Collision and cancellation spectra for concentrated energy forms

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Luminary Oddities

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Abstract

We study concentrated signed potential forms in the derivative-energy space $H_0^1(a,b)$. An explicit finite matrix retains the complete projected primitive Gram and original spline trace, with bounded norm error independent of center separation, boundary distance and rank loss. If the smallest node or boundary gap h resolves the width ε, every spectral pair instead follows the exact primitive-weighted Dirichlet Green matrix with relative error $O(\log(2+h/\varepsilon)/\sqrt{h/\varepsilon})$. This holds for a fixed number of zero-mean profiles with uniform envelopes and primitive L^2 norms bounded below, arbitrary resolved nested geometry and boundary approach. An exact trace factorization and a small shear pay trace entries that can exceed secondary spectral scales. Integrable primitives improve the relative error to $O(\sqrt{\varepsilon/h})$. Neutral fixed-order stencils give vanishing residual spectrum and critical scale $d^p/\varepsilon^{p+1/2}$; their explicit rational realizations also distinguish finite, vanishing and escaping secondary channels. Unresolved Cauchy collision and cancellation examples explain why the resolution condition cannot simply be removed. All conclusions concern fixed-order bounded-interval operators.

1 The collision question and its normalization

Fix a finite interval $I=(a,b)$ and write $\ell=b-a$. The Hilbert space and its inner product are

$$H=H_0^1(I), \qquad e(u,z)=\int_I u'\overline{z'}\,dx. \tag{1}$$

For a real bounded potential V_ε, let C_ε be defined by

$$e(C_\varepsilon u,z)=q_\varepsilon(u,z), \qquad q_\varepsilon(u,z)=\frac{1}{2}\int_I V_\varepsilon u\overline{z}\,dx. \tag{2}$$

For every fixed $\varepsilon>0$, C_ε is compact and selfadjoint, by the compact embedding of $H_0^1(I)$ into $L^2(I)$. We ask which spectral branches escape when zero-mean profiles concentrate and their centers simultaneously collide. Eigenvalues are normalized by derivative energy, not by the L^2 norm of the test function.

The separated-center analysis in [1] gives a full Dirichlet Green interaction matrix and one positive and one negative escaping branch per nonzero profile under its fixed-separation hypotheses. Those error estimates cannot be used unchanged when the Green matrix loses its smallest scales during collision. A uniform description must retain both the primitive interactions and the actual trace block. Distinctness of the moving centers alone does not preserve the old branch count.

Our first result supplies this description for arbitrary distinct interior moving centers. Its error contains no inverse minimum separation or inverse smallest Green eigenvalue. The central refinement then resolves every Green scale under only $h/\varepsilon\to\infty$, even for unequal nested gaps

and boundary approach. It factors the actual large trace and removes its principal part by a near-identity congruence; it does not assume that trace entries are smaller than every spectral channel.

Cancellation supplies a complementary small-amplitude mechanism. If $d/\varepsilon \to 0$ and the first uncancelled stencil moment has fixed order p, the entire operator is one pair with vanishing remainder, and that pair has scale

$$\eta_\varepsilon = \frac{(d/\varepsilon)^p}{\sqrt{\varepsilon}} = \frac{d^p}{\varepsilon^{p+1/2}}. \tag{3}$$

Several resolved neutral clusters inherit all the weighted Green scales with a vanishing residual. The explicit two-center Cauchy family also exhibits the genuinely unresolved regimes. In the regime $d/\varepsilon \to \infty$ with $d \to 0$, two pairs escape on different scales; at comparable widths only one pair escapes. For opposite profiles, sub-width cancellation produces the additional scale $d/\varepsilon^{3/2}$.

Relation to existing methods. Primitive energies in weak coupling and Hardy/trace decompositions have classical antecedents [2, 3]. Sobolev multipliers and Birman–Schwinger sandwiches already give negative-Sobolev norm control [4]; changing the Hilbert metric or proving a primitive-norm upper bound is not, by itself, a new result. Two-scale point-interaction limits, antiderivative rank-two constructions and fusion laws also precede this work [6, 7, 8]. They address different full-line L^2 Hamiltonian or matching problems. In periodic perforated domains, Douanla [5] uses zero effective mean, divergence compensation and a compact energy-Riesz form to obtain paired spectral levels. This is a related energy-space mechanism in a multidimensional homogenization setting. Relative congruence comparison is classical Ostrowski theory [10, 11]; flat-kernel and graded-matrix spectral hierarchies also precede this work [12, 13]. Recursive vanishing-moment cluster bases are established [14]. Here the increment is the original-form trace factorization and cellwise logarithmic/Bessel/Loewner estimates that return every resolved Green scale to the complete energy operator, together with the unrestricted receiver and cancellation laws. The elementary classical Hardy and polar-decomposition ingredients are proved below. The Dirichlet Green representation is classical, including in expository treatments such as [9].

All numbers of profiles, stencil coefficients and cancellation orders are fixed. Constants in asymptotic notation may depend on these fixed data.

2 A finite matrix uniform under collision

Let $k \geq 1$ be fixed. For each $\varepsilon > 0$, choose distinct centers $a < r_1 < \cdots < r_k < b$, with no lower bound on their gaps or distance to the endpoints. Let v_j be real measurable profiles satisfying

$$\int_{\mathbb{R}} v_j(t)\,dt = 0, \qquad |v_j(t)| \leq \frac{b_j}{1+t^2}. \tag{4}$$

We allow zero profiles and set

$$L_j(t) = \int_{-\infty}^{t} v_j(s)\,ds, \quad c_j = \sup_{t\in\mathbb{R}} |tL_j(t)|, \quad K_j(x) = \varepsilon^{-1} L_j((x-r_j)/\varepsilon). \tag{5}$$

The zero mean and inverse-square envelope imply $c_j < \infty$: integrate from the nearer infinity on either tail, and use boundedness on compact sets. We take

$$V_\varepsilon(x) = \sum_{j=1}^{k} \varepsilon^{-2} v_j((x-r_j)/\varepsilon) + W_\varepsilon(x), \qquad \|W_\varepsilon\|_\infty \leq W_*. \tag{6}$$

Let $H_* = \{u \in H : u(r_j) = 0 \ (1 \le j \le k)\}$ and $T = H_*^{\perp}$, with orthogonality in energy. The Green matrix is

$$G_{ij} = G_I(r_i, r_j), \qquad G_I(x,y) = \frac{(\min(x,y) - a)(b - \max(x,y))}{\ell}. \tag{7}$$

It is positive definite for distinct interior nodes. The canonical real energy-orthonormal basis $e_1, \dots, e_k$ of T has evaluation matrix $e_l(r_j) = (G^{1/2})_{jl}$: indeed the Green representers have energy Gram G, and multiplication by $G^{-1/2}$ gives this basis.

Let Π be the $L^2(I)$ orthogonal projection onto functions with zero mean on each of the $k+1$ cells cut out by the endpoints and nodes. Thus Π subtracts the mean on each cell. Define the explicit real matrices

$$M_{ij} = \int_I (\Pi K_i)(\Pi K_j)\, dx, \tag{8}$$

$$S = \frac{1}{2}(G^{1/2} M G^{1/2})^{1/2}, \qquad A_{lm} = q_\varepsilon(e_l, e_m), \tag{9}$$

$$F_\varepsilon = \begin{pmatrix} 0 & S \\ S & A \end{pmatrix}. \tag{10}$$

The positive square root in (9) is defined also when M is singular. Both M and A are original one-dimensional integrals; the trace block A has not been replaced by a separated-center expansion.

