Collision-uniform kernel asymptotics for Báez–Duarte zero packets

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Luminary Oddities

October 25, 2026

Abstract

We determine the high-height kernel of the harmonic-step space underlying the Báez–Duarte approximation problem. For packets with unequal real parts in a fixed compact rectangle inside the critical strip, an exact arithmetic gauge gives an explicit limiting Mellin kernel. Convergence holds with every fixed number of mixed derivatives. Complex Newton–Hermite divided differences then give a relative Gram comparison uniform through arbitrary collisions, without a minimum-separation condition. The resulting native zero-jet projection bound retains the exact unequal powers and oscillatory phases of the target. We derive a two-point interference formula and a multiple-zero lower bound with a logarithmic multiplicity factor. The proof works directly with the arithmetic step kernel and pays both ends of its infinite series.

1 The arithmetic packet problem

Let S be the closed subspace of $L^2(0,1)$ consisting of functions constant on every interval

$$I_n = (1/(n+1), 1/n], \qquad n \geq 1.$$

We use the Mellin transform $\widehat{f}(s) = \int_0^1 f(t)t^{s-1}\,\mathrm{d}t$ and identify S with its Mellin image. It is a subspace of $H^2(\Omega)$, where $\Omega = \{\Re s > 1/2\}$ and the squared boundary norm is $(2\pi)^{-1}\int_{\mathbb{R}} |f(1/2 + it)|^2\,\mathrm{d}t$. The inner product is linear in its first argument. The orthogonal basis and its squared norms are

$$e_n(s) = \frac{n^{-s} - (n+1)^{-s}}{s}, \qquad \|e_n\|^2 = \frac{1}{n(n+1)}. \tag{1}$$

Thus the evaluation kernel of S is

$$K_S(s,w) = \sum_{n\geq 1} n(n+1)e_n(s)\overline{e_n(w)}. \tag{2}$$

This kernel retains the harmonic-step restriction, rather than replacing S by the full half-plane Hardy space.

The Báez–Duarte approximation problem [1, 2] concerns

$$E(s) = \frac{1}{s}, \qquad G_l(s) = \frac{(l^{-s} - l^{-1})\zeta(s)}{s} \quad (l \geq 2), \qquad V = \overline{\operatorname{span}}\{G_l : l \geq 2\} \subset S. \tag{3}$$

Write $D_\infty = \operatorname{dist}(E, V)$. Bounded zero-jet functionals on S give lower bounds for this distance. At high height their conditioning depends on the relative positions of the zeros, their multiplicities, and the phase of the target $1/s$.

The step-space zero-jet projection framework and the sharp diagonal asymptotic were developed in [3, Sections 2–5]. Weighted finite-cutoff conjugate-pair kernels and their monotonicity were studied in [4, Theorem 4]. Our question is the full packet limit, including unequal real parts and offsets that may collide as the height grows. Entrywise kernel asymptotics alone are insufficient in that regime: an evaluation Gram matrix can become arbitrarily ill-conditioned. We obtain a comparison of quadratic forms after a collision-compatible change of basis.

1.1 Main theorem

Fix an integer $m \geq 1$ and a compact rectangle

$$Q = \{q \in \mathbb{C} : b_0 \leq \Re q \leq b_1,\ |\Im q| \leq L\}, \qquad \frac{1}{2} < b_0 \leq b_1 < 1, \tag{4}$$

where $L < \infty$. For $T > 2L + 2$, put $s_q = iT + q$ and define

$$A_T(q,p) = T^{q+\bar{p}-3} s_q \overline{s_p}\, K_S(s_q, s_p), \qquad T^q = \exp(q \log T). \tag{5}$$

The logarithm of T is real. The limiting kernel is

$$\begin{aligned} H(q,p) &= \int_0^\infty x^{2-q-\bar{p}} 4 \sin^2 \frac{1}{2x}\, \mathrm{d}x \\ &= \frac{\pi}{\Gamma(4-q-\bar{p}) \sin(\pi(3-q-\bar{p})/2)}. \end{aligned} \tag{6}$$

