Closure correlations in paving, low-rank and small matroids and in complete graphs

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Abstract

Sample an independent set of a finite matroid after deleting two marked elements, with probability proportional to the product of nonnegative element weights. Çiçeksiz and Ravichandran conjectured that the events that the sample spans the first and the second marked element are nonnegatively correlated; for graphic matroids this is their connectivity conjecture for the weighted random forest. They also proposed the stronger statement that the covariance numerator has nonnegative coefficients in the weights. We prove this coefficientwise statement, and with it both conjectures, for paving matroids, for matroids of corank at most three, for every matroid of rank at most four, for every matroid on at most nine elements, and for the cycle matroid of every graph on at most nine vertices, at all nonnegative weights. For every complete graph on at least four vertices we prove both conjectures, strictly, at every common positive weight. The paving and corank cases use a circuit exchange and a two-pencil inequality in the dual. Identities for minors, direct sums and series-parallel extensions show that a smallest counterexample in a minor-closed class is simple, cosimple and connected, with a negative coefficient of full squarefree support and at most twice its rank in elements. This reduces the remaining cases to finite lists, among them the 617 simple rank-four matroids on eight elements and the two-connected graphs of minimum degree at least three on six to nine vertices. Exact integer computations settle every list, and independent programs recheck each of them.

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1 Introduction and main results

Let M be a finite matroid on E, with distinct marked elements e, f. Put $X = E \setminus \{e, f\}$. For nonnegative finite weights $\mathbf{x} = (x_g)_{g \in X}$, sample an independent set A of $M|X$ according to

$$\mathbb{P}_{\mathbf{x}}(A) = \frac{x^A}{Z(\mathbf{x})}, \qquad x^A = \prod_{g \in A} x_g, \qquad Z(\mathbf{x}) = \sum_{A \in \mathcal{I}(M|X)} x^A.$$

The empty set gives $Z \geq 1$. We study whether spanning one marked element makes spanning the other more likely. For a graphic matroid, a marked edge is in $\mathrm{cl}_M(A)$ exactly when its endpoints are connected by the sampled forest.

For $i, j \in \{0, 1\}$ define the category polynomials

$$I_{ij}(\mathbf{x}) = \sum_{\substack{A \in \mathcal{I}(M|X) \\ \mathbf{1}_{\{e \in \mathrm{cl}_M(A)\}} = i,\ \mathbf{1}_{\{f \in \mathrm{cl}_M(A)\}} = j}} x^A, \qquad \Gamma_{M;e,f} = I_{11} I_{00} - I_{10} I_{01}. \tag{1}$$

Then

$$\mathrm{Cov}_{\mathbf{x}}(\mathbf{1}_{\{e \in \mathrm{cl}_M(A)\}}, \mathbf{1}_{\{f \in \mathrm{cl}_M(A)\}}) = \frac{\Gamma_{M;e,f}(\mathbf{x})}{Z(\mathbf{x})^2}. \tag{2}$$

For $D \subseteq X$ we also write $x_D = \prod_{g \in D} x_g$. Write $P \succeq_{\mathrm{coef}} 0$ when every coefficient of P is nonnegative, and write $[x^\alpha]P$ for the coefficient of the monomial x^α in P.

The conjectures. Çiçeksiz and Ravichandran [3] posed the question for matroids and for graphs as follows.

Conjecture 7.1 of Çiçeksiz and Ravichandran ([3, Section 7])**.** *Let M be a finite matroid and let e, f be distinct elements of M. Assign a weight $z_g \geq 0$ to each element g, and sample an independent set A of $M \setminus \{e, f\}$ with probability proportional to $\prod_{g \in A} z_g$. Then* $\mathrm{Cov}(\mathbf{1}_{\{e \in \mathrm{cl}_M(A)\}}, \mathbf{1}_{\{f \in \mathrm{cl}_M(A)\}}) \geq 0$.

Conjecture 6.2 of Çiçeksiz and Ravichandran ([3, Section 6])**.** *Let G be a finite graph and let a, b, c, d be distinct vertices of G. Under the arboreal gas on G with a positive weight on each edge, $\mathbb{P}(a \leftrightarrow b,\ c \leftrightarrow d) \geq \mathbb{P}(a \leftrightarrow b)\,\mathbb{P}(c \leftrightarrow d)$.*

Here the arboreal gas samples an acyclic edge set F with probability proportional to the product of its edge weights, and $a \leftrightarrow b$ means that F contains a path from a to b. Adjoining marked edges $e = ab$ and $f = cd$ to G identifies these events with the closure events of Conjecture 7.1 for the cycle matroid $M(G{+}e{+}f)$, and (2) expresses both covariances through $\Gamma_{M;e,f}$. After their Conjecture 6.3, Çiçeksiz and Ravichandran list three strengthenings of Conjecture 6.2, the first of which asks that every monomial coefficient of the covariance numerator be nonnegative [3, Section 6]. They observe that nonnegativity of every coefficient of $\Gamma_{M;e,f}$ implies Conjecture 7.1, and they test it by exact computation on the families of [3, Table 1]. We call $\Gamma_{M;e,f} \succeq_{\mathrm{coef}} 0$ the *coefficientwise form*. Since $Z > 0$, it gives the inequality of Conjecture 7.1 for M at all nonnegative weights.

Main results. Recall that a rank-r matroid is *paving* if every circuit has at least r elements. We use the standard rank, closure, circuit, duality and simplification conventions of [12]; the corank is $|E| - \operatorname{rk} M$.

Theorem 1.1 (A circuit condition at one mark)**.** *Suppose every circuit of M containing e and avoiding f has size at least $\operatorname{rk} M$. Then $\Gamma_{M;e,f} \succeq_{\mathrm{coef}} 0$. The same conclusion holds with e and f interchanged.*

Theorem 1.2 (Corank at most three)**.** *If $|E| - \operatorname{rk} M \leq 3$, then $\Gamma_{M;e,f} \succeq_{\mathrm{coef}} 0$ for every pair of distinct marked elements.*

Theorem 1.1 applies to every paving matroid. The parallel identities proved below extend this to every matroid with paving simplification. In particular it applies to all matroids of rank at most three. Theorem 1.2 covers a complementary range: its dual argument transports the probability law to spanning sets and the events to coloop events. The next results cover every matroid of small rank or size and every small graph.

Theorem 1.3 (Rank at most four)**.** *If $\operatorname{rk} M \leq 4$, then $\Gamma_{M;e,f} \succeq_{\mathrm{coef}} 0$ for every pair of distinct marked elements.*

Theorem 1.4 (At most nine elements)**.** *If $|E| \leq 9$, then $\Gamma_{M;e,f} \succeq_{\mathrm{coef}} 0$ for every pair of distinct marked elements.*

Theorem 1.5 (Graphs on at most nine vertices)**.** *Let G be a finite multigraph, loops allowed, with at most nine vertices. Then $\Gamma_{M(G);e,f} \succeq_{\mathrm{coef}} 0$ for all distinct edges e, f of G.*

For complete graphs at a common weight we prove the conjectures for every number of vertices, with strict inequality.

Theorem 1.6 (Complete graphs at a common weight)**.** *Let $n \geq 4$ and $\lambda > 0$, and give every edge of K_n fugacity λ.*

(i) *For distinct vertices a, b, c, d, the inequality of Conjecture 6.2 holds strictly for $G = K_n$ and for $G = K_n - ab - cd$. It holds when the six edges with both ends in $\{a, b, c, d\}$ carry arbitrary nonnegative fugacities, in particular after deleting any of them, and strictly so when $n \geq 6$.*

(ii) *For distinct vertices a, b, c, the strict inequality $\mathbb{P}_G(a \leftrightarrow b,\ b \leftrightarrow c) > \mathbb{P}_G(a \leftrightarrow b)\,\mathbb{P}_G(b \leftrightarrow c)$ holds when the three edges with both ends in $\{a, b, c\}$ carry arbitrary nonnegative fugacities, in particular for $G = K_n$ and $G = K_n - ab - bc$.*

(iii) *For the graphic matroid $M(K_n)$, every pair of distinct marks and a common positive weight on the unmarked elements, the covariance in* (2) *is positive.*

Corollary 1.7 (Connectivity correlations)**.** *Let G be a finite multigraph with at most nine vertices and nonnegative finite edge weights, and sample a forest F of G with probability proportional to the product of its edge weights. Then for all vertices a, b, c, d, distinct or not,*

$$\mathbb{P}(a \leftrightarrow_F b,\ c \leftrightarrow_F d) \ \geq\ \mathbb{P}(a \leftrightarrow_F b)\,\mathbb{P}(c \leftrightarrow_F d).$$

For every matroid of rank at most four or with at most nine elements, and for all nonnegative weights, the two closure events in (2) *are nonnegatively correlated.*

How far the conjectures are answered. Theorems 1.3–1.5 and Corollary 1.7 answer Conjecture 7.1 of [3] for every matroid of rank at most four and for every matroid on at most nine elements, and Conjecture 6.2 of [3] for every graph on at most nine vertices. In each case they prove the coefficientwise strengthening proposed in [3, Sections 6 and 7], at all nonnegative weights and, for graphs, for arbitrary vertices a, b, c, d. Theorems 1.1 and 1.2 and Corollary 3.2 give the same conclusion for paving matroids, for matroids with paving simplification and for matroids of corank at most three. The identities of Section 3 carry it to every minor of a matroid assembled from these classes by series and parallel extensions and direct sums; Corollary 7.1 records the graph case, which includes every subdivision of every graph on at most nine vertices. Theorem 1.6 answers the equal-weight case of Conjecture 6.2 for every complete graph K_n with $n \geq 4$, strictly, and the equal-weight case of Conjecture 7.1 for every graphic matroid $M(K_n)$ with $n \geq 4$, which for $n \geq 5$ is neither paving nor of corank at most three.

These statements extend the evidence of [3]. For a connected graph $H = G - e - f$ at a common fugacity λ, [3, Proposition 6.1] expands the covariance of the two connectivity events as $\tau_2^{\times}(H; e, f)\,\tau(H)^{-1}\lambda^{-1} + O(\lambda^{-2})$, where $\tau(H)$ counts spanning trees and $\tau_2^{\times}(H; e, f)$ counts spanning two-component forests separating both endpoint pairs. The covariance is therefore positive for all large λ when $\tau_2^{\times}(H; e, f) > 0$, and it vanishes identically otherwise. Çiçeksiz and Ravichandran also verified coefficientwise positivity for K_4 [3, Example 6.4], and they tested it on uniform matroids, finite geometries, the matroids S_8 and V_8, their duals and random families [3, Table 1]. Theorems 1.1–1.5 hold at every nonnegative weight and for every graph and matroid in the stated ranges, and Theorem 1.6 holds on every complete graph at every common positive weight.

For example, Corollary 1.7 applies to the arboreal gas on K_9, on $K_{4,5}$ and on the cube graph Q_3 with arbitrary edge weights, and Corollary 7.1 extends it to every graph obtained from these by subdividing edges and adding parallel edges. On graphs where the negative-correlation inequality of Kahn is also known, the connectivity covariance lies between zero and an explicit joint-bridge probability (Remark 8.2).

Relation to earlier correlation inequalities. The ordinary edge-Rayleigh difference compares inclusion of marked elements in a sampled independent set. The present events instead ask whether a set avoiding the marks spans them. This distinction can be seen algebraically. Put $a = I_{00}, b = I_{10}, c = I_{01}, d = I_{11}$ and $Z = a + b + c + d$. Let j generate the category-00 sets for which adjoining e puts f in closure. The full independent-set polynomial, with marked weights s, t, is

$$F(s,t) = Z + s(a+c) + t(a+b) + st(a-j).$$

Its marked-edge Rayleigh difference is

$$F_s F_t - F F_{st} = (a+c)(a+b) - Z(a-j) = Zj - \Gamma_{M;e,f}. \tag{3}$$

For graphic matroids this is the decomposition of the Rayleigh difference into a crossing term and the connectivity covariance found by Çiçeksiz and Ravichandran [3, Corollary 3.3]; Section 8 returns to it. Erickson [5, Lemma 4.0.5 and Theorem 4.4.1] develops paving exchange counts and proves a coefficientwise lower comparison of the independent-set and basis Rayleigh differences. Equation (3) shows that closure positivity requires an upper bound $F_sF_t - FF_{st} \preceq_{\mathrm{coef}} Zj$, whereas a lower Rayleigh bound gives an upper bound on $\Gamma_{M;e,f}$. We therefore construct a monomial-preserving injection between the closure categories using classical fundamental circuits. For the same reason, Rayleigh-type theorems for low-rank matroids bound $\Gamma_{M;e,f}$ from above. For example, Zhao and Zhou [20] prove that every matroid of rank or corank at most three is Z-Rayleigh, so its independent-set polynomial is Rayleigh by the limit $q \to 0^+$; by (3) this gives $\Gamma_{M;e,f} \leq Zj$ pointwise for positive weights, which complements the positivity proved here. The corank proof uses a two-pencil counting argument in the rank-three dual; Zhao and Zhou [20] use the same bipartite incidence graph of lines through two fixed elements for the Rayleigh direction. Minor and coefficient reduction, also present in [5], organize the finite reductions of Section 5.