Theorem 2.1 (Uniform primitive model)**.** *There is an isometry $J_\varepsilon : \mathbb{C}^{2k} \to H$ such that*

$$\|C_\varepsilon - J_\varepsilon F_\varepsilon J_\varepsilon^*\| \le 2\sum_j b_j + 2\sum_j c_j + \frac{W_* \ell^2}{\pi^2}. \tag{11}$$

The bound is independent of ε, every node gap and every distance to the boundary. At most k eigenvalues of either sign lie outside a fixed bounded interval. Every diverging spectral branch is therefore described to additive $O(1)$ by the full matrix (10).

Rank loss and cancellation are included, and the entries of A may carry essential collision scales.

Lemma 2.2 (One-sided Hardy estimate)**.** *If $f \in H^1(0,h)$ and $f(0) = 0$, then*

$$\int_0^h \frac{|f(x)|^2}{x^2}\, dx \le 4 \int_0^h |f'(x)|^2\, dx. \tag{12}$$

Its constant is independent of h and no condition at h is needed.

Proof. For smooth functions vanishing at zero, integration by parts, followed by Cauchy–Schwarz, gives

$$\int_0^h \frac{|f|^2}{x^2}\, dx = -\frac{|f(h)|^2}{h} + 2\,\mathrm{Re}\int_0^h \frac{f'\overline{f}}{x}\, dx \le 2\,\|f'\|_2 \left(\int_0^h \frac{|f|^2}{x^2}\, dx\right)^{1/2}.$$

Approximation in H^1 and Fatou's lemma extend the estimate. Applying it on each side of a marked point also gives $\|(u - u(r))/(x - r)\|_2 \le 2\,\|u'\|_2$ for arbitrary $u \in H^1(I)$. □

Proof of Theorem 2.1. For $u \in H_*$, apply Lemma 2.2 at every node and use Poincare's inequality. The compression $B = P_* C_\varepsilon P_*$ satisfies

$$\|B\| \le 2\sum_j b_j + \frac{W_* \ell^2}{2\pi^2}. \tag{13}$$

This is a single simultaneous trace-zero compression of codimension k. Minmax immediately gives the claimed bounded-spectrum count.

Since $K_j' = \varepsilon^{-2} v_j((x-r_j)/\varepsilon)$, integration by parts gives the complete identity

$$q_\varepsilon(u,z) = -\frac{1}{2}\sum_j \int_I K_j(u'\overline{z} + u\overline{z'})\,dx + \frac{1}{2}\int_I W_\varepsilon u\overline{z}\,dx. \tag{14}$$

Endpoint terms vanish because $u, z \in H_0^1(I)$. The fixed-ε boundedness of the potentials justifies extension from smooth tests. For $z \in H_*$, write $u = u(r_j) + (u - u(r_j))$ in the second summand. The principal term is

$$-\frac{1}{2}\sum_j u(r_j)\int_I K_j\overline{z'}\,dx.$$

The two remaining primitive terms obey, individually,

$$\left|\int_I K_j u'\overline{z}\,dx\right| \le 2c_j \left\|u'\right\|_2 \left\|z'\right\|_2,$$
$$\left|\int_I K_j(u-u(r_j))\overline{z'}\,dx\right| \le 2c_j \left\|u'\right\|_2 \left\|z'\right\|_2.$$

These estimates use $|K_j(x)| \le c_j/|x-r_j|$ and the value-zero Hardy inequality. They remain uniform when cells shrink.

The derivative map from H_* onto the cell-mean-zero subspace of $L^2(I)$ is an isometry. Surjectivity follows by integrating each cell derivative from its left endpoint; zero mean makes its right endpoint value zero, so the resulting functions sew continuously. Consequently the actual Riesz vector $R_j \in H_*$ of the functional $z \mapsto \int K_j\overline{z'}$ has derivative $R_j' = \Pi K_j$. Let $R : \mathbb{C}^k \to H_*$ have these columns. The trace-to-H_* coupling in $H_* \oplus T$ is

$$T_\varepsilon = -\frac{1}{2}RG^{1/2} + E_\varepsilon, \qquad \|E_\varepsilon\| \le 2\sum_j c_j + \frac{W_*\ell^2}{2\pi^2}. \tag{15}$$

All cell means and off-diagonal primitive entries remain in $R^*R = M$. The principal coupling $H_\varepsilon = -RG^{1/2}/2$ has $H_\varepsilon^* H_\varepsilon = S^2$. Its polar partial isometry can be extended across its finite-dimensional kernel into the infinite-dimensional space H_*, giving an isometry $U : \mathbb{C}^k \to H_*$ with $H_\varepsilon = US$. No inverse of S is used.

Let J_ε be U on its first k coordinates and the canonical trace basis on its last k. The two ranges are orthogonal. In $H_* \oplus T$, the residual operator has block form

$$\begin{pmatrix} B & E_\varepsilon \\ E_\varepsilon^* & 0 \end{pmatrix}.$$

Its norm is at most $\|B\| + \|E_\varepsilon\|$, which gives (11). Compact selfadjoint eigenvalue perturbation then gives the asserted additive comparison. □

Remark 2.3. If a family of combined profiles has both the envelope constant in (4) and $\sup|tL(t)|$ bounded by $O(\rho)$, with zero background, the same proof bounds its compression and coupling remainder by $O(\rho)$. A small trace estimate must still be proved for its actual trace block. This refinement will give a vanishing whole-operator error, not merely a bounded one.

3 Every resolved spectral scale

The full model of Theorem 2.1 remains meaningful without resolution or full primitive rank. When each layer is resolved, a stronger conclusion is possible even for nonintegrable primitives. The key is to factor and pay the original trace, rather than compare its absolute norm with the smallest coupling.

Let k be fixed and allow real profiles $v_{j,\varepsilon}$ to vary with ε, subject to the uniform hypotheses

$$\int_{\mathbb{R}} v_{j,\varepsilon} = 0, \qquad |v_{j,\varepsilon}(t)| \le \frac{A}{1+t^2}, \qquad \nu_{j,\varepsilon} := \|L_{j,\varepsilon}\|_2 \ge \nu_* > 0, \quad L_{j,\varepsilon}(t) = \int_{-\infty}^{t} v_{j,\varepsilon}. \tag{16}$$

Fixed nonzero profiles are included. Their envelopes and zero means give $|L_{j,\varepsilon}(t)| \le C/(1+|t|)$ uniformly, so the primitive norms also have a uniform upper bound. Choose arbitrary distinct moving interior centers and put

$$h = \min_{0\le i\le k}(r_{i+1} - r_i), \qquad r_0 = a, \quad r_{k+1} = b, \qquad s = h/\varepsilon \longrightarrow \infty. \tag{17}$$

Take an arbitrary common amplitude $\rho = \rho_\varepsilon > 0$ and zero background:

$$V_\varepsilon(x) = \rho \sum_{j=1}^{k} \varepsilon^{-2} v_{j,\varepsilon}((x - r_j)/\varepsilon). \tag{18}$$

Let $N = \operatorname{diag}(\nu_{1,\varepsilon}, \dots, \nu_{k,\varepsilon})$, and list the eigenvalues of the exact weighted Green matrix as $\gamma_1 \ge \cdots \ge \gamma_k > 0$, where $\gamma_j = \lambda_j(NGN)$.