For nodes $\mathbf{q} = (q_1, \dots, q_m) \in Q^m$, let $\Delta_j g = [q_1, \dots, q_j]g$ be the complex Newton divided difference, with its Hermite interpretation at repeated nodes. In particular, $[a, \dots, a]g = g^{(j-1)}(a)/(j-1)!$. For a sesquiholomorphic kernel, $\overline{\Delta_k}$ denotes the anti-holomorphic divided difference in the second variable, with conjugate nodes and conjugate coefficients. Set

$$\mathsf{G}_T(\mathbf{q})_{jk} = \Delta_j^{(q)} \overline{\Delta_k^{(p)}} A_T(q,p), \qquad \mathsf{G}(\mathbf{q})_{jk} = \Delta_j^{(q)} \overline{\Delta_k^{(p)}} H(q,p). \tag{7}$$

Theorem 1.1 (Collision-uniform packet kernel)**.** *For every fixed $r, k \geq 0$,*

$$\sup_{q,p \in Q} \left| \partial_q^r \partial_{\bar{p}}^k (A_T - H)(q,p) \right| \longrightarrow 0 \qquad (T \to \infty). \tag{8}$$

The matrices $\mathsf{G}(\mathbf{q})$ are positive definite with a common positive smallest-eigenvalue bound on Q^m. There is $\eta_T \to 0$, depending only on Q, m, such that uniformly for all $\mathbf{q} \in Q^m$,

$$(1 - \eta_T)\mathsf{G}(\mathbf{q}) \preceq \mathsf{G}_T(\mathbf{q}) \preceq (1 + \eta_T)\mathsf{G}(\mathbf{q}). \tag{9}$$

Arbitrary partial and complete collisions are included.

The native target vector is

$$z_T(\mathbf{q})_j = [q_1, \dots, q_j] T^q. \tag{10}$$

Corollary 1.2 (Zero-packet distance bound)**.** *Suppose the points $iT + q_j$ are zeros of ζ, with each repeated point listed at most its multiplicity. For all sufficiently large T,*

$$D_\infty^2 \geq T^{-3} z_T^* \mathsf{G}_T^{-1} z_T \geq \frac{T^{-3}}{1 + \eta_T} z_T^* \mathsf{G}^{-1} z_T. \tag{11}$$

The same η_T works for every such packet in Q^m.

All parameters in Q and m are fixed in these statements. In particular, no separation is imposed between the nodes, whereas the size of the packet and its compact real-part range remain specified.

1.2 Related analytic interfaces

The kernel-translation operator of Manzur–Noor [5, Sections 4–5] provides a different quantitative zero obstruction and an invariant-subspace criterion. The shifted-Hardy approximation of Manzur–Noor–Quintero [6, Section 3] also concerns the Báez–Duarte generators, with a different approximation norm. Our result addresses the high-height geometry of the exact kernel (2). The one-point exponent and constant are inherited from [3, Section 5, Remark 4]; the full kernel, mixed-derivative uniformity, collision-uniform relative comparison and exact packet target are the extensions proved below. The divided-difference integral and the positive moment-matrix argument are classical tools.

2 The native kernel and finite zero-jet projection

For $\rho \in \Omega$, the Mellin evaluation functional is represented in $L^2(0,1)$ by $t^{\bar\rho-1}$, of squared norm $1/(2\Re\rho-1)$. Its projection onto S is the restricted representer r_ρ. This proves boundedness of evaluation. Higher jets are bounded as well, since $t^{\bar\rho-1}(\log t)^j \in L^2(0,1)$. Expanding in the orthogonal basis (1) proves (2), with local convergence of all required derivatives. Alternatively, these follow directly by projecting the bounded full-space jet functionals.

For completeness, G_l belongs to S. In the original variable it is the negative Mellin transform of

$$\rho_{1/l}(t) = \{1/(lt)\} - l^{-1}\{1/t\}.$$

If $1/t = n+\theta$, $0 \le \theta < 1$, then $\rho_{1/l}(t) = \{n/l\}$, so it is constant on each I_n and bounded. The Mellin identity follows first where the defining zeta integrals converge and then by continuation. Each zero of ζ therefore forces the corresponding jets of every G_l to vanish.

We use the following elementary projection principle, with the evaluation-row convention of (2).