Organization. Sections 2–4 prove Theorems 1.1 and 1.2 and the exact identities for parallel and series extensions, minors and direct sums. Section 5 shows that a smallest counterexample in a minor-closed class is simple, cosimple and connected, has a negative coefficient of full squarefree support, and has rank at least four, corank at least four and at most twice its rank in elements. This reduces Theorems 1.3–1.5 to finite lists, which Sections 6 and 7 check. Section 8 treats the Rayleigh direction, Section 9 proves Theorem 1.6, and Appendix A keeps the binary certificate of edition 2 as an independent check of the coefficient computations. Edition 3 adds Theorems 1.3–1.6, Corollary 1.7 and Sections 5–9 to edition 2 (`https://doi.org/10.5281/zenodo.24407844`); in particular, Section 9 proves the equal-weight case of the forest-connectivity conjecture of Çiçeksiz and Ravichandran on every complete graph K_n with $n \geq 4$, and the equal-weight case of their closure-correlation conjecture for every graphic matroid $M(K_n)$ with $n \geq 4$, with strict inequality (Theorem 1.6).

2 A recoverable circuit exchange

Proof of Theorem 1.1. If a mark is a loop, its indicator is constant and $\Gamma = 0$. If either exclusive category is empty, $I_{10}I_{01} = 0$ and the conclusion is immediate. Otherwise let $r = \operatorname{rk} M$, fix a total order on X, and take

$$A \in \mathcal{I}(M|X), \quad e \in \operatorname{cl}(A),\ f \notin \operatorname{cl}(A), \qquad B \in \mathcal{I}(M|X), \quad f \in \operatorname{cl}(B),\ e \notin \operatorname{cl}(B).$$

We construct an injection from these ordered pairs to category 11 times category 00, preserving the multiset union and hence every monomial.

The independent set A spans e, so $A \cup \{e\}$ contains a unique circuit through e. By hypothesis its size is at least r. Since A does not span f, it is nonspanning, and therefore $|A| \leq r - 1$. It follows that

$$|A| = r - 1, \qquad A \cup \{e\} \text{ is a circuit.}$$

In particular $\operatorname{cl}(A)$ is a hyperplane.

Let K_f be the unique circuit in $B \cup \{f\}$. There exists $v \in K_f \setminus \{f\}$ outside $\operatorname{cl}(A)$: otherwise $f \in \operatorname{cl}(K_f \setminus \{f\}) \subseteq \operatorname{cl}(A)$, a contradiction. Choose the least such v and set

$$C = A \cup \{v\}, \qquad D = B \setminus \{v\}.$$

The set C is an independent r-set, hence a basis and a category-11 set. The subset D of B is independent and does not span e. It does not span f either: a circuit in $D \cup \{f\}$ would be a second circuit in $B \cup \{f\}$ avoiding v, whereas its unique circuit is K_f. Thus D belongs to category 00.

Since $v \notin \mathrm{cl}(A)$, it is not in A, and $x^A x^B = x^C x^D$, including any squared variables from $A \cap B$. To recover the input, take the unique fundamental e-circuit in $C \cup \{e\}$. It equals $A \cup \{e\}$, so it determines A, then the singleton $C \setminus A = \{v\}$, and finally $B = D \cup \{v\}$. This recovery proves injectivity, and hence the coefficientwise inequality. □

The hypothesis concerns only circuits through one mark that avoid the other. Thus short circuits elsewhere in M, and short circuits through both marks, are permitted. The exchange uses the fundamental f-circuit of B and allows $|B| < r - 1$.

Corollary 2.1 (Strictness for paving matroids)**.** *Let M be paving of rank $r \geq 3$, with strictly positive unmarked weights. Then the covariance in* (2) *is positive if and only if $E \setminus \{e, f\}$ spans M.*

Proof. If the deletion does not span, at least one mark lies outside its closure, so that mark's indicator is constantly false. If it spans, choose a basis B in the deletion. Its degree-r monomial has coefficient at least one in $I_{11}I_{00}$, using the empty set in the second factor. Each nonzero exclusive category polynomial has degree $r-1$, by the circuit-size argument above. Thus $I_{10}I_{01}$ has degree $2r-2 > r$ and cannot cancel that coefficient. All other coefficients of Γ are nonnegative. □

3 Parallel extensions and coefficient locality

A marked loop gives $\Gamma = 0$. Parallel nonloop marks have identical closure indicators, so $I_{10} = I_{01} = 0$. Unmarked loops have no effect on the independent-set polynomials.

Lemma 3.1 (Parallel extensions)**.** *Coefficientwise closure positivity is preserved by adding a parallel copy of any marked or unmarked element.*

Proof. A copy of an unmarked nonloop g, with weight t, replaces x_g by $x_g + t$ in each category polynomial and in Γ. This preserves nonnegative coefficients.

For a new unmarked copy of e, use a, b, c, d as above. Split $a = j + k$, where j counts category-00 sets for which adjoining e puts f in closure. A set can accept the new copy precisely when it does not already span e. The new categories are consequently

$$a' = a, \qquad c' = c, \qquad b' = b + tk, \qquad d' = d + t(c + j).$$

Hence

$$\Gamma' = \Gamma + t((c + j)a - kc) = \Gamma + tj(a + c) \succeq_{\mathrm{coef}} \Gamma. \tag{4}$$

The case of a copy of f is symmetric. Loop copies are loops. □

Corollary 3.2. *If the simplification of M is paving, then $\Gamma_{M;e,f} \succeq_{\mathrm{coef}} 0$ for every marked pair. In particular the assertion holds when* $\mathrm{rk}\, M \leq 3$.

Proof. For nonparallel nonloop marks choose a simplification retaining both marks. Apply Theorem 1.1 and restore parallel classes by Lemma 3.1, followed by loops. The other marked cases were handled above. Every simple rank-three matroid is paving; the same conclusion in smaller ranks follows from the loop and parallel cases or directly from Theorem 1.1. □

Lemma 3.3 (Coefficient locality)**.** *Let $I, T \subseteq X$ be disjoint. If I is dependent, then $[x_I^2 x_T]\Gamma_{M;e,f} = 0$. If I is independent, then*

$$[x_I^2 x_T]\Gamma_{M;e,f} = [x_T]\Gamma_{(M/I)|(T\cup\{e,f\});e,f}. \tag{5}$$

Consequently coefficientwise closure positivity is preserved by deletion and contraction of unmarked elements. Any negative coefficient of a rank-r marked matroid has a marked minor witness on at most $2r$ elements.

Proof. Exponent two forces every element of I into both independent factors. This is impossible when I is dependent. Otherwise the two factors are $I\cup U, I\cup V$, where (U, V) is an ordered partition of T. Independence passes to M/I, and $h \in \mathrm{cl}_M(I \cup U)$ is equivalent to $h \in \mathrm{cl}_{M/I}(U)$ for $h = e, f$. Deleting the remaining unmarked elements preserves these ranks and closure tests. This proves (5) with each ordered factorization counted once.

Deletion is the constant coefficient in a variable. Contraction of a nonloop g is its squared coefficient. Contraction of a loop equals deletion, so positivity is preserved in that case as well.

If a coefficient is negative, a contributing pair from $I_{10}I_{01}$ exists. Both factors are independent and nonspanning, so each has size at most $r - 1$. Thus $2|I| + |T| \leq 2r - 2$. The witness in (5) has $|T| + 2 \leq 2r - 2|I| \leq 2r$ elements. □

Lemma 3.4 (Direct sums)**.** *If the marks belong to different components of M, then $\Gamma = 0$. If both belong to a component M_0, then $\Gamma_M = \Gamma_{M_0} Z^2_{M\setminus E(M_0)}$.*

Proof. Independent sets and their weights factor over components, and closure of a marked element depends only on its component. The assertions follow by factoring the category polynomials. □

Proposition 3.5 (Series extensions)**.** *The all-pair coefficientwise closure-positivity property is preserved by a series extension.*

Proof. Let M' replace an element g by the series pair g, h, using the dual of parallel extension. Circuits avoiding g are unchanged; a circuit containing g replaces it by $\{g, h\}$. This description also covers an old loop, while extension at a coloop gives two coloops.

First take marks outside the series pair. Write each old category as $I_{ij} = a_{ij} + zb_{ij}$, where $z = x_g$, and write $\Gamma_M(z) = G_0 + zG_1 + z^2G_2$. With $s = x_g, t = x_h$, selecting neither or one of the series elements has the independence and marked closure statuses of its old part avoiding g; selecting both has those of the old part with g adjoined. Hence

$$I'_{ij} = (1 + s + t)a_{ij} + stb_{ij}, \qquad \Gamma_{M'} = (1 + s + t)^2G_0 + st(1 + s + t)G_1 + s^2t^2G_2.$$

All prefactors have nonnegative coefficients.

If $g = e$ is marked and f is another old mark, put $(a, b, c, d) = (I_{00}, I_{10}, I_{01}, I_{11})$ and $t = x_h$. Without h, the mark e cannot be closed, so the old categories collapse according to their f-status. With h, every old independent set avoiding the marks can accept it, and both old statuses are restored. Thus

$$a' = (1 + t)a + b, \quad b' = tb, \quad c' = (1 + t)c + d, \quad d' = td, \qquad \Gamma_{M';e,f} = t(1 + t)\Gamma_{M;e,f}.$$

The case with h marked is symmetric. If both series elements are marked, a set avoiding both cannot close either and $\Gamma = 0$. These cases exhaust the marked pairs. □

Together the identities prove positivity for every minor of a matroid obtained by series and parallel extensions and direct sums from the all-pair-positive seed classes established here. For the all-pair property under parallel extension, note that exchanging the two copies is an automorphism, so a pair containing the new copy reduces to Lemma 3.1, and two parallel marks give $I_{10} = I_{01} = 0$. These operations enlarge the literal seed classes. For example, extend one element of $U_{4,8}$ twice in series. The resulting simple rank-six, ten-element matroid has corank four and retains five-element circuits on the seven untouched elements. It is therefore nonpaving, while Proposition 3.5 proves all-pair closure positivity. Contracting the two new elements recovers its nonbinary minor $U_{4,8}$.

4 Two pencils in the dual

We first isolate the coefficient count needed for the dual experiment. Let G be a finite simple bipartite graph with vertex classes L, H and edge variables w_g. Define

$$Z_G = \prod_{g \in E(G)} (1 + w_g) - 1, \qquad K_G = \sum_g w_g,$$
$$R_G = \sum_{u \in L} \left(\prod_{g \ni u} (1 + w_g) - 1 \right), \qquad C_G = \sum_{v \in H} \left(\prod_{g \ni v} (1 + w_g) - 1 \right).$$

Thus R_G and C_G generate nonempty subsets of individual row-stars and column-stars, while K_G generates single edges.

Lemma 4.1 (Two-star concentration)**.** *For every such graph, $K_G Z_G - R_G C_G \succeq_{\mathrm{coef}} 0$.*

Proof. Each factor is multiaffine. A term of $R_G C_G$ containing w_g^2 has its two factors on the row and column of g. Their only common edge is g, so every other variable has a forced factor. Its coefficient is at most one, and $K_G Z_G$ supplies one. Two different squared variables cannot occur in either product.