Theorem 3.1 (Uniform resolved Green law)**.** *For all sufficiently large s, the original compact operator has at least k eigenvalues of each sign. Listed by decreasing positive value and decreasing negative magnitude, its first k satisfy*

$$\lambda_j^{\pm} = \pm \frac{\rho}{2\sqrt{\varepsilon}} \sqrt{\gamma_j} \left[1 + O\left(\frac{\log(2+s)}{\sqrt{s}}\right)\right], \qquad 1 \le j \le k. \tag{19}$$

Every remaining eigenvalue has absolute value at most $C_0\rho$. Constants depend only on I, k, A, ν_*, not on the moving geometry, profiles or amplitude. The smallest displayed magnitude is at least $c\rho\sqrt{s}$, so the relative conclusion holds even for rapidly vanishing ρ. If all primitive L^1 norms are uniformly bounded, the relative error improves to $O(s^{-1/2})$. In this refinement the constant may also depend on the uniform L^1 bound.

With $\rho = 1$, all k pairs escape. With $\rho \to 0$, the residual spectrum vanishes, whereas principal pairs may escape, have finite limits or vanish. There is no comparable-gap condition and no requirement $\varepsilon/h^2 \to 0$. Unbounded relative amplitudes are not covered by declaring a common ρ: the normalized profiles must still satisfy (16).

We first isolate an intrinsic trace payment that does not require resolution, a positive primitive Gram, or the lower norm bound. Use the actual splitting $H = H_* \oplus T$ from Section 2, write $E = G^{1/2}$, and set

$$K_j(x) = \rho\varepsilon^{-1} L_{j,\varepsilon}((x - r_j)/\varepsilon), \qquad R_j' = \Pi K_j, \qquad R = (R_j), \qquad M = R^*R.$$

Let $H_c = -RE/2$ and $B_{jl} = \int_I K_j e_l'\, dx$.

Lemma 3.2 (Compatible trace payment, including rank loss)**.** *Under the zero-mean uniform envelope and zero background, with arbitrary distinct interior centers, the original trace satisfies*

$$A = A_{\mathrm{pr}} + A_{\mathrm{rem}}, \qquad A_{\mathrm{pr}} = -\tfrac{1}{2}(EB + B^*E), \qquad \|A_{\mathrm{rem}}\| \le C\rho. \tag{20}$$

One has $\operatorname{ran} B \subseteq \operatorname{ran} M$*. With the Moore–Penrose inverse, define*

$$X = -RM^{\dagger}B, \quad \eta = \left\|(M^{\dagger})^{1/2}B\right\| = \|X\|, \quad P = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}, \quad F = \begin{pmatrix} 0 & H_c \\ H_c^* & A_{\mathrm{pr}} \end{pmatrix}. \tag{21}$$

Then $\|C_\varepsilon - F\| \le C\rho$ *and exactly*

$$P^*FP = F_{\mathrm{pair}} := \begin{pmatrix} 0 & H_c \\ H_c^* & 0 \end{pmatrix}. \tag{22}$$

If $q = \operatorname{rank} M$ *and* $\sigma_1 \ge \cdots \ge \sigma_q > 0$ *are the nonzero singular values of* H_c*,* F *has* q *eigenvalues of each sign. For every* $\eta \ge 0$*, their positive values and negative magnitudes satisfy*

$$(1+\eta)^{-2}\sigma_j \le \Lambda_j^{\pm}(F) \le (1+\eta)^2\sigma_j. \tag{23}$$

For the original C_ε*, the positive and negative variational sequences, completed by zeros, differ from those of* F *by at most* $C\rho$*. Every other eigenvalue is* $O(\rho)$*. These are parameterwise statements, permitting varying rank and* $q = 0$*. Along parameters with* $q > 0$*,* $\eta \to 0$ *and* $\sigma_q/\rho \to \infty$*, all* q *actual pairs exist and satisfy* $\Lambda_j^{\pm}(C_\varepsilon) = \sigma_j[1 + O(\eta + \rho/\sigma_q)]$*.*

Proof. In the integration-by-parts formula for A_{lm}, replace every undifferentiated spline value by its value at r_j plus its difference from that value. The nodal terms give $EB + B^*E$ exactly. Each remaining entry is bounded by $C\rho\,\|e_l'\|_2\,\|e_m'\|_2$, using $|K_j| \le C\rho/|x - r_j|$ and the value-difference Hardy bound. This proves (20). The same Hardy argument in Section 2 pays the compression and coupling errors by $C\rho$, giving the asserted original-operator error.

If $a \in \ker M$, then $\sum_j a_j K_j$ is constant almost everywhere on each cell, since its cell-mean-zero projection has zero norm. Actual primitives are absolutely continuous for each fixed parameter, so continuity across the nodes makes these constants equal on the whole interval. Pairing with e_l' gives zero because $e_l(a) = e_l(b) = 0$. Thus $\ker M \subseteq \ker B^*$, proving range consistency. This uses actual primitives, not arbitrary unrelated matrices.

Consequently $R^*X = -MM^{\dagger}B = -B$. Also $R(M^{\dagger})^{1/2}$ is an isometry on $\operatorname{ran} M$, proving the exact norm identity in (21). Matrix multiplication now proves (22), including singular M. The operators act on $\operatorname{ran} R \oplus T$ and trivially outside this finite-dimensional support. The shear is invertible for every η, and both $\|P\|$ and $\|P^{-1}\|$ are at most $1 + \eta$, since its off-diagonal part squares to zero.

The paired operator has exactly q signed singular pairs. Invertible congruence preserves inertia. The positive-eigenvalue max–min principle, applied with numerator $\langle F_{\mathrm{pair}}v, v\rangle$ and denominator $\|Pv\|^2$, gives (23); applying it to $-F$ gives negative magnitudes. This is classical signed congruence comparison [10, 11], not preservation of exact eigenvalues. Compact selfadjoint norm perturbation gives the zero-completed variational comparisons and residual bound for C_ε. The dominance condition makes the lower bounds positive and gives the stated relative conclusion. Without it, additive variational bounds do not assert the existence of nonzero signed eigenvalues. □

The lemma proves automatic compatibility. The resolved estimates below provide that additional quantitative fact.

Proof of Theorem 3.1. Use the notation of Lemma 3.2 and let $P_0 = I - \Pi$ project onto cell-constant functions. All bounds below are uniform.

Cell means and the trace functional. For an adjacent cell of length $l \geq h$,

$$\left|\int_{\text{cell}} K_j\right| \leq C\rho \log(1 + l/\varepsilon), \qquad \frac{1}{l}\left|\int_{\text{cell}} K_j\right|^2 \leq \frac{C\rho^2 \log^2(1+s)}{h}.$$

Indeed $\log^2(1+x)/x$ is decreasing for all sufficiently large x. For a nonadjacent cell, let $q \geq h$ be its nearer distance from r_j. Its integral is bounded by $C\rho \log(1 + l/q)$; its squared-mean contribution is at most

$$\frac{C\rho^2}{q}\frac{\log^2(1+l/q)}{l/q} \leq C\rho^2/h,$$

since the scalar ratio is bounded on $(0, \infty)$. With only $k+1$ cells this proves

$$\|P_0 K_j\|_2^2 \leq C\rho^2 \log^2(2+s)/h. \tag{24}$$

The derivatives e_l' are an orthonormal family of cell-constant functions. Bessel's inequality therefore gives

$$\sum_l |B_{jl}|^2 \leq \|P_0 K_j\|_2^2, \qquad \|B\| \leq C\rho \log(2+s)/\sqrt{h}. \tag{25}$$

The same projection loss thus pays the original trace functional. No uncontrolled $\log(1/\varepsilon)/\sqrt{h}$ is substituted.

