Chollet's inequality for positive semidefinite matrices of rank at most four

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October 22, 2026
Archive edition 3

Abstract

We prove Chollet's inequality in every matrix order for pairs of complex Hermitian positive semidefinite matrices when both factors have rank at most four. A controlled-loss extension of the occupation-border comparison of Li, Zhao and Luo propagates the strict reserve in an established singular-five bound with factor $3/5$. We optimize the scalar balancing parameter, sharpen the resulting factors in orders seven through nine, and obtain an exact factor below 0.959 in order ten. The all-order conclusion combines this continuation with Rodtes' order-four theorem and Ma's large-order theorem. We include the complete interpretation and positivity proof of the inherited principal-minor sum-of-squares certificate, together with its unchanged data and standalone verifier. The coefficient-one inequality is strict for positive diagonal in orders at least five. The certificate and quantitative base are prior inputs; the continuation optimizes two explicitly defined coefficientwise losses.

Keywords: permanent; positive semidefinite matrix; Hadamard product; Chollet inequality; rank four; border comparison.
Mathematics Subject Classification (2020): 15A15; 15A45.

1 Introduction and main results

For a matrix $A = (a_{ij})$ of order n, put

$$\operatorname{per} A = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} a_{i,\sigma(i)}, \qquad P(A) = \operatorname{per} A, \qquad Q(A) = \operatorname{per}(A \circ \overline{A}).$$

Here $\circ$ denotes entrywise multiplication and $\overline{A}$ entrywise conjugation. All positive semidefinite matrices in this paper are complex Hermitian. Chollet's conjecture asks whether $\operatorname{per}(A \circ B) \le (\operatorname{per} A)(\operatorname{per} B)$ for all such pairs [1].

Theorem 1.1. *Let $A \ge 0$ have rank at most four and order $5 \le n \le 10$. Then*

$$Q(A) \le c_n P(A)^2, \qquad c_5 = c_6 = \frac{3}{5}, \qquad c_n = \frac{3}{5} \prod_{m=6}^{n-1} \frac{2}{(m+1)^{2/m}} \quad (7 \le n \le 10).$$

In particular

$$c_{10} = \frac{48}{5\, 7^{1/3} 8^{2/7} \sqrt{3}\, 10^{2/9}} < \frac{959}{1000} < 1.$$

Each inequality is strict if all diagonal entries are positive. If a diagonal entry is zero, $P(A) = Q(A) = 0$.

*Earlier versions of this paper are available in the Luminary Research Archive at `https://research.luminaryoddities.com/chollets-inequality-positive-semidefinite-matrices-rank-at-most-four`. The singular-five certificate originates in the author's earlier order-seven preprint, cited below.

The paired conclusion below restricts both factors to rank at most four. The proof spends a strict reserve from the singular-five base by balancing two border losses; the large-order theorem is a separately credited input.

The order-five bound and its exact certificate are from the author's earlier order-seven paper [2]. The full-domain closure in orders eight and nine and the sharper quantitative estimate in order ten follow from one optimized continuation. Its order-seven factor also improves the previous $192/245$. No tight-frame, spectral, row-cap or real-entry condition is imposed.

Corollary 1.2 (All orders, rank at most four)**.** *For every $n \geq 1$ and every pair $A, B \geq 0$ of order n with* rank $A \leq 4$ *and* rank $B \leq 4$*,*

$$\operatorname{per}(A \circ B) \leq (\operatorname{per} A)(\operatorname{per} B).$$

For $5 \leq n \leq 10$, the stronger coefficient c_n in Theorem 1.1 applies. For $n \geq 11$, Ma's coefficient $\rho_4 = 999991742359/10^{12} < 1$ applies. The coefficient-one inequality is strict for $n \geq 5$ if both diagonals are positive. If either matrix has a zero diagonal entry, both sides vanish.