For squarefree support S of size $k \geq 2$, the coefficient in $K_G Z_G$ is k. The other product counts ordered nonempty partitions $S = A \sqcup B$ with A in a row and B in a column. For $k = 2$ there are two. For $k \geq 3$, if S is a row-star, the column factor is a singleton, giving k partitions; the column-star case is symmetric. Otherwise S contains disjoint edges $(u, w), (z, v)$. A row and a column covering both edges must be either (u, v) or (z, w). The edges covered by both of these cross-pairs are exactly those two disjoint edges. A third edge therefore rules out one of the pairs. For the remaining pair every edge is forced except its possible intersection edge, giving at most two partitions, and hence at most k. Constant and squarefree linear coefficients are zero in both products. These cases cover all monomials. □

Proof of Theorem 1.2. Let $N = M^*$ have rank $c \leq 3$. For $A \subseteq X$ put $B = X \setminus A$ and $S = B \cup \{e, f\}$. The dual rank formula gives

$$A \in \mathcal{I}(M) \iff S \text{ spans } N, \qquad h \in \mathrm{cl}_M(A) \iff h \text{ is a coloop of } N|S \tag{6}$$

for $h = e, f$ when S spans. Let $J_{ij}(\mathbf{y})$ generate these spanning complements with the indicated coloop statuses. Then

$$I_{ij}(\mathbf{x}) = x_X J_{ij}(\mathbf{x}^{-1}), \qquad \Gamma_M(\mathbf{x}) = x_X^2 (J_{11} J_{00} - J_{10} J_{01})(\mathbf{x}^{-1}).$$

All exponents in $J_{11} J_{00} - J_{10} J_{01}$ lie in $\{0, 1, 2\}$. Multiplication by x_X^2 reverses each exponent k to $2 - k$, so coefficientwise positivity is preserved. This is a polynomial identity; no reciprocal weight needs to be evaluated when a weight is zero.

If a mark is a loop of N, its coloop event is constantly false. If the marks are parallel nonloops of N, neither is a coloop of $N|S$. Thus we may assume they are independent, and $c \geq 2$. For $c = 2$ every B is admissible. Partition the unmarked nonloops of N into P parallel to e, Q parallel to f, and the remaining elements T, and denote the loops by W. In the original A variables write $p = x_P, q = x_Q, t = x_T$ and $U_D = \prod_{g \in D}(1 + x_g)$. The closure events require respectively $P \cup T$ and $Q \cup T$ to be contained in A. Direct multiplication gives

$$\Gamma_M = pqt\, U_P U_Q (U_T - t) U_W^2 \succeq_{\text{coef}} 0.$$

It remains to consider $c = 3$. Put $L_0 = \text{cl}_N\{e, f\}$. Each parallel class outside L_0 belongs to a unique line through e and a unique line through f. These two pencils define a simple bipartite graph G, with one edge for each outside parallel class. Indeed, two nonparallel outside classes on both the same e-line and the same f-line would force those lines to coincide and contain e, f and an outside element, contrary to the definition of L_0. For an outside class D_g substitute

$$w_g = \prod_{h \in D_g} (1 + y_h) - 1.$$

Its coefficients are nonnegative and it generates nonempty subsets of that class. Use Z_G, R_G, C_G, K_G after this substitution, with rows indexed by the f-pencil and columns by the e-pencil.

Partition the unmarked elements in L_0 into P parallel to e, Q parallel to f, the other nonloops T, and the loops W. Now put $U_D = \prod_{h \in D}(1 + y_h)$. Admissibility of B means choosing a nonempty outside subset, since $\{e, f\}$ has rank two. Its total polynomial is

$$J = Z_G U_P U_Q U_T U_W.$$

The event that e is a coloop means that $B \cup \{f\}$ has rank two. The outside subset must lie on one f-line; among nonloops of L_0, only those parallel to f may be selected. Thus its polynomial is $R_G U_Q U_W$. Similarly the other event has polynomial $C_G U_P U_W$. Both events occur precisely when all selected nonloops belong to one outside parallel class: the two distinct pencil lines have rank-one intersection. Consequently $J_{11} = K_G U_W$ and

$$\begin{aligned} J_{11}J_{00} - J_{10}J_{01} &= J_{11}J - (J_{11} + J_{10})(J_{11} + J_{01}) \\ &= U_P U_Q U_W^2 (K_G Z_G U_T - R_G C_G) \succeq_{\text{coef}} 0. \end{aligned}$$

For the last step, write the bracket as $(K_G Z_G - R_G C_G) + K_G Z_G (U_T - 1)$ and apply Lemma 4.1 and the nonnegative substitutions above. Exponent reversal proves the result for M. □

Corollary 4.2 (Low-corank strictness)**.** *Assume all weights are strictly positive and put $N = M^*$. If the marks are dependent in N, then $\Gamma_M = 0$. For independent marks in rank-two N, the covariance is positive precisely when N has an unmarked nonloop parallel to neither mark. For independent marks in rank-three N, use the outside-class graph G and the set T constructed in the preceding proof. The covariance is positive precisely when $T \neq \varnothing$ or the edges of G do not all share a single vertex.*

Proof. The dependent-mark and rank-two assertions follow from the displayed factorizations above. In rank three, G has an edge. If all its edges share a row, $R_G = Z_G, C_G = K_G$; the column case is symmetric. Thus $K_G Z_G - R_G C_G = 0$. Otherwise G contains disjoint edges g, h. The coefficient of $w_g^2 w_h$ in $K_G Z_G$ is one, while in $R_G C_G$ it is zero: the squared edge forces its row and column, neither containing h. Thus the nonnegative difference is nonzero. Also $K_G Z_G (U_T - 1)$ is nonzero exactly when T is nonempty. Positive parallel-class substitution and exponent reversal preserve strict positivity of these nonnegative polynomials. The covariance denominator is positive throughout. □

5 Minimal counterexamples

The identities of Section 3 reduce Theorems 1.3–1.5 to finite computations. We take a smallest marked matroid with a negative coefficient in a class closed under minors and show that it is rigid in several ways: each identity of Section 3, applied to a smaller member of the class, excludes one kind of degeneracy.

A class $\mathcal{C}$ of matroids is *minor-closed* if it is closed under isomorphism, deletion and contraction. A matroid is *all-pair positive* if $\Gamma_{M;e,f} \succeq_{\mathrm{coef}} 0$ for every pair of distinct elements e, f.

Proposition 5.1 (Structure of a minimal counterexample)**.** *Let $\mathcal{C}$ be a minor-closed class. Suppose that some $M \in \mathcal{C}$ and distinct $e, f \in E(M)$ satisfy $\Gamma_{M;e,f} \not\succeq_{\mathrm{coef}} 0$. Choose such a triple (M, e, f) with $|E(M)|$ minimal, and put $r = \operatorname{rk} M$ and $X = E(M) \setminus \{e, f\}$. Then:*

(a) the full squarefree coefficient $[x_X]\Gamma_{M;e,f}$ is negative;

(b) M is simple and connected;

(c) M is cosimple;

(d) $r \geq 4$, $|E(M)| - r \geq 4$ and $|E(M)| \leq 2r$;

(e) each mark lies in a circuit that avoids the other mark and has at least three and fewer than r elements.

Proof. (a) Choose a negative coefficient $[x_I^2 x_T]\Gamma_{M;e,f}$ with disjoint $I, T \subseteq X$. By Lemma 3.3, I is independent and the marked minor $N = (M/I)|(T \cup \{e, f\})$ satisfies $[x_T]\Gamma_{N;e,f} < 0$. Since $N \in \mathcal{C}$, minimality gives $|T| + 2 \geq |E(M)|$, so $T = X$ and $I = \varnothing$.

(b) A marked loop gives $\Gamma = 0$, and parallel nonloop marks give $\Gamma = I_{11}I_{00} \succeq_{\mathrm{coef}} 0$ (Section 3). Deleting an unmarked loop does not change Γ. If an unmarked element g is parallel to another element, then M is a parallel extension of $M \setminus g$, and Lemma 3.1 shows that $\Gamma_{M\setminus g;e,f} \not\succeq_{\mathrm{coef}} 0$. Each case contradicts minimality or the choice of (M, e, f), so M is simple. If the marks lie in different components, $\Gamma = 0$ by Lemma 3.4. Otherwise they lie in one component M_0, and $\Gamma_M = \Gamma_{M_0} Z^2$ with $Z^2 \succeq_{\mathrm{coef}} 0$, so $\Gamma_{M_0;e,f} \not\succeq_{\mathrm{coef}} 0$. The restriction M_0 belongs to $\mathcal{C}$, and minimality forces $M = M_0$.

(c) A connected matroid with at least two elements has no coloop. Suppose $\{g, h\}$ is a series pair. Then M is the series extension of M/h that replaces g by g, h. By minimality every marked pair of $M/h \in \mathcal{C}$ has nonnegative coefficients, so M/h is all-pair positive. Proposition 3.5 then makes M all-pair positive, contrary to the choice of (M, e, f).

(d) The bound $r \geq 4$ is Corollary 3.2, and the corank bound is Theorem 1.2. By (a), some pair (A, B) with $A \sqcup B = X$, A in category 10 and B in category 01 contributes to $[x_X]\Gamma_{M;e,f}$. Both sets are independent and nonspanning, so $|X| = |A| + |B| \leq 2r - 2$.

(e) This is the contrapositive of Theorem 1.1; in a simple matroid every circuit has at least three elements. □

For rank four, (a), (b), (d) and (e) give the localization of Corollary A.2, and (c) adds cosimplicity.

Corollary 5.2 (Finite reductions)**.**

(i) Suppose $[x_X]\Gamma_{N;e,f} \geq 0$ for every simple and cosimple rank-four matroid N on eight elements and all distinct $e, f \in E(N)$, where $X = E(N) \setminus \{e, f\}$. Then Theorem 1.3 holds.

(ii) Assume Theorem 1.3. Suppose $[x_X]\Gamma_{N;e,f} \geq 0$ for every simple and cosimple rank-five matroid N on nine elements and all distinct $e, f \in E(N)$. Then Theorem 1.4 holds.

(iii) Assume Theorem 1.3, and let $n_0 \geq 5$. Suppose that for every n with $6 \leq n \leq n_0$, every simple, 2-connected and 3-edge-connected graph H on n vertices with $n+3 \leq |E(H)| \leq 2n-2$, and all distinct edges e, f of H, the full squarefree coefficient $[x_{E(H)\setminus\{e,f\}}]\Gamma_{M(H);e,f}$ is nonnegative. Then $\Gamma_{M(G);e,f} \succeq_{\mathrm{coef}} 0$ for every finite multigraph G with at most n_0 vertices and all distinct edges e, f of G.

Proof. (i) Matroids of rank at most four form a minor-closed class. By Proposition 5.1(d), a minimal counterexample has rank four and $8 \leq |E| \leq 8$ elements. It is simple and cosimple by (b) and (c), and its full squarefree coefficient is negative by (a), contrary to the hypothesis.

(ii) Matroids on at most nine elements form a minor-closed class. By Theorem 1.3, a minimal counterexample has rank $r \geq 5$, and by (d) it has at least $r+4 \geq 9$ elements. Hence $r = 5$ and $|E| = 9$, and (a)–(c) contradict the hypothesis.

(iii) Cycle matroids of multigraphs with at most n_0 vertices form a minor-closed class: deletion keeps the vertices, and contraction of an edge identifies its ends or, for a loop, deletes it. Let $M(G^*)$ be a minimal counterexample, with G^* chosen without isolated vertices, and let n be its number of vertices. By (b), G^* has no loops and no parallel edges, and any two of its edges lie on a common cycle. Hence G^* is connected and has no cut vertex, since two edges in different blocks at a cut vertex lie on no common cycle. Thus G^* is 2-connected and $r = n-1$. Theorem 1.3 gives $r \geq 5$, so $n \geq 6$, and (d) gives $n+3 \leq |E(G^*)| \leq 2n-2$. The cocircuits of a cycle matroid are the bonds of the graph, and a connected graph with an edge cut of at most two edges has a bond of at most two edges. By (c), G^* is therefore 3-edge-connected. Now (a) contradicts the hypothesis. □

Remark 5.3. A 3-edge-connected graph has minimum degree at least three, so the graphs in (iii) have at least $\lceil 3n/2 \rceil$ edges, a bound that is at least $n+3$ for $n \geq 6$. The certificates of Section 7 check the larger list of simple 2-connected graphs with minimum degree at least three and $\lceil 3n/2 \rceil \leq |E| \leq 2n-2$. For $n = 9$ this list has 1,654 graphs, while 33,481 two-connected graphs on nine vertices have between 12 and 16 edges.

