rational complex c, if this enclosure is $\mathcal{Z}$, the entire residual tail in (1) is at most

$$(M+1)^{-2a} \left[\sup_{Z\in\mathcal{Z}} |1 - cZ| + |c| L_{M+1}(q) \right]^2. \tag{7}$$

No zeta evaluation or uncomputed outside-height energy enters this bound. Euler acceleration itself is classical.

2.3 The classical zero-free disk

Proposition 2.2. *For $0 < r < 1$, a trial at the center λ satisfying $\mathcal{M}_{a,T}(A) < 1 - r^2$ certifies that*

$$\left| \frac{\lambda - \mu}{\lambda + \bar{\mu} - 1} \right| < r \tag{8}$$

contains no zero μ of ζ.

Proof. Let U be the finite span of the functions $W\zeta A$ used for the approximation. Since $W = ak_\lambda$, the trial implies $\|P_U k_\lambda\|^2 > r^2 \|k_\lambda\|^2$. If μ is a zero, every $u \in U$ vanishes at μ. The half-plane Blaschke factor $B_\mu(s) = (s - \mu)/(s + \bar{\mu} - 1)$ has boundary modulus one; division by it preserves the Hardy norm of such a u. Therefore $|u(\lambda)| \le |B_\mu(\lambda)|\, \|u\| \|k_\lambda\|$. Taking the supremum over unit vectors of U gives $\|P_U k_\lambda\|^2 / \|k_\lambda\|^2 \le |B_\mu(\lambda)|^2$, contradicting (8). This is the kernel-projection argument of Nikolski and Delaunay–Fricain–Mosaki–Robert [3, Section 5]. □

The accompanying interval checker uses $a = 1/4$, $T = 1000$, $M = 20000$, $K = 8$ and $c = (10240377 - 14768460i)/10^8$ in (1) with $A = c(1 - 2^{1-s})$. Outward-rounded evaluation of its finite head and (7) proves $\mathcal{M}_{a,T}(A) < 3/4$ at the center and on the stated small interval box in the input contract. With $r = 1/2$ at the exact center, Proposition 2.2 gives

$$|s - (11/12 + 1000i)| < 1/3.$$

The disk is inside established verified-height coverage. Its role here is a reproducible full-norm example of the classical criterion. For a proposed left boundary $\mu = \beta + iT$ and center $\lambda = \alpha + iT$, the arithmetic requirement is precisely $1 - \mathcal{M}(\lambda) > [(\alpha - \beta)/(\alpha + \beta - 1)]^2$.

3 Why a compact boundary band is insufficient

Lemma 3.1 (Anchored compact-band density)**.** *For any nondegenerate bounded interval $I = [A, B]$ and real σ,*

$$\overline{\operatorname{span}}\{n^{-\sigma - it} - n^{-1} : n \ge 2\} = L^2(I,\, \mathrm{d}t).$$

Proof. If h is orthogonal to this span, its Fourier–Laplace transform $F(z) = \int_I e^{-itz} \overline{h(t)}\, \mathrm{d}t$ is entire with $|F(z)| \le \|h\|_1 e^{K|z|}$, $K = \max(|A|, |B|)$. Orthogonality gives $F(\log n) = F(0) e^{-(1-\sigma)\log n}$. The entire exponential-type function $D(z) = F(z) - F(0) e^{-(1-\sigma)z}$ thus vanishes at every $\log n$. If nonzero, choose z_0 with $D(z_0) \ne 0$. Jensen's formula on radius-$2R$ disks about z_0 bounds the number of zeros within radius R by $O(R)$: the logarithmic maximum is $O(R)$ and each such zero contributes at least $\log 2$. The exponentially many distinct $\log n$ in those disks contradict this bound. Hence $D = 0$. On the real axis F is bounded; if $\sigma \ne 1$ this forces $F(0) = 0$ by taking the appropriate real infinite direction. For $\sigma = 1$, Riemann–Lebesgue gives the same conclusion. Fourier uniqueness now implies $h = 0$. □

On the critical boundary, multiplication by $b(t) = \zeta(1/2 + it)/(1/2 + it)$ is bounded on I and nonzero almost everywhere. Its range in $L^2(I)$ is dense, since a vector orthogonal to its range must vanish wherever $b \ne 0$. Lemma 3.1 therefore gives unconditionally

$$\inf_{\text{finite } c} \int_I \left| 1/s - \sum c_n G_n(s) \right|^2 \mathrm{d}t = 0, \qquad s = 1/2 + it.$$

This is a boundary height band, not a physical prefix of step cells. Paid physical-prefix transfer results such as [7] retain different errors and are not contradicted by this density.