Lemma 2.1. *Let $\Lambda_1, \dots, \Lambda_d$ be independent bounded linear functionals on a complex Hilbert space, annihilating a closed subspace V. If r_j is the Riesz representer of Λ_j, put $\Gamma_{jk} = \langle r_k, r_j\rangle$ and $c_j = \Lambda_j(E)$. Then*

$$\operatorname{dist}(E,V)^2 \ge c^*\Gamma^{-1}c.$$

Proof. The representers lie in $V^\perp$. If the projection of E onto their span is $\sum_k a_k r_k$, its functional values give $\Gamma a = c$. Its squared norm is $a^*\Gamma a = c^*\Gamma^{-1}c$. Projection onto this subspace cannot exceed projection onto $V^\perp$. □

This argument uses only finitely many bounded functionals. It does not require a description of the entire orthogonal complement of V.

3 The exact arithmetic gauge and its smooth limit

For $x = n/T$, define

$$F_T(q,x) = \sqrt{1+1/n}\, x^{1-q}\big[1-(1+1/n)^{-iT-q}\big]. \tag{12}$$

Expansion of (2) gives the exact identity

$$A_T(q,p) = \frac{1}{T}\sum_{n\ge1} F_T(q,n/T)\overline{F_T(p,n/T)}. \tag{13}$$

In particular, the factors $s_q\overline{s_p}$ in (5) have not been approximated by T^2.

Lemma 3.1. *For every fixed integer $r \geq 0$, constants depending on Q, r give*

$$\begin{aligned} |\partial_q^r F_T(q,x)| &\leq C_r x^{1-b_1}(1+|\log x|)^r && (x \leq 1), \\ |\partial_q^r F_T(q,x)| &\leq C_r x^{-b_0}(1+\log x)^r && (x \geq 1), \end{aligned} \tag{14}$$

uniformly for $q \in Q$ and $T > 2L+2$.

Proof. Put $a_n = \log(1+1/n)$. The bracket in (12) has modulus at most two, since $\Re(iT+q) > 0$, and its positive derivatives have modulus at most a_n^j. On the small-x range use $a_n \leq \log 2$ and $\sqrt{1+1/n} \leq \sqrt{2}$. On the large-$x$ range, the integral identity for $1-e^{-z}$, $\Re z \geq 0$, yields

$$|1 - e^{-(iT+q)a_n}| \leq |iT+q|a_n \leq CT/n = C/x.$$

For $j \geq 1$, $a_n^j \leq C_j/n \leq C_j/x$ there. Differentiating x^{1-q} supplies only powers of $\log x$. Finally use $x^{1-\Re q} \leq x^{1-b_1}$ for $x \leq 1$ and $x^{-\Re q} \leq x^{-b_0}$ for $x \geq 1$. □

Proof of (8). On any fixed $\varepsilon \leq x \leq M$, uniformly in $q \in Q$,

$$F_T(q,x) \longrightarrow F(q,x) = x^{1-q}(1-e^{-i/x}),$$

with every fixed number of q derivatives. Indeed $T\log(1+1/n) \to 1/x$ uniformly on this range, while the derivatives of the bracket of positive order introduce powers of $\log(1+1/n) = O(1/T)$. Uniform Riemann sums on the compact interval therefore converge, including mixed derivatives in the two kernel slots.

We verify the infinite tails separately. By Lemma 3.1, a mixed derivative of orders r, k of the summand product is bounded at zero by

$$Cx^{2-2b_1}(1+|\log x|)^{r+k},$$

and at infinity by

$$Cx^{-2b_0}(1+\log x)^{r+k}.$$

Both are integrable. On a sufficiently small interval the first envelope is increasing, so its mesh sum through ε is bounded by its integral through 2ε; an empty mesh contributes zero. On a sufficiently late tail the second envelope is decreasing, and its mesh sum is bounded by an integral beginning at $M/2$. These bounds tend to zero uniformly as $\varepsilon \downarrow 0$, $M \to \infty$. They justify the termwise derivatives of (13) and the differentiated limiting integral. Choosing the tails first and then taking the compact Riemann limit proves (8).