6 The matroid certificates

We verify the hypotheses of Corollary 5.2(i) and (ii). All computations use exact integer arithmetic; the section on reproducibility lists the programs.

6.1 Rank four on eight elements

A rank-r matroid is stored by its set of nonbases. A simple rank-r matroid on $n+1 > r$ elements has an element that is not a coloop, and deleting it leaves a simple rank-r matroid on n elements. Every simple rank-r matroid on $n+1$ elements is therefore a rank-preserving single-element extension of a simple one on n elements. By the theorem of Crapo [4] (see also [12, Section 7.2]), these extensions correspond to the linear subclasses $\mathcal{H}$ of the hyperplanes, namely the sets of hyperplanes that contain none, exactly one or all of the hyperplanes through each flat of rank $r-2$. In the extension $M+p$, an $(r-1)$-set S forms a nonbasis with p exactly when S is dependent or $\mathrm{cl}(S) \in \mathcal{H}$. The extension is simple exactly when $\mathcal{H}$ is not the set of all hyperplanes, which would make p a loop, and does not contain all hyperplanes through a point q, which would make p parallel to q.

Starting from the free matroid of rank r, the first program, which identifies isomorphic matroids with nauty [10] on the bipartite element–nonbasis incidence graph, produced $1, 3, 11, 49, 617$ simple rank-four matroids on $4, \ldots, 8$ elements and $1, 2, 4, 9, 23, 68$ simple rank-three matroids on $3, \ldots, 8$ elements. These numbers agree with the catalogue of Mayhew and Royle [13], as read in Section 6.2

below, and with the sequences A058733 and A058731 of [11]. The second program uses its own exact canonical forms (colour refinement followed by all permutations within cells). It reproduced both sequences, generated all 30,033 labeled simple single-element extensions of the 49 rank-four matroids on seven elements, and verified the submodularity of the rank function of each.

The first program expands $\Gamma_{M;e,f}$ completely, keying each monomial by its squared and its linear part; Table 1 summarizes the result. In rank four on eight elements, the full squarefree coefficients of the $17{,}276 = 617 \cdot 28$ marked pairs take values in $\{0, 2, 3, \ldots, 20\}$; the value 0 occurs 365 times and the value 1 never occurs. The second program computed the full squarefree coefficient of each of the 30,033 labeled matroids at each of its 28 marked pairs; the minimum is 0. The coefficient routine of the first program also reproduces the distribution of Table 4 over the 1716 binary configurations of Appendix A.

Simple matroids	Number	Marked pairs	Monomials examined	Minimum
Rank 3, at most 8 elements	107	2,577	393,491	0
Rank 4, at most 8 elements	681	18,506	4,646,527	0

Table 1: Every coefficient of $\Gamma_{M;e,f}$ for every simple matroid of rank three or four on at most eight elements and every marked pair. The monomials examined are those occurring in $I_{11}I_{00}$ or $I_{10}I_{01}$.

Proof of Theorem 1.3. Both programs verify the hypothesis of Corollary 5.2(i) for all simple rank-four matroids on eight elements, cosimple or not. By Table 1, every coefficient, and not only the full squarefree one, is nonnegative for every simple matroid of rank at most four on at most eight elements. □

6.2 Rank five on nine elements

Mayhew and Royle [9] enumerated the matroids on at most nine elements, and their catalogue [13] lists the rank function of each of the 385,369 nonempty matroids on at most nine elements, 383,172 of them on exactly nine. The first program read the catalogue and found 1, 383, 185,981, 188,936, 1,092, 66, 7 and 1 simple matroids on nine elements of ranks $2, \ldots, 9$; the rank-four number agrees with A058733. For each of the 188,936 simple rank-five matroids and each of its 36 marked pairs, it computed the full squarefree coefficient as a signed count over the 2^7 ordered partitions of X. The minimum over all 6,801,696 marked pairs is 0. Applied to the 617 simple rank-four matroids on eight elements of the catalogue, the same routine again gives minimum 0.

The single-element extension method of Section 6.1, run in rank five with the second program, reproduces the numbers $1, 4, 22, 217$ of simple rank-five matroids on $5, \ldots, 8$ elements found in the catalogue. The second program then checks the rank-five list independently. A simple and cosimple rank-five matroid N on nine elements has no coloop, so for each marked pair (e, f) the deletion $N \setminus e$ is a simple rank-five matroid on eight elements, and N is a labeled single-element extension of one of the 217 representatives with new element e. The program generates all 21,258,245 simple labeled single-element extensions of the 217 representatives, verifies the matroid axioms for each, and computes the full squarefree coefficient at all 170,065,960 marked pairs through the new element; the minimum is 0.

Proof of Theorem 1.4. The catalogue computation verifies the hypothesis of Corollary 5.2(ii), the extension computation verifies it again, and Theorem 1.3 holds. □

7 Forest connectivity

For graphic matroids the closure events are the connectivity events of the weighted random forest. We verify the hypothesis of Corollary 5.2(iii) for $n_0 = 9$, deduce Theorem 1.5 and Corollary 1.7, and extend them to subdivisions.

7.1 Graphs on at most nine vertices

For a graph H and distinct edges e, f put $X = E(H) \setminus \{e, f\}$. The full squarefree coefficient is the signed count

$$[x_X]\Gamma_{M(H);e,f} = \#\{U \subseteq X : \ U \in \mathcal{I}_{11},\ X \setminus U \in \mathcal{I}_{00}\} - \#\{U \subseteq X : \ U \in \mathcal{I}_{10},\ X \setminus U \in \mathcal{I}_{01}\},$$

where $\mathcal{I}_{ij}$ is the set of forests of $H - e - f$ in category ij. For a forest U, an edge h lies in $\mathrm{cl}(U)$ exactly when $U + h$ contains a cycle. Each program tabulates the acyclicity of every edge subset of H and evaluates this count for every pair of edges.

Three programs produced the graph lists of Table 2. The first lists, with nauty's `geng`, all 2-connected graphs on $n = 6, \dots, 9$ vertices with $n + 3$ to $2n - 2$ edges, the list needed without Proposition 5.1(c), and a second routine through the general matroid code agrees with it on all 1,044 marked pairs at $n = 6$. The second program generates the same lists without nauty, by adding a vertex of degree two or three to every connected graph on $n-1$ vertices and identifying graphs by exact canonical forms. It reproduces the 11,117 connected graphs on eight vertices of A001349 [11] and the numbers 433, 1,729, 4,796, 9,981 and 16,542 of 2-connected graphs on nine vertices with $12, \dots, 16$ edges of A339071 [11], and it recomputed all 3,632,921 marked pairs at $n = 9$. The third program generates only the graphs of minimum degree at least three, for $n = 6, \dots, 9$. Such a graph with at most $2n - 2$ edges has a vertex of degree three whose deletion leaves a connected graph with minimum degree at least two and at most three vertices of degree two, and that graph arises in turn from a graph on $n - 2$ vertices by adding a vertex of degree two or three. The program identifies graphs by an exact individualization–refinement canonical form and reproduces the numbers $1, 2, 4, 11, 34, 156, 1044$ of all graphs on at most seven vertices of A000088 [11]. Its lists coincide with the minimum-degree sublists of the second program.

	2-connected, $n+3 \le \lvert E\rvert \le 2n-2$			minimum degree ≥ 3, $\lceil 3n/2\rceil \le \lvert E\rvert \le 2n-2$			
n	graphs	marked pairs	min.	graphs	marked pairs	min.	$\lvert E\rvert$
6	26	1,044	0	6	252	0	9–10
7	226	12,964	0	21	1,342	0	11–12
8	2,446	195,925	0	171	15,007	0	12–14
9	33,481	3,632,921	0	1,654	193,600	0	14–16

Table 2: The graph certificates for Corollary 5.2(iii): the minimum of the full squarefree coefficient over all marked pairs. The last column gives the edge range of the minimum-degree lists.

Proof of Theorem 1.5. By Remark 5.3, the graphs in the hypothesis of Corollary 5.2(iii) belong to the minimum-degree lists of Table 2, whose coefficients are nonnegative for $n = 6, \dots, 9$. Combine Theorem 1.3 and Corollary 5.2(iii) with $n_0 = 9$. □

Proof of Corollary 1.7. If $a = b$ or $c = d$, one event is sure and the covariance vanishes. Otherwise adjoin to G new edges $e = ab$ and $f = cd$, parallel to existing edges if necessary. This does not change the number of vertices. The forests of G are the independent sets of $M(G + e + f) \,|\, E(G)$,

and $e \in \mathrm{cl}(F)$ holds exactly when $a \leftrightarrow_F b$. Equation (2) and Theorem 1.5 give the inequality, also when $\{a,b\} = \{c,d\}$. The matroid statement follows in the same way from Theorems 1.3 and 1.4. □

Corollary A.3 is the case of at most five vertices.

Corollary 7.1 (Blocks and subdivisions)**.** *Let G be a finite multigraph each of whose blocks is obtained from a graph on at most nine vertices by subdividing edges and adding parallel edges. Then $\Gamma_{M(G);e,f} \succeq_{\mathrm{coef}} 0$ for all distinct edges e, f of G, and the same holds for every minor of G. In particular, the inequality of Corollary 1.7 holds on every such graph, for example on every subdivision of K_9.*

Proof. The cycle matroid of G is the direct sum of the cycle matroids of its blocks, since every cycle lies in one block. By Lemma 3.4 it suffices to treat a single block B. Subdividing an edge is a series extension of the cycle matroid, and adding a parallel edge is a parallel extension. The graph on at most nine vertices from which B arises is all-pair positive by Theorem 1.5, and Proposition 3.5 and the remark after it preserve this property along both operations. Minors follow from Lemma 3.3, and the probabilistic statement from (2) as in the proof of Corollary 1.7. □

8 The Rayleigh direction

By (3), the marked Rayleigh difference $F_s F_t - F F_{st}$ of the full independent-set polynomial equals the crossing term Zj minus the closure numerator. For a graph G with edges $e = ab$ and $f = cd$ and $H = G - e - f$, Çiçeksiz and Ravichandran [3, Corollary 3.3] write this as

$$\frac{F_s F_t - F F_{st}}{Z_H^2} = \mathbb{P}_H(J_{ef}) - \mathrm{Cov}_H\big(\mathbf{1}_{\{a \leftrightarrow b\}}, \mathbf{1}_{\{c \leftrightarrow d\}}\big), \tag{7}$$

where J_{ef} is the event that each of e and f, but not both, can be added to the sampled forest without creating a cycle, so that $\mathbb{P}_H(J_{ef}) = j/Z_H$. The negative-correlation conjecture of Kahn [7] (see also Grimmett and Winkler [6]), stated in [3, Conjecture 1.2] at a common fugacity, asserts that the left side is nonnegative. Its form for arbitrary positive weights asks that the cycle matroid be I-Rayleigh, in the terminology of Erickson [5]; this property is studied by Wagner [19] and by Semple and Welsh [16]. In contrast with Theorem 1.5, the Rayleigh difference is not coefficientwise nonnegative.

Lemma 8.1 (Restriction)**.** *Let N be a restriction of M containing e and f. For every monomial supported on $E(N)\backslash\{e,f\}$, its coefficient in the marked Rayleigh difference of M equals its coefficient in that of N.*

Proof. Setting $x_g = 0$ for $g \notin E(N)$ turns the independent-set polynomial of M into that of N and commutes with the derivatives in s and t; it keeps exactly the monomials supported on $E(N)$. □

Çiçeksiz and Ravichandran [3, Example 6.4] expand the Rayleigh difference of K_4 with $e = 12$ and $f = 34$ as

$$(z_{13}z_{24} - z_{14}z_{23})^2 + z_{13}z_{24}(1 + z_{13} + z_{24}) + z_{14}z_{23}(1 + z_{14} + z_{23}),$$

whose coefficient of $z_{13}z_{14}z_{23}z_{24}$ is -2, while the corresponding coefficient of the connectivity numerator is 4. By Lemma 8.1, whenever $e = ab$ and $f = cd$ are opposite edges of a K_4 subgraph of a graph G, the coefficient of $x_{ac}x_{ad}x_{bc}x_{bd}$ in the Rayleigh difference of G is -2. In particular

every pair of disjoint edges of K_n, $n \geq 4$, has a negative Rayleigh coefficient. Table 3 records all 2-connected simple graphs on four, five and six vertices. On four vertices the pairs with a negative coefficient are exactly the three opposite pairs of K_4; on five and six vertices, 20 and 507 of them are not opposite in any K_4 subgraph. The closure numerator Γ has no negative coefficient on any of these pairs, in accordance with Theorem 1.5.

n	Graphs	Marked pairs	Negative pairs	K_4-opposite pairs	Minimum
4	3	31	3	3	-2
5	10	240	44	24	-4
6	56	2,496	696	189	-12

Table 3: Negative coefficients of the forest Rayleigh difference $F_sF_t - FF_{st}$ over all pairs of distinct edges of 2-connected simple graphs on n vertices. Every K_4-opposite pair is negative.