Proposition 3.2 (An explicit native countermodel)**.** *Let* $\gamma = 2^{3/4}$*,* $Q(s) = 1-\gamma 2^{-s}$ *and* $H_k = QG_k$*. Then* $H_k \in S$ *and, for* $\rho = 3/4 + 2\pi i/\log 2$*,*

$$\operatorname{dist}_S(E, \overline{\operatorname{span}}\{H_k : k \geq 2\})^2 > 1/202.$$

Nevertheless the distance on every bounded boundary band is zero.

Proof. Mellin multiplication by 2^{-s} is the physical dilation $f(t) \mapsto f(2t)1_{(0,1/2]}(t)$. It preserves the harmonic-step space: its nth cell value is $f_{\lfloor n/2 \rfloor}$, with $f_0 = 0$. Thus $(H_k)_n = -\{n/k\} + \gamma\{\lfloor n/2 \rfloor/k\}$. Every $H_k(\rho) = 0$, whereas $E(\rho) = 1/\rho$. Bounded native evaluation gives the squared distance lower $1/(|\rho|^2 K_S(\rho,\rho))$. Here $K_S(\rho,\rho) \leq 2$ and $|\rho|^2 < 101$, since $\pi < 22/7$ and $\log 2 > 2/3$ imply $2\pi/\log 2 < 10$. This proves the strict rational floor.

On $\Re s = 1/2$, $|\gamma 2^{-s}| = 2^{1/4} > 1$, so Q has no boundary zeros. The multiplier $Q\zeta/s$ is bounded and nonzero almost everywhere on each compact band. The same density proof therefore applies to H_k. This countermodel retains the native target, metric and bounded off-line evaluation; its additional Dirichlet factor deliberately introduces the known off-line zero. □

For a second view of the loss, define

$$J_I(s) = \frac{1}{\pi}\int_A^B \frac{\mathrm{d}t}{s - 1/2 - it}, \qquad x = \Re s - 1/2,$$
$$\omega_I(s) = \Re J_I(s) = \frac{1}{\pi}\left[\arctan\frac{B - \Im s}{x} - \arctan\frac{A - \Im s}{x}\right].$$

For interior s, $0 < \omega_I(s) < 1$; its boundary values are one on I and zero outside, almost everywhere. $O_m(s) = e^{-mJ_I(s)}$ is bounded analytic, with boundary modulus e^{-m} on I and one outside. For $s_0 \in \Omega$ and real $c > 0$, the Hardy functions

$$f_m(s) = \frac{O_m(s)}{O_m(s_0)}\frac{s_0 + c}{s + c}$$

satisfy $f_m(s_0) = 1$, while their band norms are a fixed positive constant times $e^{-2m(1-\omega_I(s_0))} \to 0$. Evaluation is therefore unbounded in the band-only norm.

4 The exact exterior-weight payment

Let $0 < \epsilon \leq 1$ and give the boundary weight value one on I and ϵ outside. The outer multiplier

$$W_\epsilon(s) = \exp\left(\frac{\log\epsilon}{2}(1 - J_I(s))\right)$$

and its inverse are bounded analytic. Its boundary squared modulus is that weight, and $|W_\epsilon(s)|^2 = \epsilon^{1-\omega_I(s)}$. Multiplication by W_ϵ is an isometry from the weighted Hardy norm onto ordinary H^2. Consequently its exact ambient evaluation-kernel diagonal is

$$K_\epsilon(s,s) = \frac{\epsilon^{-(1-\omega_I(s))}}{2\Re s - 1}. \tag{9}$$

The weighted native subspace can have a smaller kernel.