All primitive overlaps and boundary tails. The whole-line diagonal is $\rho^2 \nu_{j,\varepsilon}^2/\varepsilon$. Omitted boundary square tails are $O(\rho^2/h)$. For $S > 0$, the elementary envelope convolution is

$$\int_{\mathbb{R}} \frac{dt}{(1+|t|)(1+|t-S|)} = 2\log(1+S)\left(\frac{1}{S} + \frac{1}{S+2}\right). \tag{26}$$

Split at $0, S$: the two exterior integrals are each $\log(1+S)/S$, and partial fractions give the interior term. For distinct centers $S = |r_i - r_j|/\varepsilon \geq s$, this bounds every mixed entry by $C(\rho^2/\varepsilon)\log(2+s)/s$. Subtracting the positive Gram of the cell means costs at most its trace, controlled by (24). Consequently

$$M = \frac{\rho^2}{\varepsilon}[N^2 + O(\beta)], \qquad \beta = \frac{\log^2(2+s)}{s} \to 0. \tag{27}$$

The error is in matrix norm. Since $N \geq \nu_* I$, this is a relative Loewner comparison with $(\rho^2/\varepsilon)N^2$. In particular $M > 0$ and $\left\|M^{-1/2}\right\| \leq C\sqrt{\varepsilon}/\rho$. The comparison is made before Green congruence, so it does not lose small Green directions.

The resolved intrinsic criterion. Now $M > 0$, so the intrinsic shear is $X = -RM^{-1}B$. The map $RM^{-1/2}$ is an isometry, and

$$\|X\| \leq \left\|M^{-1/2}\right\| \|B\| \leq C\log(2+s)/\sqrt{s} = o(1).$$

Lemma 3.2 therefore pays the complete trace multiplicatively, without requiring its absolute norm to be small relative to the last coupling.

Green constants and the whole-operator return. The nonzero eigenvalues of F_{pair} are precisely the signed singular values of H_c, and $H_c^* H_c = EME/4$. Equation (27) compares these squares relatively with $(\rho^2/(4\varepsilon))EN^2E$. The spectra of EN^2E and NGN agree, being the two products of NE and its adjoint. The exact inverse Green identity is

$$x^* G^{-1} x = \sum_{i=0}^{k} \frac{|x_{i+1} - x_i|^2}{r_{i+1} - r_i} \le \frac{4}{h} \|x\|^2, \qquad x_0 = x_{k+1} = 0. \tag{28}$$

Indeed the minimum-energy interpolant is linear on each cell, so its energy is the displayed sum. Thus $G \ge (h/4)I$ and $NGN \ge (\nu_*^2 h/4)I$. Every singular value of H_c is at least $c\rho\sqrt{s}$. The original $O(\rho)$ perturbation is therefore relatively $O(s^{-1/2})$ on every pair. It also bounds all remaining eigenvalues because F has rank $2k$. This proves (19) and the residual claim.

If the primitive L^1 norms are bounded, each cell integral is $O(\rho)$; hence (24) improves to $C\rho^2/h$. The mixed overlap is $O(1/S)$: split into regions closer to each center, use inverse-linear decay for the distant factor and the L^1 bound for the other. Thus $\beta = 1/s$ in (27), $\|B\| \le C\rho/\sqrt{h}$, and the shear and return errors are $O(s^{-1/2})$. □

The weighted Green hierarchy itself is classical finite-matrix geometry, related to flat-kernel asymptotics [12, 13]. The new issue here is its uniform return to every spectral channel of the original energy form, including the full trace and shrinking boundary cells. When h is comparable to ε, leading primitive overlaps can persist; Theorem 2.1, rather than the simplified Green law, remains the applicable general receiver.

4 Fixed-order moment cancellation

Fix an interior point $r_0 \in I$ and an integer $p \ge 1$. Put $\tau = G_I(r_0, r_0) > 0$. Let $L \not\equiv 0$ be real and belong to $C^{p+2}(\mathbb{R})$, with

$$|L^{(m)}(t)| \le A_m (1 + |t|)^{-m-1}, \qquad 0 \le m \le p + 2. \tag{29}$$

Take fixed real shifts $a_1, \dots, a_{m_0}$ and coefficients $c_1, \dots, c_{m_0}$ satisfying

$$\mu_m := \sum_j c_j a_j^m = 0 \quad (0 \le m < p), \qquad \mu_p \ne 0. \tag{30}$$

Repeated shifts may first be combined. For $d > 0$ and $\varepsilon > 0$, assume $\varepsilon \to 0$ and $\delta = d/\varepsilon \to 0$. In (2), set

$$V_\varepsilon(x) = \varepsilon^{-2} \sum_j c_j L'((x - r_0 - d a_j)/\varepsilon), \qquad \Psi_\delta(t) = \sum_j c_j L(t - \delta a_j). \tag{31}$$

There is no background in this section. Each translated profile has zero mean since L vanishes at both infinities.

Theorem 4.1 (Paired moment-cancellation law)**.** *There are isometries $J_\varepsilon : \mathbb{C}^2 \to H$ such that, for*

$$s_\varepsilon = \frac{\sqrt{\tau}}{2\sqrt{\varepsilon}} \|\Psi_\delta\|_2, \qquad \kappa_p = \frac{\sqrt{\tau}}{2} \frac{|\mu_p|}{p!} \left\|L^{(p)}\right\|_2 > 0, \tag{32}$$

one has

$$\left\| C_\varepsilon - J_\varepsilon \begin{pmatrix} 0 & s_\varepsilon \\ s_\varepsilon & 0 \end{pmatrix} J_\varepsilon^* \right\| \le C\delta^p, \qquad s_\varepsilon = \kappa_p \eta_\varepsilon (1 + O(\delta)), \tag{33}$$

where η_ε is given by (3). In particular, the largest positive and most negative eigenvalues satisfy

$$\lambda_\varepsilon^+ = s_\varepsilon + O(\delta^p), \qquad \lambda_\varepsilon^- = -s_\varepsilon + O(\delta^p), \tag{34}$$

and every other eigenvalue has absolute value $O(\delta^p)$. The constants depend only on the fixed interval, point, order, stencil and derivative bounds, not on d, ε sufficiently small.