Remark 8.2 (Two-sided bounds). For a graph G on at most nine vertices, Theorem 1.5 and (7) give

$$0 \leq \mathrm{Cov}_H\big(\mathbf{1}_{\{a\leftrightarrow b\}}, \mathbf{1}_{\{c\leftrightarrow d\}}\big) \leq \mathbb{P}_H(J_{ef})$$

at all positive weights whenever G is I-Rayleigh. Both bounds therefore hold for every simple graph on at most seven vertices, which is I-Rayleigh by [5, Corollary 5.3.2], and for every simple graph on at most nine vertices with a vertex cover of at most three vertices, such as $K_{3,6}$, which is I-Rayleigh by [15, Theorem 7.1]. Further classes on which the upper bound holds are collected in [15, Table 1].

9 Complete graphs at every common weight

For a finite graph G with positive edge fugacities $\mathbf{w} = (w_g)$, let Z_G be its forest generating polynomial and $Z_G(\cdot)$ its restriction to the forests in an event, and put

$$\Gamma_G(a,b;c,d) = Z_G\, Z_G(a \leftrightarrow b,\ c \leftrightarrow d) - Z_G(a \leftrightarrow b)\, Z_G(c \leftrightarrow d).$$

Conjecture 6.2, recalled in the introduction, asks that for distinct vertices a, b, c, d

$$\mathbb{P}_G(a \leftrightarrow b,\ c \leftrightarrow d) \ \geq\ \mathbb{P}_G(a \leftrightarrow b)\,\mathbb{P}_G(c \leftrightarrow d), \qquad \text{equivalently} \qquad \Gamma_G(a,b;c,d) \geq 0. \tag{8}$$

Of the three levels listed after it in [3, Section 6] we use two: (L2) after setting every $w_g = \lambda$, every coefficient of $\Gamma_G(a,b;c,d)$ in λ is nonnegative; (L3) $\Gamma_G(a,b;c,d) \geq 0$ at every common fugacity $\lambda > 0$, which is the equal-fugacity case of Conjecture 6.2. Adjacent marks ab, bc in Conjecture 7.1 give the events $a \leftrightarrow b$ and $b \leftrightarrow c$ in the same way as disjoint marks give those of (8).

This section proves Theorem 1.6. By that theorem the equal-fugacity case (L3) of Conjecture 6.2 holds for every complete graph, and the equal-weight case of Conjecture 7.1 holds for the graphic matroid of every complete graph. Theorem 1.6 covers every common fugacity on complete graphs. Theorem 9.2 below also allows arbitrary fugacities on the edges among the marked vertices and a separate fugacity for the edges from each marked vertex to the remaining vertices. For (iii) this means that the unmarked edges among the endpoints of the marks may carry arbitrary nonnegative weights; the covariance is then nonnegative, and positive when the marks share a vertex or $n \geq 6$. For $n \leq 9$ the nonstrict inequalities of Theorem 1.6, in the coefficientwise form and at arbitrary nonnegative weights, are contained in Theorem 1.5, and for $n \leq 5$ already in Corollaries A.1 and A.3. Theorem 1.6 extends them to every n at a common weight and adds strictness. For $n \geq 10$ the matroid $M(K_n)$ is simple and non-paving, with rank $n-1 \geq 9$, corank $\binom{n-1}{2} \geq 36$ and $\binom{n}{2} \geq 45$

elements, so it lies outside the ranges of Theorems 1.1, 1.2, 1.3, 1.4 and 1.5, and of Corollary 7.1: a K_n minor of a graph in that corollary lies in one block, since K_n is 2-connected, and there a branch set containing no vertex of the underlying graph on at most nine vertices lies on one subdivided edge and meets at most two other branch sets, so $n \le 9$. The large-fugacity result of [3, Proposition 6.1] is recalled in the introduction. For complete graphs with $4 \le n \le 40$ we also record level (L2) (Proposition 9.15).

Route of the proof. Fix the marked vertices and call the remaining vertices the background. Each tree of the forest inside the background meets the marked vertices through edges whose fugacities depend only on the marked endpoint, so contracting it leaves a single site joined to the marked vertices (Lemma 9.4). In the fermionic representation of unrooted forests of Caracciolo, Jacobsen, Saleur, Sokal and Sportiello [1], these sites can be integrated out, and the partition function becomes a linear functional of the polynomial $\prod_i(1+\rho_i t)$ of the site masses (Proposition 9.7). Averaging over the background replaces this polynomial by its mean, the one-hub polynomial of the background. For a complete background it is an affine image of a combination of two consecutive Hermite polynomials, so its roots are real (Lemma 9.8), and a deterministic family of positive sites has the same partition functions (Corollary 9.9). These steps are the complete-background case of the rank-one replacement [15, Theorem 4.2 and Section 4] and of the real-rootedness of the one-hub polynomial [15, Corollary 4.1]; the Hermite form for complete graphs is in [14, Section 13.1]. We include short proofs to keep the section self-contained. The new ingredients are the following. For three marked vertices the replacement preserves the whole connectivity law (Proposition 9.10). For four marked vertices one crosswise term is quadratic in the masses, and the correction it produces is a covariance of two statistics of the background forest (Proposition 9.13). This covariance is nonnegative by an induction on the number of vertices (Proposition 9.14). The remaining inequalities for finitely many sites reduce to at most five active sites and are checked exactly (Proposition 9.11).

9.1 Complete attachments

Throughout this section, graphs are finite and loopless and may have parallel edges; each edge g carries a weight $w_g \ge 0$, and an edge of weight zero may be omitted. A forest is an acyclic edge set, $w(F) = \prod_{g\in F} w_g$, $Z_G = \sum_F w(F)$, and $Z_G(\mathcal{E})$ restricts the sum to forests in the event $\mathcal{E}$. For a partition θ of a set of vertices, G/θ identifies the vertices of each block, deletes the resulting loops and keeps parallel edges. Thus the forests of G/θ are the forests of G in which these identifications create no cycle. If π is a partition of a vertex set R, $W_\pi(G)$ denotes the total weight of the forests of G whose trees induce the partition π on R.

Definition 9.1 (Complete attachments)**.** Let R be a finite set of marked vertices, $m \ge 0$, and V a set of m further vertices. Let $\lambda > 0$, let $s = (s_h)_{h\in R}$ be nonnegative scales with $|s| = \sum_h s_h > 0$, and let Q be a weighted graph on R. The graph $K_m^R(\lambda, s, Q)$ has vertex set $R \cup V$, the edges of Q, an edge of weight λ between any two vertices of V, and an edge of weight s_h between $h \in R$ and each $v \in V$. We write $Z_m = Z_{K_m}$ for the forest polynomial of K_m at weight λ.

With $s_h = \lambda$ and Q the complete graph on R at weight λ, the graph $K_m^R(\lambda, s, Q)$ is $K_{m+|R|}$ at weight λ. For a partition θ of R, the quotient $K_m^R(\lambda, s, Q)/\theta$ is again a complete attachment, with marked set R/θ, scales $s_B = \sum_{h\in B} s_h$ and the quotient multigraph Q/θ.

Theorem 9.2 (Complete attachments)**.** *Let $G = K_m^R(\lambda, s, Q)$.*

(a) *If $R = \{1,2,3,4\}$, then $\Gamma_G(1,2;3,4) \ge 0$.*

(b) *If $R = \{1,2,3\}$, then $\Gamma_G(1,2;2,3) \ge 0$.*

9.2 Background trees as sites

Definition 9.3 (Site graphs)**.** For a finite list $\rho = (\rho_1, \dots, \rho_k)$ of nonnegative masses, the site graph $\mathcal{S}_\rho = \mathcal{S}_\rho^R(s, Q)$ consists of Q together with k further vertices $1', \dots, k'$, the sites, and an edge of weight $s_h\rho_i$ between $h \in R$ and i' for all h and i. Put

$$p_\rho(t) = \prod_{i=1}^{k}(1 + \rho_i t), \qquad \alpha(\rho) = \sum_i \rho_i, \qquad e_2(\rho) = \sum_{i<l} \rho_i\rho_l.$$

For a forest J of K_m, $\rho(J)$ is the list of the sizes (numbers of vertices) of the trees of J, isolated vertices included; thus $\alpha(\rho(J)) = m$.

Lemma 9.4 (Sites)**.** *Let $G = K_m^R(\lambda, s, Q)$ and let J be a forest of K_m. For a forest F of G with $F \cap E(K_m) = J$, contracting each tree of J to a single vertex gives a forest F' of $\mathcal{S}_{\rho(J)}$, and for every function φ of F',*

$$\sum_{F:\, F\cap E(K_m)=J} w(F)\,\varphi(F') = \lambda^{|J|} \sum_{F'} w(F')\,\varphi(F'),$$

the right-hand sum running over all forests F' of $\mathcal{S}_{\rho(J)}$. Moreover F and F' induce the same partition of R, and two marked vertices are joined in F by a path through no other marked vertex exactly when they are adjacent in F' or adjacent to a common site of F'.

Proof. Write $F = J \cup X'$, where X' avoids $E(K_m)$. By the contraction rule for the graphic matroid [12, Section 3.1], $J \cup X'$ is a forest exactly when X' is a forest of G/J, the graph in which each tree of J is contracted to a point. The edges of G/J outside $E(K_m)$ are those of Q and, for each $h \in R$ and each tree C of J, a parallel class of $|C|$ edges of weight s_h from h to the point C. A forest uses at most one edge of a parallel class, and the class has total weight $s_h|C|$, which is the weight of the edge from h to the site of C in $\mathcal{S}_{\rho(J)}$. The background edges outside J are absent from F. This proves the identity and the statements on R, since the trees of J contain no marked vertex. □

9.3 A fermionic forest integral

Let $\mathcal{V}$ be a finite set and let $\mathfrak{G}_\mathcal{V}$ be the real Grassmann algebra generated by $\psi_v, \bar\psi_v$ $(v \in \mathcal{V})$. Put $\tau_v = \bar\psi_v\psi_v$. Even elements commute and $\tau_v^2 = 0$. For $\mathcal{V}' \subseteq \mathcal{V}$ and $\Upsilon \in \mathfrak{G}_\mathcal{V}$, the integral $\int_{\mathcal{V}'} \Upsilon \in \mathfrak{G}_{\mathcal{V}\setminus\mathcal{V}'}$ is defined by writing $\Upsilon = \Upsilon_0 \prod_{v\in\mathcal{V}'} \tau_v + \Upsilon_1$, where $\Upsilon_0 \in \mathfrak{G}_{\mathcal{V}\setminus\mathcal{V}'}$ and every monomial of Υ_1 misses a generator indexed by $\mathcal{V}'$, and setting $\int_{\mathcal{V}'} \Upsilon = \Upsilon_0$; we write $\int = \int_\mathcal{V}$. Since the τ_v are even, $\int_\mathcal{V} = \int_{\mathcal{V}\setminus\mathcal{V}'} \int_{\mathcal{V}'}$. For a real matrix $Y = (Y_{uv})$ indexed by $\mathcal{V}$ and $V_0 \subseteq \mathcal{V}$,

$$\int \exp\Big(\sum_{u,v} \bar\psi_u Y_{uv} \psi_v\Big) \prod_{v\in V_0} \tau_v = \det Y[\mathcal{V} \setminus V_0],$$

the principal minor on $\mathcal{V} \setminus V_0$: the factors τ_v, $v \in V_0$, remove every term containing a generator at V_0, and the coefficient of $\prod_{v\notin V_0} \tau_v$ in the exponential of the remaining quadratic form is the Leibniz expansion of that minor.