If a residual has target value $1/\rho$ at a zero, its complete weighted energy must therefore be at least

$$\frac{(2\Re\rho - 1)\epsilon^{1-\omega_I(\rho)}}{|\rho|^2}. \tag{10}$$

For fixed certified unweighted energies $U_I, U_{\mathrm{ext}} > 0$, with the $1/(2\pi)$ normalization, the scalar tradeoff is exact:

$$\epsilon_* = \frac{(1-\omega)U_I}{\omega U_{\mathrm{ext}}},$$
$$\min_{\epsilon>0} \frac{U_I + \epsilon U_{\mathrm{ext}}}{\epsilon^{1-\omega}} = \frac{U_I^{\omega} U_{\mathrm{ext}}^{1-\omega}}{\omega^{\omega}(1-\omega)^{1-\omega}}. \tag{11}$$

Differentiate the positive expression to find its unique stationary point; it decreases before that point and increases after it. For the allowed range $\epsilon \le 1$, use ϵ_* only if $\epsilon_* \le 1$, and otherwise use $\epsilon = 1$. Zero-energy endpoints are obtained by limits. These constants are sharp in ambient Hardy space: the weighted evaluation kernel attains equality, and its unweighted energy ratio is $U_I/U_{\mathrm{ext}} = \epsilon\omega/(1-\omega)$. Indeed the unweighted kernel at ρ distributes its boundary mass in harmonic-measure proportions $\omega, 1-\omega$; division by the outer multiplier changes the exterior mass by $1/\epsilon$. Thus local improvement cannot retain an unchanged evaluation signal while its exterior weight tends to zero.

5 A prefix subtraction changes the remaining signal

Fix $M \ge N$ and minimize the residual norm on the first M step cells over V_N. Let g be a minimizer and write $r = E - g = h + t$, where h is its head restriction and t its tail. The prefix normal equations give $h \in V_N^{\perp}$ even in the full native norm, since h is head-supported. They also give $\langle E, h\rangle = \|h\|^2 =: Q$. Disjoint support gives

$$\|E - g\|^2 = Q + \|t\|^2.$$

Assume $Q > 0$ and put $v = h(\rho)$. At a hypothetical zero ρ, $g(\rho) = 0$, so the target after projecting against h is $1/\rho - v = t(\rho)$. The native evaluation representer has squared norm $K_S(\rho,\rho)$; its component perpendicular to h has squared norm $K_S(\rho,\rho) - |v|^2/Q$. Thus, when this denominator is positive,

$$d_N^2 \ge Q + \frac{|1/\rho - v|^2}{K_S(\rho,\rho) - |v|^2/Q}. \tag{12}$$

This follows by the orthogonal two-functional projection; if the denominator vanishes one keeps the one-dimensional projection and does not divide by zero.

Let K_{tail} denote the step-tail evaluation kernel. The head projection satisfies $|v|^2/Q \le K_{\mathrm{head}}$, so the denominator in (12) is at least K_{tail}. Tail Cauchy–Schwarz then gives, under the zero premise,

$$\frac{|1/\rho - v|^2}{K_S(\rho,\rho) - |v|^2/Q} = \frac{|t(\rho)|^2}{K_S(\rho,\rho) - |v|^2/Q} \le \|t\|^2.$$

The baseline subtraction is legitimate, but the remaining target and its Schur denominator both change. A smaller excess by itself is not an additional zero signal.

For example, $N = M = 2$ has optimizer $g_n = 2\{n/2\}$. The head residual is $(0,1)$ and $Q = 1/6$. The full squared norm is $1-\log 2$, and its exact tail excess is $5/6-\log 2$. Direct Mellin transformation gives

$$\frac{1 - 2^{-s} + 3^{-s}}{s} = \frac{(1-2^{1-s})\zeta(s)}{s} + \widehat{t}(s).$$

This makes the changed signal visible in a completely explicit case. The finite-family normal equations and all tails remain in the original native metric throughout this calculation.

Reproduction. The companion interval program verifies the complete point/box Poisson upper using rational input coefficients and (5)–(7). The localization checker verifies original step jumps of the countermodel, exact exterior scalar identities and interval bounds for the displayed rational distance floor. The infinite density, Hardy membership and all-node claims rest on the proofs above, not on finite sampling.

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Data and code availability. The source package accompanying the preprint contains the complete proofs, verification programs and input data. `https://research.luminaryoddities.com/collision-uniform-kernel-asymptotics-baez-duarte-zero-packets/v/4`.

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