Proof. Choose δ small enough that $\delta \max_j |a_j| \le 1$. For $t \in \mathbb{R}$ and $0 \le s \le 1$, $1 + |t - s\delta a_j| \ge (1 + |t|)/2$. Taylor's formula with integral remainder and (30) give

$$\Psi_\delta(t) = \frac{(-\delta)^p}{(p-1)!} \sum_j c_j a_j^p \int_0^1 (1-s)^{p-1} L^{(p)}(t - s\delta a_j)\, ds.$$

The same formula applied to L' proves

$$|\Psi_\delta(t)| \le C\delta^p (1 + |t|)^{-p-1}, \qquad |\Psi_\delta'(t)| \le C\delta^p (1 + |t|)^{-p-2}, \tag{35}$$

$$\Psi_\delta(t) = \frac{(-\delta)^p \mu_p}{p!} L^{(p)}(t) + O\big(\delta^{p+1}(1 + |t|)^{-p-2}\big). \tag{36}$$

Consequently

$$\|\Psi_\delta\|_1 + \sup_t |t\Psi_\delta(t)| \le C\delta^p, \qquad |\Psi_\delta'(t)| \le \frac{C\delta^p}{1 + t^2}. \tag{37}$$

The L^1 estimate uses $p \ge 1$. The remainder in (36) is $O(\delta^{p+1})$ in L^2, giving the asymptotic for s_ε. Decay and $L \ne 0$ imply $L^{(p)} \not\equiv 0$, since otherwise L would be a decaying polynomial.

Use the single marked point r_0, with unit-energy Green tent $g = G_I(\cdot, r_0)/\sqrt{\tau}$. Thus $g(r_0) = \sqrt{\tau}$ and $H = H_* \oplus \operatorname{span}\{g\}$, where now $H_* = \{u : u(r_0) = 0\}$. Regard the whole stencil as one profile Ψ_δ' and put $K(x) = \varepsilon^{-1}\Psi_\delta((x - r_0)/\varepsilon)$. Remark 2.3 and (37) give the complete compression and coupling estimates

$$\|B\| \le C\delta^p, \qquad T_\varepsilon = -\frac{\sqrt{\tau}}{2} R + E_\varepsilon, \qquad \|E_\varepsilon\| \le C\delta^p, \tag{38}$$

where $R' = \Pi K$ and Π subtracts the mean on each of the two fixed half-cells. The original trace is also small: by (14),

$$q_\varepsilon(g, g) = -\int_I K g g'\, dx = O(\delta^p), \tag{39}$$

because gg' is bounded and $\|K\|_1 \le \|\Psi_\delta\|_1$. No logarithmic trace term has been suppressed; cancellation has made the combined primitive integrable.

The omitted whole-line square tails in the norm of K are at most $C\delta^{2p}\varepsilon^{2p}$ by (35). Each half-cell integral of K is $O(\delta^p)$, so subtracting its mean costs $O(\delta^{2p})$ in squared norm; both cell lengths are fixed. Therefore

$$\left| \|R'\|_2^2 - \varepsilon^{-1} \|\Psi_\delta\|_2^2 \right| \le C\delta^{2p}, \qquad \left| \|R'\|_2 - \varepsilon^{-1/2} \|\Psi_\delta\|_2 \right| \le C\delta^p. \tag{40}$$

The second bound follows from $|\sqrt{A} - \sqrt{B}| \le \sqrt{|A - B|}$. The rank-one polar coupling from (38), together with (39) and (40), proves the whole-operator model (33). Minmax gives (34) and the residual-spectrum bound. Since $s_\varepsilon/\delta^p \sim \kappa_p/\sqrt{\varepsilon}$, the pair is relatively separated from the error even when it tends to zero. □

Corollary 4.2 (Critical separation)**.** *Under Theorem 4.1, $\|C_\varepsilon\| \to 0$ if and only if $\eta_\varepsilon \to 0$. If $\eta_\varepsilon \to \eta \in (0, \infty)$, precisely two eigenvalues have nonzero limits $\pm\kappa_p\eta$, and the rest tend uniformly to zero. If $\eta_\varepsilon \to \infty$, precisely one branch in each sign escapes, with leading terms $\pm\kappa_p\eta_\varepsilon$, while the remaining spectrum tends uniformly to zero. Thus the critical separation is $d \asymp \varepsilon^{1+1/(2p)}$. For arbitrary oscillating parameters the pointwise norm model, rather than a limiting regime, remains valid.*

4.1 Several neutral clusters and explicit contrast constants

Corollary 4.3 (Resolved neutral clusters)**.** *Use the fixed L, stencil and p of Theorem 4.1. Let k cluster centers satisfy (17), and take fixed nonzero real weights α_j. For*

$$V_\varepsilon(x)=\sum_{j=1}^{k}\alpha_j\varepsilon^{-2}\Psi'_\delta((x-r_j)/\varepsilon),\qquad \rho=\delta^p,\quad \nu=\frac{|\mu_p|}{p!}\left\|L^{(p)}\right\|_2,\quad D_\alpha=\operatorname{diag}(\alpha_1,\dots,\alpha_k),$$

the first k positive and negative magnitudes satisfy

$$\lambda_j^{\pm}=\pm\frac{\rho\nu}{2\sqrt{\varepsilon}}\sqrt{\lambda_j(D_\alpha G D_\alpha)}\,[1+O(\delta+\sqrt{\varepsilon/h})]. \tag{41}$$

Every remaining eigenvalue is $O(\rho)$. All data, including k,p and the weights, are fixed; only the declared scales and centers vary.

Proof. Normalize the profile at node j as $v_{j,\varepsilon}=\alpha_j\Psi'_\delta/\rho$. Equations (35)–(37) give uniform envelopes, uniformly bounded primitive L^1 norms and primitive norms $|\alpha_j|(\nu+O(\delta))$ bounded below. Apply the L^1 version of Theorem 3.1. Relative diagonal congruence pays the $O(\delta)$ changes of these norms. Diagonal sign conjugation shows that the weighted Green eigenvalues with weights $|\alpha_j|$ equal those with signed D_α. This proves every channel, including vanishing ones. □

The constants of a single collapsing outer cluster can also be made explicit. This is classical finite Green geometry; its transfer to the original operator is supplied by (41). Let $r_j=r_*+Rb_j$, with fixed ordered distinct b_j, fixed interior r_*, $R\to 0$ and $\varepsilon/R\to 0$. Put

$$\alpha=(\alpha_j)_{j=1}^k,\quad P_\alpha=I-\frac{\alpha\alpha^*}{\|\alpha\|^2},\quad K_{ij}=-\tfrac12|b_i-b_j|,\quad \tau=G_I(r_*,r_*).$$

Let $\mu_1\ge\cdots\ge\mu_{k-1}>0$ be the eigenvalues of $P_\alpha D_\alpha K D_\alpha P_\alpha$ on $\alpha^\perp$. Then the exact eigenvalues of $D_\alpha G D_\alpha$ obey

$$\gamma_1=\tau\|\alpha\|^2+O(R),\qquad \gamma_{i+1}=R\mu_i+O(R^2),\quad 1\le i<k. \tag{42}$$

For $k=1$ the contrast assertion is empty. To prove the formula, expand the explicit Green kernel:

$$G_{ij}=\tau+R\left[\left(\tfrac12-\frac{r_*-a}{\ell}\right)(b_i+b_j)-\frac{|b_i-b_j|}{2}\right]-\frac{R^2b_ib_j}{\ell}.$$

After weighted congruence, the constant matrix is $\tau\alpha\alpha^*$ and the affine terms vanish on $\alpha^\perp$. For $x\in\alpha^\perp$ put $z=D_\alpha x$; then $\sum_j z_j=0$ and

$$-\frac12\sum_{i,j}\overline{z_i}|b_i-b_j|z_j=\sum_{m=1}^{k-1}(b_{m+1}-b_m)\left|\sum_{j\le m}z_j\right|^2>0\quad(z\ne0). \tag{43}$$

The identity follows by expressing each distance as the sum of the gaps it crosses. Strictness follows successively from the prefix sums, and proves positivity of the μ_i. In the decomposition $\operatorname{span}(\alpha)\oplus\alpha^\perp$, an orthogonal rotation equal to $I+O(R)$ eliminates the order-R off-diagonal block against the isolated eigenvalue $\tau\|\alpha\|^2$. The remaining contrast block is $R(P_\alpha D_\alpha K D_\alpha P_\alpha)+O(R^2)$. Minmax proves (42), including repeated contrast eigenvalues; this is the familiar flat-kernel mechanism [12, 13].