For a weighted graph G on $\mathcal{V}$ put

$$\mathcal{A}_G = \sum_{v\in\mathcal{V}} \tau_v + \sum_{g=uv\in E(G)} w_g\big((\bar\psi_u - \bar\psi_v)(\psi_u - \psi_v) - \tau_u\tau_v\big).$$

Lemma 9.5 (Unrooted forests)**.** $Z_G = \int \exp(\mathcal{A}_G)$.

Proof. This is the fermionic representation of unrooted spanning forests of [1]; we include the short derivation. The edge differences in $\mathcal{A}_G$ sum to $\sum_{u,v} \bar\psi_u L_{uv} \psi_v$ with L the weighted Laplacian of G. The remaining terms are even and $\tau_v^2 = 0$, so

$$\exp\Big(\sum_v \tau_v - \sum_{g=uv} w_g \tau_u \tau_v\Big) = \sum_{(V_0,\mathcal{M})} (-1)^{|\mathcal{M}|} \prod_{g\in\mathcal{M}} w_g \prod_{v\in V_0 \cup V(\mathcal{M})} \tau_v,$$

summed over sets V_0 of vertices and sets $\mathcal{M}$ of edges forming a matching whose vertex set $V(\mathcal{M})$ is disjoint from V_0. By the principal-minors matrix-tree theorem [2], $\det L[\mathcal{V} \setminus (V_0 \cup V(\mathcal{M}))]$ is the total weight of the spanning forests in which every tree contains exactly one vertex of $V_0 \cup V(\mathcal{M})$. Join the two trees rooted at the ends of each edge of $\mathcal{M}$ by that edge. This is a bijection between the triples $(V_0, \mathcal{M}, \text{rooted forest})$ and the spanning forests whose trees are each marked by one vertex or by one edge. Every tree thus receives the total sign weight (number of vertices) minus (number of edges), which is one, and the integral equals $\sum_F w(F) = Z_G$. □

Lemma 9.6 (Elimination of sites)**.** *For $\mathcal{S}_\rho = \mathcal{S}_\rho^R(s, Q)$ put*

$$T_s = \sum_{h\in R} s_h \tau_h, \quad \Theta = |s| - T_s, \quad \beta = \sum_{h\in R} s_h \bar\psi_h, \quad \gamma = \sum_{h\in R} s_h \psi_h, \quad \Omega = \beta\gamma.$$

Then

$$Z_{\mathcal{S}_\rho} = \int_R \exp(\mathcal{A}_Q + \alpha(\rho) T_s) \prod_{i=1}^{k} (1 + \rho_i \Theta - \rho_i^2 \Omega).$$

Proof. Since $(\bar\psi_h - \bar\psi_{i'})(\psi_h - \psi_{i'}) = \tau_h + \tau_{i'} - \bar\psi_h \psi_{i'} - \bar\psi_{i'} \psi_h$, the action of $\mathcal{S}_\rho$ is $\mathcal{A}_Q + \alpha(\rho) T_s + \sum_i \mathcal{X}_i$ with

$$\mathcal{X}_i = (1 + \rho_i \Theta)\tau_{i'} - \rho_i \beta \psi_{i'} - \rho_i \bar\psi_{i'} \gamma.$$

The $\mathcal{X}_i$ are even, commute with each other and with $\mathcal{A}_Q$, and $\mathcal{X}_i$ involves only the generators at i' among the sites. Hence $\int_{\{1',\dots,k'\}} \exp(\sum_i \mathcal{X}_i) = \prod_i \int_{\{i'\}} \exp(\mathcal{X}_i)$. Since $\mathcal{X}_i^3 = 0$, $\exp(\mathcal{X}_i) = 1 + \mathcal{X}_i + \mathcal{X}_i^2/2$, and the coefficient of $\tau_{i'}$ is $1 + \rho_i\Theta$ from $\mathcal{X}_i$; from $\mathcal{X}_i^2/2$ it is the product of the two odd terms, $\frac{1}{2}\rho_i^2(\beta\psi_{i'}\bar\psi_{i'}\gamma + \bar\psi_{i'}\gamma\beta\psi_{i'}) = -\rho_i^2 \beta\gamma\, \tau_{i'}$, because $\psi_{i'}\bar\psi_{i'} = -\tau_{i'}$ and $\gamma\beta = -\beta\gamma$. Lemma 9.5 for $\mathcal{S}_\rho$ and integration over the sites first give the claim. □

For $\alpha \geq 0$ let $\mathcal{P}_\alpha$ be the linear space of real polynomials p with $p'(0) = \alpha p(0)$, and let $(\Delta_\alpha p)(t) = (\alpha p(t) - p'(t))/t$, a polynomial for $p \in \mathcal{P}_\alpha$. Every p_ρ with $\alpha(\rho) = \alpha$ lies in $\mathcal{P}_\alpha$.

Proposition 9.7 (Linear response)**.** *For fixed R, s, Q and $\alpha \geq 0$, the map*

$$\Lambda_\alpha[p] = \int_R \exp(\mathcal{A}_Q + \alpha T_s)\,(p(\Theta) - \Omega\,(\Delta_\alpha p)(\Theta))$$

is linear on $\mathcal{P}_\alpha$, and $Z_{\mathcal{S}_\rho} = \Lambda_\alpha[p_\rho]$ for every list ρ with $\alpha(\rho) = \alpha$.

Proof. Linearity is clear. Since $\beta^2 = \gamma^2 = 0$ we have $\Omega^2 = 0$, and all the factors commute, so

$$\prod_i (1 + \rho_i\Theta - \rho_i^2\Omega) = p_\rho(\Theta) - \Omega \sum_i \rho_i^2 \prod_{l\neq i} (1 + \rho_l \Theta).$$

The polynomial identity $t\sum_i \rho_i^2 \prod_{l\neq i}(1 + \rho_l t) = \alpha p_\rho(t) - p_\rho'(t)$ holds because $\alpha p_\rho(t) = \sum_i \rho_i (1 + \rho_i t) \prod_{l\neq i}(1+\rho_l t)$ and $p_\rho'(t) = \sum_i \rho_i \prod_{l\neq i}(1+\rho_l t)$. Substituting Θ and Lemma 9.6 prove the claim. □

Lemma 9.6 and Proposition 9.7 are the case needed here of the elimination in [15, Section 4].

9.4 Real roots and positive sites

Let

$$P_m(t) = \frac{1}{Z_m} \sum_J \lambda^{|J|} p_{\rho(J)}(t),$$

the sum over the forests J of K_m. Equivalently, $Z_m P_m(t)$ is the forest polynomial of K_m with one extra vertex joined to every vertex by an edge of weight t, since that vertex sends at most one edge to each tree. We write $\mathbb{E}_m$ for expectation under the forest law of K_m at weight λ; thus $P_m(t) = \mathbb{E}_m\, p_\rho(t)$ with $\rho = \rho(J)$.

Lemma 9.8 (Real roots)**.** *Put $\mu = 1/\lambda$ and $\mathcal{H}_j(\zeta) = j!\,[\nu^j] \exp(\zeta\nu - \mu\nu^2/2)$. Then*

$$Z_m P_m(t) = \lambda^m \big(\mathcal{H}_m(\zeta) - m\,\mathcal{H}_{m-1}(\zeta)\big), \qquad \zeta = m + \mu(1+t).$$

All m roots of P_m are real and negative. Hence $P_m(t) = \prod_{i=1}^m (1+\sigma_i t)$ with $\sigma_i > 0$ and $\sum_i \sigma_i = m$.

Proof. By Cayley's formula and the exponential formula [17, Section 5.1], $\sum_{m\ge0} Z_m P_m(t) \xi^m/m! = \exp\big(\sum_{r\ge1} r^{r-2}\lambda^{r-1}(1+tr)\xi^r/r!\big)$. Let $T(z) = \sum_{r\ge1} r^{r-1} z^r/r!$, so that $T = ze^T$, and $U(z) = \sum_{r\ge1} r^{r-2} z^r/r!$. Both zU' and $z(T - T^2/2)' = zT'(1-T)$ equal T, and both U and $T - T^2/2$ vanish at 0, so $U = T - T^2/2$. With $z = \lambda\xi$ the exponent is $\mu(U(z) + tT(z))$, and therefore $Z_m P_m(t) = \lambda^m m!\,[z^m] \exp\big(\mu(T - T^2/2 + tT)\big)$. Lagrange inversion for $T = ze^T$ [17, Section 5.4] turns $[z^m]$ of a power series in T into $[\nu^m]$ of the same series in ν multiplied by $e^{m\nu}(1-\nu)$, and $m![\nu^m](1-\nu)e^{\zeta\nu - \mu\nu^2/2} = \mathcal{H}_m(\zeta) - m\mathcal{H}_{m-1}(\zeta)$.

Since $e^{\varkappa\omega - \omega^2/2} = \sum_j \mathrm{He}_j(\varkappa)\omega^j/j!$ for the probabilists' Hermite polynomials He_j, we have $\mathcal{H}_j(\zeta) = \mu^{j/2}\mathrm{He}_j(\zeta/\sqrt{\mu})$, and $\mathcal{H}_m - m\mathcal{H}_{m-1}$ is a positive multiple of $f = \mathrm{He}_m - \delta\,\mathrm{He}_{m-1}$ at $\varkappa = \zeta/\sqrt{\mu}$, with $\delta = m/\sqrt{\mu}$. The zeros of He_m and He_{m-1} are real and simple and strictly interlace [18]; Szegő treats the physicists' polynomials $2^{j/2}\mathrm{He}_j(\sqrt{2}\,\varkappa)$, whose zeros are those of He_j divided by $\sqrt{2}$. Let $\varkappa_1 < \cdots < \varkappa_{m-1}$ be the zeros of He_{m-1}. Exactly $m-i$ zeros of He_m exceed $\varkappa_i$, so $f(\varkappa_i) = \mathrm{He}_m(\varkappa_i)$ has sign $(-1)^{m-i}$. Hence f has a zero in each interval $(\varkappa_i, \varkappa_{i+1})$, one in $(\varkappa_{m-1}, \infty)$ and one in $(-\infty, \varkappa_1)$; for $m = 1$, f is linear. So f has m real zeros. As ζ is an increasing affine function of t, P_m has m real roots. Its coefficients are nonnegative, its constant term is one and its leading coefficient is $1/Z_m > 0$, which comes from the empty forest; so the roots are negative. Finally $\sum_i \sigma_i = P_m'(0) = \mathbb{E}_m\,\alpha(\rho) = m$. □

Lemma 9.8 is the complete-graph case of the real-rootedness in [15, Corollary 4.1], which holds for every background; the same Hermite form appears for the degree polynomial of a vertex of K_{m+1} in [14, Section 13.1]. We call $\sigma = \sigma^{(m)} = (\sigma_1, \dots, \sigma_m)$ the root masses of K_m. Since $P_m = p_\sigma$,

$$\mathbb{E}_m\, p_\rho(t) = p_\sigma(t), \qquad \mathbb{E}_m\, e_2(\rho) = e_2(\sigma), \qquad \alpha(\rho) = \alpha(\sigma) = m. \tag{9}$$

Corollary 9.9 (Positive sites)**.** *Let $G = K_m^R(\lambda, s, Q)$ and $\sigma = \sigma^{(m)}$. For every partition θ of R, $Z_{G/\theta} = Z_m\, Z_{\mathcal{S}_\sigma/\theta}$.*

Proof. Since G/θ is a complete attachment with site graph $\mathcal{S}_\sigma/\theta$, it suffices to take θ discrete. Lemma 9.4 with $\varphi \equiv 1$, Proposition 9.7 and (9) give $Z_G = \sum_J \lambda^{|J|}\Lambda_m[p_{\rho(J)}] = Z_m\Lambda_m[P_m] = Z_m\Lambda_m[p_\sigma] = Z_m Z_{\mathcal{S}_\sigma}$. □

Corollary 9.9 is the complete-background case of [15, Theorem 4.2], applied to the quotients of Q.