To evaluate the integral, put $u = 1/x$ and $\alpha = 3-q-\bar{p}$, so $1 < \Re\alpha < 2$. The Mellin identity

$$2\int_0^\infty (1-\cos u)u^{-1-\alpha}\,\mathrm{d}u = \frac{\pi}{\Gamma(1+\alpha)\sin(\pi\alpha/2)}, \qquad 0 < \Re\alpha < 2,$$

gives (6). One derivation integrates by parts and uses the sine Mellin integral, first on an overlapping convergence strip and then by analytic continuation. Absolute convergence of the displayed $1-\cos u$ integral throughout this strip also identifies the same analytic continuation directly. □

4 Collisions and relative inversion

The complex Hermite–Genocchi formula is

$$[q_1,\dots,q_j]g=\int_{\substack{\theta_l\ge 0\\ \sum\theta_l=1}} g^{(j-1)}\left(\sum_l \theta_l q_l\right)\,\mathrm{d}\theta, \tag{15}$$

where simplex measure has total mass $1/(j-1)!$. Repeated fundamental-theorem integration proves the formula for complex nodes as well as real ones; continuity gives every Hermite endpoint. The convexity of Q keeps all arguments inside Q.

Applying (15) in both slots of (8), with orders at most $m-1$, gives

$$\sup_{\mathbf{q}\in Q^m}\|\mathsf{G}_T(\mathbf{q})-\mathsf{G}(\mathbf{q})\|\longrightarrow 0. \tag{16}$$

The conjugation of the second divided-difference coefficients is essential here.

Lemma 4.1. *The limiting Gram matrix $\mathsf{G}(\mathbf{q})$ has a common positive eigenvalue floor on Q^m.*

Proof. Set $\psi_j(x)=[q_1,\dots,q_j]x^{-q}$. Then

$$\mathsf{G}(\mathbf{q})_{jk}=\int_0^\infty \psi_j(x)\overline{\psi_k(x)}x^2 4\sin^2\frac{1}{2x}\,\mathrm{d}x. \tag{17}$$

The base measure has infinite mass, but all the displayed products are integrable by the preceding envelopes. In $y=\log x$, the expansion of e^{-qy} gives

$$\psi_j(e^y)=\frac{(-y)^{j-1}}{(j-1)!}+O(y^j).$$

An order-$j-1$ divided difference kills q^k for $k<j-1$ and sends q^{j-1} to one, including at collisions. The triangular leading orders therefore make $\psi_1,\dots,\psi_m$ linearly independent for every tuple. A nontrivial linear combination is an entire exponential polynomial in y and cannot vanish on a set of positive measure. Since the weight in (17) is positive off a discrete set, the matrix is positive definite.

The simplex formula and the uniform integrable logarithmic envelopes make the matrix continuous on Q^m. Its least eigenvalue thus has a positive minimum on this compact set. □

Completion of Theorem 1.1. Let $c>0$ be the floor in Lemma 4.1, and let ϵ_T be the supremum in (16). For every vector v,

$$|v^*(\mathsf{G}_T-\mathsf{G})v|\le\epsilon_T\|v\|^2\le(\epsilon_T/c)v^*\mathsf{G}v.$$

Taking $\eta_T=\epsilon_T/c$ proves (9). □

Proof of Corollary 1.2. Use the native functionals

$$\Lambda_j(f)=[q_1,\dots,q_j](s_qT^qf(s_q)). \tag{18}$$

Their Gram matrix is exactly $T^3\mathsf{G}_T$, by (5). Their values on $E=1/s$ are exactly z_T from (10). The gauge s_qT^q is analytic and nonzero on a neighborhood of Q, and multiplication by it is triangular and invertible on each finite jet space. The derivative of s_q with respect to q is one. Thus repeated nodes in (18) are invertible combinations of the native Mellin jets, with the stated Hermite factorials.