Consequently one spectral pair in Corollary 4.3 has magnitude asymptotic to $\rho\nu\sqrt{\tau}\,\|\alpha\|/(2\sqrt{\varepsilon})$, and the other $k-1$ have magnitudes $\rho\nu\sqrt{R\mu_i}/(2\sqrt{\varepsilon})$. For two equal weights at $r_j=\pm R/2$ on

$(-1,1)$, the exact Green eigenvalues are $1-R/2$ and $(R/2)(1-R/2)$, so the two magnitudes are asymptotic to

$$\frac{\rho\nu}{2\sqrt{\varepsilon}}, \qquad \frac{\rho\nu\sqrt{R}}{2\sqrt{2\varepsilon}}.$$

The primary pair can escape while the secondary pair vanishes, has a finite nonzero limit or escapes. All are larger than the paid $O(\rho)$ residual on their own relative scales.

4.2 An explicit rational family at every fixed order

Take

$$L(t) = -\frac{2t}{1+t^2}, \qquad v(t) = L'(t) = \frac{2(t^2-1)}{(1+t^2)^2}, \tag{44}$$

and the centered pth finite difference

$$a_j = j - p/2, \qquad c_j = (-1)^j \binom{p}{j}, \qquad 0 \le j \le p. \tag{45}$$

The classical finite-difference identities give $\mu_m = 0$ for $m < p$ and $\mu_p = (-1)^p p!$. The recurrence

$$L^{(m)}(t) = \frac{P_m(t)}{(1+t^2)^{m+1}}, \quad P_0 = -2t, \quad P_{m+1} = (1+t^2)P_m' - 2(m+1)tP_m \tag{46}$$

proves (29) for every fixed order. Its leading numerator coefficient is $-2(-1)^m m!$. With Fourier convention $\widehat{L}(\xi) = \int L(t)e^{-it\xi}\,dt$, the classical transform of $(1+t^2)^{-1}$ gives, in L^2, $\widehat{L}(\xi) = 2\pi i\,\mathrm{sgn}(\xi)e^{-|\xi|}$. Plancherel yields

$$\left\|L^{(p)}\right\|_2^2 = \frac{1}{2\pi}\int_{\mathbb{R}} 4\pi^2|\xi|^{2p}e^{-2|\xi|}\,d\xi = 2^{1-2p}\pi(2p)!. \tag{47}$$

On $I = (-1,1)$ with $r_0 = 0$, $\tau = 1/2$, hence

$$\kappa_p = 2^{-p-1}\sqrt{\pi(2p)!}. \tag{48}$$

For every fixed p, taking $d = \varepsilon^{1+3/(4p)}$ makes the complete operator norm tend to zero, whereas $d = \varepsilon^{1+1/(4p)}$ gives one escape per sign. These are exact regimes of an explicit family, not numerical eigenvalue observations. Appendix A gives the beta-integral verification of (47) and the supporting portable programs.

5 Two-center collision regimes

On $I = (-1,1)$ take the profile (44) and consider

$$V_{\varepsilon,d}^{\sigma}(x) = \varepsilon^{-2}\left[v\left(\frac{x+d/2}{\varepsilon}\right) + \sigma v\left(\frac{x-d/2}{\varepsilon}\right)\right], \qquad \sigma \in \{1,-1\}. \tag{49}$$

Throughout this section $d > 0$, $d \to 0$, $\varepsilon \to 0$, and $\delta = d/\varepsilon$. The background is initially zero. A bounded background changes the operator by $O(1)$ in norm and hence preserves the diverging leading terms and bounded-bulk assertions below.

Lemma 5.1 (Primitive correlation)**.** *For the rational primitive L,*

$$\|L\|_2^2 = 2\pi, \qquad \int_{\mathbb{R}} L(t+\delta/2)L(t-\delta/2)\,dt = \frac{8\pi}{\delta^2+4}. \tag{50}$$

Consequently, for $L_\delta^\sigma(t) = L(t+\delta/2) + \sigma L(t-\delta/2)$,

$$\left\|L_\delta^+\right\|_2^2 = 4\pi\left(1+\frac{4}{\delta^2+4}\right), \qquad \left\|L_\delta^-\right\|_2^2 = \frac{4\pi\delta^2}{\delta^2+4}. \tag{51}$$

Proof. The norm follows from (47) with $p=0$. For the correlation write $h=\delta/2$; the rational integrand is

$$\frac{4(z^2-h^2)}{((z+h)^2+1)((z-h)^2+1)}.$$

For $h\neq 0$, direct substitution shows that the sum of its residues at the two upper-half-plane poles $h+i, -h+i$ is $-i/(1+h^2)$. The closing semicircle contributes zero by quadratic decay, so its integral is $2\pi/(1+h^2)$. At $h=0$, dominated convergence gives the same formula. Expansion of the squared sum and difference proves (51). □

Theorem 5.2 (Comparable or sub-width centers)**.** *For every fixed $D<\infty$, uniformly for $0<\delta\le D$ in the same-sign case, there is exactly one escaping eigenvalue per sign, with*

$$\lambda_\varepsilon^\pm = \pm\sqrt{\frac{\pi}{2\varepsilon}\left(1+\frac{4}{\delta^2+4}\right)} + o(\varepsilon^{-1/2}). \tag{52}$$

Every other eigenvalue is uniformly bounded. For opposite signs, the corresponding statement holds uniformly on every $0<\delta_0\le\delta\le D$, with

$$\lambda_\varepsilon^\pm = \pm\sqrt{\frac{\pi}{2\varepsilon}}\frac{\delta}{\sqrt{\delta^2+4}} + o(\varepsilon^{-1/2}). \tag{53}$$

Proof. Combine the profiles about the single midpoint. Their zero-mean inverse-square envelope and primitive Hardy constants are uniform for $\delta\le D$. The one-node compression in Theorem 2.1 is therefore uniformly bounded. Its unit-energy trace vector is $g(x)=(1-|x|)/\sqrt{2}$ and $g(0)=1/\sqrt{2}$. The principal coupling has norm

$$\frac{1}{2\sqrt{2}}\,\|\Pi K\|_2\,, \qquad K(x)=\varepsilon^{-1}L_\delta^\sigma(x/\varepsilon),$$

where Π subtracts the means on $(-1,0),(0,1)$. The omitted square tails outside $(-1,1)$ cost $O(1)$ since the combined primitive is $O((1+|t|)^{-1})$. Each half-cell integral costs at most $O(\log(1/\varepsilon))$, so subtracting its mean costs $O(\log^2(1/\varepsilon))$ in squared norm. Thus

$$\|\Pi K\|_2^2 = \varepsilon^{-1}\,\|L_\delta^\sigma\|_2^2 + O(\log^2(1/\varepsilon)).$$

By (14), the actual tent trace is $O(\log(1/\varepsilon))$ and the actual coupling differs from the principal one by $O(1)$. When the combined primitive norm is bounded away from zero, the two eigenvalues of the complete two-by-two block have leading terms $\pm\,\|L_\delta^\sigma\|_2/(2\sqrt{2\varepsilon})$. The codimension-one bound controls every other eigenvalue. Now use (51). Its lower bound is uniform for same signs on $[0,D]$ and for opposite signs on $[\delta_0,D]$. □

In particular, for same signs and $d=o(\varepsilon)$ the coefficient in (52) tends to $\sqrt{\pi}$, whereas the two-center separated description had two pairs. For opposite signs with $d=o(\varepsilon)$ one must resolve the smaller combined primitive; Theorem 5.2 is not being extended beyond its stated uniformity range.