Proposition 9.10 (Three marked vertices)**.** *If $|R| \le 3$, then $W_\pi(G) = Z_m W_\pi(\mathcal{S}_\sigma)$ for every partition π of R, with G and σ as in Corollary 9.9.*

Proof. For $R = \{1,2,3\}$, write W_0 for the discrete partition, W_{ij} for $\{ij,k\}$ and W_{123} for the single block. A forest of G is a forest of $G/\{ij\}$ exactly when $i \not\leftrightarrow j$, and a forest of $G/\{123\}$ exactly when its partition is discrete. Hence

$$Z_{G/\{123\}} = W_0, \quad Z_{G/\{12\}} = W_0 + W_{13} + W_{23}, \quad W_{12} = \tfrac{1}{2}(Z_{G/\{13\}} + Z_{G/\{23\}} - Z_{G/\{12\}} - W_0),$$

the same formulas hold for W_{13}, W_{23}, and $W_{123} = Z_G - W_0 - W_{12} - W_{13} - W_{23}$. The same relations hold in $\mathcal{S}_\sigma$, so Corollary 9.9 proves the claim. For $|R| \le 2$ the quotient relations are simpler and the argument is the same. □

9.5 Finite site inequalities

A site i' is active in a forest of a site graph if it has degree at least two there. Removing the edges at inactive sites leaves the active part of the forest. An inactive site has degree at most one, so its edge creates no cycle and changes neither the partition of R nor the adjacency of marked vertices to common sites; summed over its choices it contributes $1+|s|\rho_i$. A forest whose partition of R has b blocks has at most $|R| - b$ active sites. Indeed, consider a tree of the forest containing r marked vertices and a active sites. The marked vertices and active sites of this tree span a forest with at least $2a$ edges at the active sites, and with at most $r + a - 1$ edges, so $a \le r - 1$; summing over the blocks gives the bound.

Consider a functional $\Phi(\rho) = \sum_{(F_1,F_2)} \epsilon(F_1,F_2)\, w(F_1)w(F_2)$, where F_1, F_2 range over the forests of two site graphs on the same sites, with marked sets R_1, R_2 and scales $s^{(1)}, s^{(2)}$, and the sign $\epsilon \in \{-1,0,1\}$ depends only on the active parts. Grouping the pairs by the set D of sites active in F_1 or F_2 gives

$$\Phi(\rho) = \sum_D \prod_{i \notin D} (1 + |s^{(1)}|\rho_i)(1 + |s^{(2)}|\rho_i)\ \Phi^{\mathrm{irr}}_D(\rho), \tag{10}$$

where Φ^{irr}_D sums $\epsilon\, w(F_1^\circ)w(F_2^\circ)$ over pairs of active parts F_1°, F_2° with active sites exactly D in total, multiplied by $1 + |s^{(j)}|\rho_i$ for every $i \in D$ that is inactive in the jth factor. By the symmetry of the sites, Φ^{irr}_D depends only on ρ_D and is obtained from the polynomial $\Phi^{\mathrm{irr}}_{[k]}$ of the model with $k = |D|$ sites by relabelling.

Proposition 9.11 (Site inequalities)**.** *Let ρ be any finite list of nonnegative masses.*

(a) *For $R = \{1,2,3,4\}$, arbitrary scales and an arbitrary weighted graph Q on R, $\Gamma_{\mathcal{S}_\rho}(1,2;3,4)$ is a polynomial with nonnegative coefficients in the masses, the scales and the weights of Q.*

(b) *The same holds for $R = \{1,2,3\}$ and $\Gamma_{\mathcal{S}_\rho}(1,2;2,3)$.*

(c) *Let $\mathcal{S}^{\mathrm{pair}}_\rho$ be the site graph on $R = \{u,v\}$ with scales (λ,λ) and an edge uv of weight λ, and $\mathcal{S}^{\mathrm{hub}}_\rho$ the site graph on $R = \{u,v,o\}$ with scales (λ,λ,η) and edges uv, uo, vo of weights λ,η,η. Let N_{uv} denote the event that the forest uses neither the edge uv nor two edges from u and v to a common site. Then*

$$\Phi_{\mathrm{hub}}(\rho) = Z_{\mathcal{S}^{\mathrm{pair}}_\rho}\, Z_{\mathcal{S}^{\mathrm{hub}}_\rho}(N_{uv}) - Z_{\mathcal{S}^{\mathrm{pair}}_\rho}(u \not\leftrightarrow v)\, Z_{\mathcal{S}^{\mathrm{hub}}_\rho}$$

is a polynomial with nonnegative coefficients in λ, η and the masses.

Proof. In (a) write $\Gamma = I_{11}I_{00} - I_{10}I_{01}$, where I_{ij} is the weight of the forests with $\mathbf{1}_{\{1\leftrightarrow 2\}} = i$ and $\mathbf{1}_{\{3\leftrightarrow 4\}} = j$; in (b) use the analogous representation with the events $1 \leftrightarrow 2$ and $2 \leftrightarrow 3$. Each of the three functionals then has the form above, with ϵ determined by the partitions of R and,

in (c), by the event N_{uv}, all of which depend only on the active parts. By the bound on active sites, in (a) a forest of I_{11} has at most three active sites and a forest of the other categories, which have at least two blocks, at most two, so $\Phi_D^{\mathrm{irr}} = 0$ for $|D| \geq 6$. In (b) and (c) the corresponding bound is $|D| \leq 3$. By (10) it suffices to show that $\Phi_{[k]}^{\mathrm{irr}}$ has nonnegative coefficients for $k \leq 5$, with independent indeterminates for the weights of Q and for the scales in (a) and (b). The program `v1_site_certificates.py` enumerates the active parts, forms these polynomials with integer arithmetic and finds no negative coefficient; it also confirms $\Phi_{[k]}^{\mathrm{irr}} = 0$ for $k = 6$ in (a) and $k \geq 4$ in (b) and (c). The site-free terms are $\Phi_{\varnothing}^{\mathrm{irr}} = \Gamma_Q$ in (a) and (b), and $\lambda\eta^2$ in (c). □

9.6 Four marked vertices

Lemma 9.12 (Pair sums)**.** *For a list ρ with $\alpha = \alpha(\rho)$ and $p = p_\rho$, put $S_1(t) = \sum_i \rho_i^2 \prod_{l\neq i}(1+\rho_l t)$ and $S_2(t) = \sum_{i\neq j} \rho_i^2\rho_j^2 \prod_{l\neq i,j}(1+\rho_l t)$. Then*

$$tS_1(t) = \alpha p(t) - p'(t), \qquad t^3 S_2(t) = t\big(2e_2(\rho)p(t) - 2\alpha p'(t) + p''(t)\big) + 2\big(\alpha p(t) - p'(t)\big).$$

Proof. The first identity was shown in the proof of Proposition 9.7. Write $p_{(i)} = \prod_{l\neq i}(1+\rho_l t)$ and $p_{(ij)} = \prod_{l\neq i,j}(1+\rho_l t)$. Since $\sum_{j\neq i}\rho_j(1+\rho_j t)p_{(ij)} = (\alpha - \rho_i)p_{(i)}$ and $S_1' = \sum_{i\neq j}\rho_i^2\rho_j p_{(ij)}$,

$$S_1' + tS_2 = \alpha S_1 - \sum_i \rho_i^3 p_{(i)}, \qquad \sum_i \rho_i^3 p_{(i)} = \frac{(\alpha^2 - 2e_2(\rho))p - S_1}{t},$$

the second because $\sum_i \rho_i^2(1+\rho_i t)p_{(i)} = (\sum_i \rho_i^2)p$. Substituting $S_1 = (\alpha p - p')/t$ and $S_1' = (\alpha p' - p'')/t - (\alpha p - p')/t^2$ and multiplying by t^2 gives the second identity. □

Thus S_1 is a linear function of p_ρ on $\mathcal{P}_\alpha$, and S_2 differs from such a function by $2e_2(\rho)p_\rho(t)/t^2$. For $m \geq 0$ and $\eta \geq 0$ define

$$\Psi_m(\eta) = \mathrm{Cov}_m\big(e_2(\rho),\, p_\rho(\eta)\big), \qquad \rho = \rho(J),$$

the covariance under the forest law of K_m at weight λ. Note that $e_2(\rho(J))$ is the number of pairs of vertices of K_m in different trees of J.

Proposition 9.13 (Four marked vertices)**.** *Let $G = K_m^R(\lambda, s, Q)$ with $R = \{1,2,3,4\}$, and $\sigma = \sigma^{(m)}$. Then*

$$\Gamma_G(1,2;3,4) = Z_m^2\Big(\Gamma_{\mathcal{S}_\sigma}(1,2;3,4) + Z_{\mathcal{S}_\sigma}\,\frac{4s_1s_2s_3s_4}{|s|^2}\,\Psi_m(|s|)\Big).$$

Proof. Put $I_{00} = Z_G(1 \not\leftrightarrow 2, 3 \not\leftrightarrow 4)$. Expanding the four categories gives $\Gamma_G(1,2;3,4) = Z_G I_{00} - Z_G(1 \not\leftrightarrow 2)Z_G(3 \not\leftrightarrow 4)$, the covariance of the complementary events. Here $Z_G(1 \not\leftrightarrow 2) = Z_{G/\{12\}}$, $Z_G(3 \not\leftrightarrow 4) = Z_{G/\{34\}}$, and $I_{00} = Z_{G/\{12,34\}} + j(G)$ with $j = W_{\{13,24\}} + W_{\{14,23\}}$, because a forest with $1 \not\leftrightarrow 2$ and $3 \not\leftrightarrow 4$ becomes cyclic in $G/\{12,34\}$ exactly when it joins 1 to 3 and 2 to 4, or 1 to 4 and 2 to 3. For the graphic matroid of G with added marks $e = 12$ and $f = 34$, $j(G)$ is the polynomial j of (3). The same relations hold in $\mathcal{S}_\sigma$, so by Corollary 9.9 it remains to compute $j(G)$.

In a forest of $\mathcal{S}_\rho$ with partition $\{13,24\}$, the pair 13 is merged by an edge of Q or by a site adjacent to 1 and 3 only, the pair 24 likewise by a different edge or site, and every other site is inactive. With q_{ij} the total weight of the edges of Q between i and j, this gives

$$\begin{aligned} j(\mathcal{S}_\rho) = {} & p_\rho(|s|)(q_{13}q_{24} + q_{14}q_{23}) + (q_{13}s_2s_4 + q_{24}s_1s_3 + q_{14}s_2s_3 + q_{23}s_1s_4)\,S_1(|s|) \\ & + 2s_1s_2s_3s_4\,S_2(|s|). \end{aligned}$$

By Lemma 9.12, $j(\mathcal{S}_\rho) = \Lambda'[p_\rho] + (4s_1s_2s_3s_4/|s|^2)\, e_2(\rho)p_\rho(|s|)$ for a linear map Λ' on $\mathcal{P}_m$. Lemma 9.4 gives $j(G) = \sum_J \lambda^{|J|} j(\mathcal{S}_{\rho(J)})$, and by (9)

$$j(G) = Z_m\Big(j(\mathcal{S}_\sigma) + \frac{4s_1s_2s_3s_4}{|s|^2}\,(\mathbb{E}_m[e_2(\rho)p_\rho(|s|)] - e_2(\sigma)p_\sigma(|s|))\Big) = Z_m\Big(j(\mathcal{S}_\sigma) + \frac{4s_1s_2s_3s_4}{|s|^2}\,\Psi_m(|s|)\Big).$$

Inserting this into $Z_G I_{00} - Z_G(1 \not\leftrightarrow 2)Z_G(3 \not\leftrightarrow 4)$ proves the formula. □

9.7 The background covariance

Proposition 9.14 (Background covariance)**.** *For all $m \geq 0$, $\lambda > 0$ and $\eta \geq 0$, $\Psi_m(\eta) \geq 0$.*

Proof. We use induction on m. For $m \leq 1$, $e_2(\rho) = 0$. Let $m \geq 2$ and fix two vertices u, v of K_m. The law of the forest is invariant under permutations of the vertices, and $e_2(\rho)$ counts the pairs in different trees, so

$$\Psi_m(\eta) = \tbinom{m}{2}\,\mathrm{Cov}_m(\mathbf{1}_{\{u \not\leftrightarrow v\}},\, p_\rho(\eta)).$$

Let K_m^+ be K_m with an extra vertex o joined to every vertex by an edge of weight η. Since $p_{\rho(J)}(\eta)$ is the total weight of the edge sets at o that extend J to a forest, $Z_m\mathbb{E}_m[\mathbf{1}_{\{u \not\leftrightarrow v\}}p_\rho(\eta)] = Z_{K_m^+}(N_{uv})$, where for forests F of K_m^+ the event N_{uv} means that u and v are not joined in $F \setminus o$. Thus

$$\frac{Z_m^2\,\Psi_m(\eta)}{\binom{m}{2}} = Z_m\, Z_{K_m^+}(N_{uv}) - Z_m(u \not\leftrightarrow v)\, Z_{K_m^+}. \tag{11}$$

Regard K_m and K_m^+ as complete attachments over the background K_{m-2} on the other vertices: K_m with marked set $\{u, v\}$, scales (λ, λ) and $Q = \{uv\}$, and K_m^+ with marked set $\{u, v, o\}$, scales (λ, λ, η) and edges uv, uo, vo of weights λ, η, η. Their site graphs are $\mathcal{S}_\rho^{\mathrm{pair}}$ and $\mathcal{S}_\rho^{\mathrm{hub}}$ of Proposition 9.11(c). Let $\sigma' = \sigma^{(m-2)}$. By Corollary 9.9, $Z_m = Z_{m-2}Z_{\mathcal{S}_{\sigma'}^{\mathrm{pair}}}$, $Z_m(u \not\leftrightarrow v) = Z_{m-2}Z_{\mathcal{S}_{\sigma'}^{\mathrm{pair}}}(u \not\leftrightarrow v)$ and $Z_{K_m^+} = Z_{m-2}Z_{\mathcal{S}_{\sigma'}^{\mathrm{hub}}}$.