Under the zero-packet hypothesis these functionals annihilate all G_l and hence V. The positive definiteness from (9) makes them independent for large T. Lemma 2.1 gives the first inequality of (11). Inverting the upper inequality of (9) gives the second. The same proof applies if the complete off-line zero set is infinite, since only the listed finite packet enters the projection. □

5 Phase interference and multiple zeros

For a shared real part, write $q = \beta + i\tau$ and

$$H_\beta(a) = \frac{\pi}{\Gamma(4 - 2\beta - ia)\sin(\pi(3 - 2\beta - ia)/2)}. \tag{19}$$

Then $H(q,p) = H_\beta(\tau - \sigma)$ and $T^q = T^\beta e^{i\tau \log T}$. For distinct offsets τ_j, cancellation of the invertible Newton change of basis in (11) gives

$$D_\infty^2 \geq \frac{T^{2\beta-3}}{1+\eta_T} z^* \mathsf{H}^{-1} z, \quad \mathsf{H}_{jk} = H_\beta(\tau_j - \tau_k), \quad z_j = e^{i\tau_j \log T}. \tag{20}$$

The relative error is uniform even when the minimum separation tends to zero. At coincident nodes, (11), rather than the inverse of a singular evaluation matrix, defines the endpoint.

Corollary 5.1 (Two-point interference)**.** *Put $h = H_\beta(0)$ and $\ell = \log T$. For two distinct zeros $\beta + iT$, $\beta + i(T+\delta)$ in a bounded-offset packet, the limiting factor in* (20) *is*

$$z^* \mathsf{H}^{-1} z = \frac{2h - 2\Re\big(H_\beta(-\delta)e^{i\delta\ell}\big)}{h^2 - |H_\beta(\delta)|^2}. \tag{21}$$

The denominator is positive for $\delta \neq 0$.

Proof. The matrix is $\left(\begin{smallmatrix} h & H_\beta(-\delta) \\ H_\beta(\delta) & h \end{smallmatrix}\right)$, positive definite by independence of its two limiting features. Direct Hermitian inversion gives (21). □

The target phase in (21) is part of the receiving formula. In particular, close distinct zeros are not replaced by a multiple zero merely from their geometric proximity.

For a fixed multiplicity m, set $r = m - 1$ and

$$\mu_j(\beta) = \int_0^\infty (\log x)^j x^{2-2\beta} 4\sin^2\frac{1}{2x}\,\mathrm{d}x, \qquad M_\beta = (\mu_{a+b}(\beta))_{a,b=0}^{r}. \tag{22}$$

Every moment exists and M_β is positive definite.

Corollary 5.2 (Multiplicity factor)**.** *Along a sequence of zeros $\rho = \beta + iT$ of multiplicity at least m, with $T \to \infty$ and β in a fixed compact subinterval of $(1/2, 1)$,*

$$D_\infty^2 \geq (1 - o(1))(M_\beta^{-1})_{r,r} T^{2\beta-3} (\log T)^{2m-2}. \tag{23}$$

The error is uniform in that real-part interval.

Proof. At fully repeated offset zero, the limiting Newton Gram and target are

$$\mathsf{G}_\beta = A M_\beta A^*, \qquad A_{aa} = \frac{(-i)^a}{a!}, \qquad z_a = \frac{(i\ell)^a}{a!}.$$

Consequently

$$z^* \mathsf{G}_\beta^{-1} z = (M_\beta^{-1})_{r,r} \ell^{2r} + O(\ell^{2r-1})$$

for $m \geq 2$; for $m = 1$ it equals $1/\mu_0(\beta)$. The factorials cancel in the highest-degree term. The moment matrix and its inverse vary continuously on a compact real-part interval and have positive eigenvalue floors there, so the estimate and its positive leading coefficient are uniform. Apply (11) in the shared-beta chart. □

The coefficient $(M_\beta^{-1})_{r,r}$ is also the reciprocal squared norm of the monic degree-r polynomial orthogonal to lower powers for the positive measure in (22). This follows from the Schur complement of the lower-order moment block. When $m = 1$, (23) recovers the sharp single-zero constant of [3, Section 5, Remark 4]. Monic square norms also retain collision geometry in the signed-measure packet estimates of [7]. Here the measure is the native positive logarithmic measure in (22), and the receiving functionals are Mellin jets; the signed-measure recovery theorem is not used as an input.

Algebraic controls. The accompanying standard-library program checks the leading Newton identities at rational complex nodes, mixed and complete collisions, the native-jet factorial convention and the two-point Hermitian inverse identity. Its exact finite controls supplement the analytic proof; the derivative limit and uniform positive floor are established above independently of computation.

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