5.1 Separated microscopic layers inside a collapsing cluster

Theorem 5.3 (Two distinct escape scales)**.** *If $\delta \to \infty$ and $d \to 0$, then for either sign $\sigma = \pm 1$ exactly two eigenvalues in each sign escape. Their two magnitudes are asymptotic to*

$$s_e = \sqrt{\frac{\pi}{2\varepsilon}}, \qquad s_o = \frac{\sqrt{\pi\delta}}{2}. \tag{54}$$

All remaining eigenvalues are uniformly bounded. For $\sigma = 1$, the two pairs belong respectively to the even and odd sectors. For $\sigma = -1$ the operator interchanges parity.

Proof. Apply Theorem 3.1 with profiles v and σv, $\rho = 1$ and $\nu_1 = \nu_2 = \sqrt{2\pi}$. For the two nodes $\pm d/2$, the smallest cell length is $h = d$ eventually and $s = d/\varepsilon = \delta \to \infty$. The exact Green eigenvalues are $1 - d/2$ and $(d/2)(1 - d/2)$. The relative law therefore gives both magnitudes in (54) and the bounded residual. Reflection preserves the same-sign operator and its even/odd trace channels give these respective scales. For opposite signs reflection anticommutes with the operator, so only the paired spectrum is asserted. A bounded background adds an $O(1)$ perturbation. □

5.2 Why a large mixed trace cannot simply be deleted

The general trace shear has a useful explicit predecessor in this opposite-sign example. It illustrates the difference between a paid coupling and an invalid absolute error estimate.

Put $R = d/2$. The unit-energy even and odd trace splines have nodal values of absolute size

$$\gamma_e^2 = (1 - R)/2, \qquad \gamma_o^2 = R(1 - R)/2. \tag{55}$$

The even spline is constant on $[-R, R]$ and linear to zero outside; the odd spline is linear through zero inside and linear to the Dirichlet endpoints outside. Choose the even spline positive inside and the odd spline positive at R. Direct integration of derivative squares gives (55). Write $K_l(x) = \varepsilon^{-1}L((x+R)/\varepsilon)$ and $K_r(x) = \varepsilon^{-1}L((x-R)/\varepsilon)$, with Π subtracting every cell mean. In the opposite-sign operator, the even trace couples through $K_l - K_r$ and the odd trace through $K_l + K_r$. The two coupling vectors have opposite parity and are orthogonal. Their actual principal norms obey

$$\widetilde{s}_e = \frac{\gamma_e}{2}\,\|\Pi(K_l - K_r)\|_2 \sim s_e, \qquad \widetilde{s}_o = \frac{\gamma_o}{2}\,\|\Pi(K_l + K_r)\|_2 \sim s_o.$$

These conclusions also follow directly from the full projected Gram estimate (27); all cell means are retained.

The original trace diagonals vanish by parity, but its mixed entry $a_\varepsilon = q_\varepsilon(e_e, e_o)$ need not vanish. The exact central primitive integral is

$$\int_{-R}^{R} (K_l - K_r)\,dx = -2\log(1 + \delta^2). \tag{56}$$

Inside this cell the derivative of $e_e e_o$ is $O(d^{-1/2})$. Outside it is $O(\sqrt{d})$, and the integral of $|K_l - K_r|$ over each outer cell is $O(1 + \log\delta)$. For this bound split distance from the adjacent node into $(0, \varepsilon), (\varepsilon, d), (d, 1)$: their costs are $O(1), O(\log\delta), O(1)$, using cancellation $O(d/x^2)$ on the last range. Integration by parts in the original trace yields

$$|a_\varepsilon| \le C(1 + \log\delta)/\sqrt{d} = o(s_e). \tag{57}$$

This estimate need not be $o(s_o)$, so it does not authorize deleting the trace relative to the secondary scale. In fact the inner product derivative is exactly $\gamma_e\gamma_o/R$, and the same outer bound gives

$$a_\varepsilon = \frac{\gamma_e\gamma_o}{R}\log(1+\delta^2) + O\big(\sqrt{d}(1+\log\delta)\big) \sim \sqrt{2/d}\log\delta.$$

For instance $d = \varepsilon^{3/4}$ makes $a_\varepsilon/s_o \to \infty$ while $a_\varepsilon/s_e \to 0$.

Reordering the complete four-dimensional principal model into its parity sectors gives off-diagonal block

$$B_\varepsilon = \begin{pmatrix} 0 & \widetilde{s}_o \\ \widetilde{s}_e & a_\varepsilon \end{pmatrix}. \tag{58}$$

The associated selfadjoint matrix has positive and negative singular values of B_ε. Since $\widetilde{s}_o/\widetilde{s}_e \to 0$ and $a_\varepsilon/\widetilde{s}_e \to 0$, its largest singular value is $\widetilde{s}_e(1+o(1))$. The product of the two singular values is exactly $|\det B_\varepsilon| = \widetilde{s}_e\widetilde{s}_o$. The smaller is therefore $\widetilde{s}_o(1+o(1))$, even if the available trace bound is much larger than s_o. The uniform additive $O(1)$ original-operator error is negligible on both diverging scales. Lemma 3.2 replaces this special determinant payment by one compatible trace shear for the full fixed-dimensional model, including primitive rank loss.

5.3 The sub-width cancellation threshold

Theorem 4.1 with $p = 1$ already determines the sharp opposite-sign scale as $\delta \to 0$. For this profile there is also a useful exact-norm refinement.

Proposition 5.4 (Dipole cancellation)**.** *If $\sigma = -1$ and $0 < \delta \le 1$, the complete operator admits a rank-two model with norm error $O(\delta)$ and paired entry*

$$s_\varepsilon = \sqrt{\frac{\pi}{2\varepsilon}}\,\frac{\delta}{\sqrt{\delta^2+4}}. \tag{59}$$

All other eigenvalues have absolute value $O(\delta)$. For $\delta \to 0$, put $t_\varepsilon = d/\varepsilon^{3/2}$. The norm-zero, finite-pair and escaping-pair regimes are respectively $t_\varepsilon \to 0$, $t_\varepsilon \to t \in (0,\infty)$ and $t_\varepsilon \to \infty$. The paired coefficient is $\sqrt{\pi}/(2\sqrt{2})$.