By Lemma 9.4, N_{uv} in K_m^+ corresponds to the event of the same name in $\mathcal{S}_\rho^{\mathrm{hub}}$. A forest of $\mathcal{S}_\rho^{\mathrm{hub}}$ lies in this event exactly when its partition of $\{u, v, o\}$ is not $\{uv, o\}$ and, if it is the single block, u and v are joined to o by two different edges or sites. Writing $\kappa = 2\lambda + \eta$,

$$Z_{\mathcal{S}_\rho^{\mathrm{hub}}}(N_{uv}) = W_{u|v|o} + W_{uo|v} + W_{vo|u} + \Xi(\rho),$$

where $W_{u|v|o}$, $W_{uo|v}$, $W_{vo|u}$ are the weights of the indicated partitions of $\{u, v, o\}$ in $\mathcal{S}_\rho^{\mathrm{hub}}$ and, with S_1, S_2 as in Lemma 9.12,

$$\Xi(\rho) = \eta^2 p_\rho(\kappa) + 2\lambda\eta^2 S_1(\kappa) + \lambda^2\eta^2 S_2(\kappa).$$

Proposition 9.10 replaces the three partition weights. Lemma 9.12, Lemma 9.4 and (9), applied as in the proof of Proposition 9.13, give

$$Z_{K_m^+}(N_{uv}) = Z_{m-2}\Big(Z_{\mathcal{S}_{\sigma'}^{\mathrm{hub}}}(N_{uv}) + \frac{2\lambda^2\eta^2}{\kappa^2}\,\Psi_{m-2}(\kappa)\Big).$$

Substituting into (11),

$$\frac{Z_m^2\,\Psi_m(\eta)}{\binom{m}{2}Z_{m-2}^2} = \Phi_{\mathrm{hub}}(\sigma') + Z_{\mathcal{S}_{\sigma'}^{\mathrm{pair}}}\,\frac{2\lambda^2\eta^2}{(2\lambda+\eta)^2}\,\Psi_{m-2}(2\lambda+\eta).$$

The first term is nonnegative by Proposition 9.11(c), since σ' has positive entries, and the second by the induction hypothesis. □

9.8 Proofs of the complete-graph theorems

Proof of Theorem 9.2. Part (b) follows from Proposition 9.10: $\Gamma_G(1,2;2,3)$ is a quadratic expression in the partition weights $W_\pi(G)$, so $\Gamma_G(1,2;2,3) = Z_m^2\Gamma_{\mathcal{S}_\sigma}(1,2;2,3) \geq 0$ by Proposition 9.11(b). Part (a) follows from Proposition 9.13, Proposition 9.11(a) and Proposition 9.14. □

Proof of Theorem 1.6. Take R to be the marked vertices, $m = n - |R|$, $s_h = \lambda$ and Q the graph of the remaining edges inside R with their fugacities; Theorem 9.2 gives the nonstrict inequalities. For strictness, Propositions 9.13 and 9.14 give $\Gamma_G \geq Z_m^2\Gamma_{\mathcal{S}_\sigma}$ in (i), with equality in (ii) by Proposition 9.10. All terms of (10) are nonnegative at $\rho = \sigma$ by Proposition 9.11, so $\Gamma_{\mathcal{S}_\sigma} \geq \prod_{i\notin D}(1+|s|\sigma_i)^2\,\Phi_D^{\mathrm{irr}}(\sigma)$ for every set D of at most m sites. Each $\Phi_{[k]}^{\mathrm{irr}}$ has nonnegative coefficients, so it is positive at positive scales and masses and nonnegative weights of Q as soon as its value with all weights of Q set to 0 and all other variables set to 1 is positive. In (i) take $D = \varnothing$ for K_n and $K_n - ab - cd$: $\Gamma_{K_4}(1,2;3,4) = 8\lambda^3+30\lambda^4+32\lambda^5$ and $\Gamma_{K_4-12-34}(1,2;3,4) = 4\lambda^3+12\lambda^4+8\lambda^5$. For arbitrary nonnegative fugacities on the six inner edges and $n \geq 6$, take $|D| = 2$: the value of $\Phi_{[2]}^{\mathrm{irr}}$ described above is 720. In (ii) take $|D| = 1$, which is possible since $m = n-3 \geq 1$: the value of $\Phi_{[1]}^{\mathrm{irr}}$ described above is 4. These values are computed by `v1_site_certificates.py`. Part (iii) is (i) for $K_n - e - f$ when the marks e, f are disjoint and (ii) when they share a vertex, by (2) and the description of closure in a graphic matroid in Section 1. □

Proposition 9.15 (Coefficients at a common fugacity)**.** *For $4 \leq n \leq 40$, the polynomials $\Gamma_{K_n}(1,2;3,4)$, $\Gamma_{K_n-12-34}(1,2;3,4)$, $\Gamma_{K_n}(1,2;2,3)$ and $\Gamma_{K_n-12-23}(1,2;2,3)$ of the common fugacity λ have nonnegative coefficients. For $n \leq 9$ this is a special case of Theorem 1.5.*

Proof. A forest of K_n with the edges inside the marked set R restricted to a graph Q is $J \cup F_R$, where $F_R \subseteq Q$ and J has no edge inside R. A tree of J with r vertices, $\ell \geq 1$ of them marked, has no edge between marked vertices; there are $r^{r-\ell-1}(r-\ell)^{\ell-1}$ such trees, by the spanning tree count of the complete multipartite graph $K_{\ell,1,\dots,1}$ [8]. The program `v4_complete_graph_common_fugacity.py` sums over the partitions of R and the subsets F_R with exact integer polynomials, checks the result against direct forest enumeration for $n \leq 7$, and finds every coefficient nonnegative for $n \leq 40$. □

A The binary certificate and forest connectivity

Theorem 1.3 contains Corollary A.1. This appendix keeps the deduction of edition 2 from the reported test of PG(3, 2) and its self-contained certificate, which also checks the coefficient routines of Section 6.

Corollary A.1 (Binary consequence of the reported geometry test)**.** *Every finite binary matroid of rank at most four satisfies $\Gamma_{M;e,f} \succeq_{\mathrm{coef}} 0$ for every pair of distinct marked elements.*

The coefficient test for PG(3, 2) was reported in [3, Section 7]. Every simple binary matroid of rank at most four is a restriction of that geometry, whose distinct marked pairs are equivalent under ambient linear changes of basis. Setting deleted weights to zero and using Lemma 3.1 gives the stated consequence. The following independent certificate supplies a smaller exhaustive check using the coefficient-locality reduction.

Independent finite certificate for Corollary A.1. Ranks at most three are covered by Corollary 3.2. By Lemma 3.3, a negative coefficient in rank four would give a marked binary minor on at most eight elements. A nonempty squared support contracts to rank at most three and cannot be negative.

By Lemma 3.1, negativity would persist in a simplification retaining the two nonparallel nonloop marks. Thus a simple binary rank-four witness with at most eight elements would exist.

Represent its columns by distinct nonzero vectors of $\mathbb{F}_2^4$. A linear change of basis sends the marked vectors to $1 = (1,0,0,0)$ and $2 = (0,1,0,0)$. Extend the column set to eight distinct nonzero vectors, if necessary. The original coefficient is a constant coefficient in the added variables, so is preserved by this extension. The remaining six columns are one of the $\binom{13}{6}$ subsets of the other thirteen points. Every resulting eight-point configuration has rank four, because a rank-three binary subspace has only seven nonzero points.

The accompanying `verify.py` enumerates exactly these subsets, computes every rank by binary elimination, forms (1), and multiplies the category polynomials with integer arithmetic. It checks every coefficient, including proper supports and repeated variables. Table 4 records its full-support distribution; all 274380 nonzero coefficients checked in total are positive. This exhaustive calculation excludes the witness above and proves the corollary. □

Full-support coefficient	0	2	3	4	5	6	8	9	10	12
Configurations	36	48	192	48	48	96	436	480	312	20

Table 4: The coefficient of the product of all six unmarked variables in the 1716 binary configurations. The verifier also checks every other coefficient.

For comparison, the preceding results localize any rank-four failure to an especially concrete support.

Corollary A.2. *If a rank-four matroid has a negative closure coefficient, it has a simple nonbinary rank-four marked minor on exactly eight elements with a negative full squarefree coefficient. Each mark lies on a triangle avoiding the other.*

Proof. Use locality and simplification as above. A witness on at most seven elements has corank at most three, contradicting Theorem 1.2. Its squared support is empty, so the witness has full squarefree support. Binary witnesses are excluded by Corollary A.1. In a simple rank-four matroid the failure of the circuit condition at a mark requires a circuit of size three through it avoiding the other. □

Theorem 1.3 shows that no such minor exists.

Corollary A.3 (Five-vertex forest connectivity)**.** *Let G be a finite graph on at most five vertices, with arbitrary nonnegative finite edge weights. Sample an acyclic edge set with mass proportional to its edge-weight product. Any two endpoint-connectivity events are nonnegatively correlated.*

Proof. Add marked edges joining the two endpoint pairs, allowing parallel marked copies of existing edges. The graphic matroid is binary and has rank at most four. Its independent sets after deleting the marks are exactly the original forests. Closure of a marked edge is connectivity of its endpoints. Apply Corollary A.1 and (2). Equal endpoints give a constant event and require no additional argument. □

Reproducibility

The source package contains every program below with its output, and a README with the commands. Every computation used in a proof is an exact integer computation; `v2_identities.py` is

an additional cross-check. The programs marked (N) need Python 3 with NumPy and NetworkX and the nauty programs `geng` and `labelg`; `v2_identities.py` needs Python 3 with SymPy and mpmath; the others need only Python 3 and NumPy, and `verify.py` and `v1_site_certificates.py` only the standard library.

Program	Use
`verify.py`	Appendix A
`matroids.py`, `enumerate_simple_matroids.py`, `closure_check.py`, `crosscheck.py` (N)	Section 6, first program for rank four
`catalogue_check.py`	Section 6, first program for nine elements; it reads `matroids09_rankLine` of [13], MD5 `66777481824f5426a1469bafde130c82`
`indep_rank4.py`	Section 6, second program for rank four
`r1_rank5_all.py`, `r1_resume.py`	Section 6.2, second program for rank five, built on `indep_rank4.py`; the second script continues an interrupted run from its log
`forest_check.py`, `xval_graph.py` (N)	Section 7, first program
`indep_graphs.py`, `graphs_n9.py`, `sublists_n6_8.py`	Section 7, second program
`forest_mindeg3.py`	Section 7, third program
`rayleigh_census.py`	Table 3; the package also contains the nauty-based `rayleigh_coef.py` and `k4_criterion.py` (N), which give the same counts
`v1_site_certificates.py`	Proposition 9.11 and the strictness values in the proof of Theorem 1.6
`v2_identities.py`	independent checks of Lemmas 9.8 and 9.12, Corollary 9.9 and Propositions 9.10, 9.13 and 9.14 (rational and 50-digit arithmetic)
`v4_complete_graph_common_fugacity.py`	Proposition 9.15

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