Proof. The combined potential is odd and its primitive L_δ^- is even. The mean-value theorem and rational derivative bounds give

$$|(L_\delta^-)'(t)| + |L_\delta^-(t)| \le \frac{C\delta}{1+t^2}, \qquad \sup_t |tL_\delta^-(t)| \le C\delta.$$

The single-midpoint compression and coupling errors are therefore $O(\delta)$. Its tent trace vanishes exactly, since the potential is odd and the tent square is even. For the projected derivative coupling, the omitted square tails are $O(\delta^2\varepsilon^2)$. The two half-cell integrals are equal, and direct integration gives

$$\int_0^1 \varepsilon^{-1}L_\delta^-(x/\varepsilon)\,dx = \log\frac{1+(\varepsilon^{-1}-\delta/2)^2}{1+(\varepsilon^{-1}+\delta/2)^2} = O(\delta\varepsilon).$$

Subtracting these means also costs $O(\delta^2\varepsilon^2)$ in squared norm. The resulting coupling norm differs from the whole-line one by $O(\delta\varepsilon)$, absorbed in $O(\delta)$. Equation (51) now gives (59). The trichotomy follows since $s_\varepsilon \sim \sqrt{\pi}\,t_\varepsilon/(2\sqrt{2})$. □

For example, $d = \varepsilon^2$ gives two distinct, nonzero zero-mean profiles for every $\varepsilon > 0$, yet the complete energy operator tends to zero in norm. This refutes a branch-count statement based solely on the number of distinct moving centers. The separated-center theorem itself is unaffected: its separation hypothesis is precisely what this counterfamily violates.

6 Scope and further questions

The bounded-error matrix theorem allows arbitrary moving distinct centers, including approach to an endpoint and loss of primitive rank. The relative Green law covers arbitrary fixed resolved hierarchies and boundary approach. Comparable-layer outer clusters and boundary distances of order ε can retain leading primitive overlap. Neither p nor the number of profiles grows in these theorems.

Signed stencil coefficients are essential for positive-order moment cancellation. They do not automatically arise as positive integer multiplicities of zeros in a logarithmic-curvature form. Thus the counterexamples establish the failure of an unrestricted moving-profile branch-count extrapolation, not a negative test for the actual Riemann theta form. No full-line spectrum or polynomial-degree cost is deduced from these bounded-interval completed-space results.

For a uniformly bounded background, the extra energy-Riesz operator has uniformly bounded norm by Poincare. It preserves fixed-amplitude resolved escapes and bounded residual spectrum. For amplitude ρ, an $O(\rho)$ perturbation preserves the full resolved error order; an $o(\rho\sqrt{h/\varepsilon})$ perturbation preserves all relative pair equivalents but may enlarge the residual bulk. An arbitrary $o(1)$ perturbation can swamp a vanishing principal channel. Finite or vanishing cancellation conclusions therefore require their stated zero background or an appropriately paid perturbation.

A Exact arithmetic checks and reproducibility

The analytic proofs establish the norm and asymptotic statements. The accompanying standard-library Python programs check supporting finite identities. The manuscript proves the operator-norm and asymptotic theorems analytically. The programs use explicit exception gates, so the checks remain active under optimized Python.

`verify_constants.py` checks 199 exact rational identities: the residue correlation at seven rational separations, derivative norms through order twelve, centered finite-difference moments and physical scaling exponents, and normalized moving Green geometry. `verify_beta.py` is a separate implementation of thirteen derivative-norm computations, 152 stencil moments through order sixteen, and four spline/Green geometries. It is an independent implementation, not an unrelated proof of the analytic theorems. `verify_geometry.py` adds 375 exact rational controls on eighteen Green/stiffness geometries and fifteen trace-shear models, including nested and boundary configurations. It tests the finite algebra, not the logarithmic estimates or spectral asymptotics.

For example, if $L^{(p)}(t) = P_p(t)/(1+t^2)^{p+1}$, the recurrence (46) gives a polynomial numerator exactly. The squared norm can then be evaluated without a Fourier transform by linearity and Euler's beta integral:

$$\frac{1}{\pi}\int_{\mathbb{R}} \frac{t^{2j}}{(1+t^2)^n}\,dt = \frac{(2j)!(2n-2j-2)!}{4^{n-1}j!(n-j-1)!(n-1)!}, \qquad n \geq 1, \quad 0 \leq j < n, \quad n, j \in \mathbb{Z}. \tag{60}$$

For the nonnegative integer indices in the programs, all terms on the right are rational. This reproduces (47) by a different calculation from the Plancherel proof. The two implementations share this classical beta identity. Reader build and execution instructions are supplied in `README.md`.

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References

[1] S. Schubr, *Spectral transitions for concentrated zero-mean energy forms*, research preprint (2026), Luminary Research Archive, edition 1.

[2] B. Simon, *The bound state of weakly coupled Schrödinger operators in one and two dimensions*, Ann. Physics **97** (1976), 279–288, author-hosted text.

[3] R. Blankenbecler, M. L. Goldberger and B. Simon, *The bound states of weakly coupled long-range one-dimensional quantum Hamiltonians*, Ann. Physics **108** (1977), 69–78, author-hosted text.

[4] F. Gesztesy and R. Nichols, *Birman–Schwinger principles for Schrödinger operators with distributional potentials revisited*, Integral Equations Operator Theory **98** (2026), article 6, doi:10.1007/s00020-025-02824-8, especially Lemma 3.1 and Appendix B.2.

[5] H. Douanla, *Bulk–surface spectral homogenization with indefinite mass in periodic perforated domains*, arXiv:2610.05487v1 (2026), especially Sections 1.1–1.4 and 4.1.

[6] Y. Golovaty, *1D Schrödinger operators with short range interactions: two-scale regularization of distributional potentials*, Integral Equations Operator Theory **75** (2013), 341–362, arXiv:1202.4711v3, doi:10.1007/s00020-012-2027-z.

[7] Y. Golovaty, *Two-parametric δ'-interactions: approximation by Schrödinger operators with localized rank-two perturbations*, J. Phys. A: Math. Theor. **51** (2018), 255202, arXiv:1801.09469v2.

[8] M. Gadella, J. Mateos-Guilarte, J. M. Muñoz-Castañeda and L. M. Nieto, *Two-point one-dimensional δ–δ' interactions: non-Abelian addition law and decoupling limit*, arXiv:1505.04359 (2015), especially the matching-matrix fusion calculation.

[9] Integrable.org, *Resolvents and spectral measures*, expository notes, `https://integrable.org/foundations/resolvents-and-spectral-measures/`, accessed October 25, 2026.

[10] R.-C. Li, *Relative perturbation theory: I. Eigenvalue and singular value variations*, SIAM J. Matrix Anal. Appl. **19**(4) (1998), 956–982, author-hosted text, Section 3, relation (3.7).

[11] N. J. Higham, *Eigenvalue inequalities for Hermitian matrices*, expository article, March 9, 2021, `https://nhigham.com/2021/03/09/eigenvalue-inequalities-for-hermitian-matrices/`, Ostrowski section.

[12] S. Barthelmé and K. Usevich, *Spectral properties of kernel matrices in the flat limit*, SIAM J. Matrix Anal. Appl. **42**(1) (2021), 17–57, doi:10.1137/19M129677X, arXiv:1910.14067v3, Theorems 4.5–4.6 and Section 4.4.

[13] K. Usevich and S. Barthelmé, *Computing asymptotic eigenvectors and eigenvalues of perturbed symmetric matrices*, arXiv:2407.17047v2, Theorem 21 and Section 4.3.

[14] H. Harbrecht, M. Multerer, O. Schenk and Ch. Schwab, *Multiresolution kernel matrix algebra*, Numer. Math. **156** (2024), 1085–1114, doi:10.1007/s00211-024-01409-8, Sections 3.2–3.3.