Corollary 1.2 is a synthesis: Rodtes proves the complex order-four case [3], and Ma proves the rank-at-most-four case for every order at least ten [4, Theorem 1.2]. The optimized continuation below handles order ten directly, so only its range $n \geq 11$ is needed for our synthesis. Ma also treats tight order-nine correlation matrices, but that separate result is not used here. Theorem 1.1 supplies the missing unrestricted rank-four small orders.

Table 1: Dependencies for the all-order rank-at-most-four conclusion.

Orders	Input	Self-conjugate coefficient
$1 \leq n \leq 4$	Rodtes, with identity padding	1
$n = 5$	Singular-five certificate below	$3/5$
$6 \leq n \leq 10$	Optimized controlled-loss continuation	c_n in Theorem 1.1
$n \geq 11$	Ma, Theorem 1.2	$999991742359/10^{12}$

The identities and occupation positivity used below come from Li–Zhao–Luo [5, Proposition 3.2 and Eq. (4.1)]. Their loss-free comparison is insufficient for continuation beyond its stated range. We instead allow both coefficient comparisons to have losses and use the existing quantitative margin to pay for them. This is an elementary refinement of that mechanism. The singular-five certificate and its principal-minor sum-of-squares construction are reused from [2]; their feature framework follows Lavi [6].

Proof strategy. The quantitative base and the new continuation play distinct roles. Section 2 records the normalization and coefficient positivity that keep all comparisons valid for singular complex matrices. Section 3 proves a rank-preserving border transfer with two explicit losses. Section 4 chooses its parameters so that the inherited reserve survives through order ten, then combines these bounds with the cited small- and large-order theorems. The complete exact proof of the starting estimate is retained in Appendix A.

2 Normalization, positivity, and the starting estimate

If $a_{ii} = 0$ in a PSD matrix, the two-by-two principal minors give $|a_{ij}|^2 \leq a_{ii}a_{jj} = 0$. Its ith row and column vanish, hence $P(A) = Q(A) = 0$. Otherwise positive diagonal congruence normalizes A to a correlation matrix. For $D = \operatorname{diag}(d_i)$, $d_i > 0$,

$$P(DAD) = \Big(\prod_i d_i^2\Big)P(A), \qquad Q(DAD) = \Big(\prod_i d_i^4\Big)Q(A). \tag{1}$$

Rank is unchanged. Thus normalization preserves all self-conjugate bounds considered here without division by $P(A)$.

We recall the occupation positivity needed for the border calculation. For PSD matrices $M_1, \dots, M_\ell$ of order m, $\operatorname{per}(\sum_j t_j M_j)$ has nonnegative real coefficients. To see this, realize their Gram vectors $u_{i,j}$ in mutually orthogonal Hilbert spaces and set $v_i = \bigoplus_j \sqrt{t_j} u_{i,j}$ for $t_j \geq 0$. The unnormalized symmetric tensor

$$T(v) = \sum_{\sigma \in \mathfrak{S}_m} v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(m)} \quad \text{satisfies} \quad \|T(v)\|^2 = m! \operatorname{per}\Big(\sum_j t_j M_j\Big).$$

Group the complete tensor sums by occupation counts in the orthogonal summands. Different profiles are orthogonal, so their squared norms, divided by $m!$, are the polynomial coefficients. This argument covers singular matrices.

2.1 The quantitative starting point

The continuation uses the following established estimate. Appendix A contains its complete certificate identity, exact verification contract, and proof, so no proof from the earlier Archive paper is required.

Proposition 2.1 (Singular-five estimate [2])**.** *If $M \geq 0$ has order five and $\det M = 0$, then $Q(M) \leq (3/5)P(M)^2$, strictly when its diagonal is positive.*

The rank hypothesis in Theorem 1.1 makes every order-five input singular. Thus Proposition 2.1 supplies a strict reserve below the coefficient one that the continuation can spend at subsequent orders. The argument below quantifies that expenditure.

3 A controlled-loss border comparison

The following extension records explicitly how to spend a quantitative input margin in the border identities of [5].

Lemma 3.1. *Let $A \geq 0$ be a correlation matrix of order $m+1$, and write*

$$A = \begin{pmatrix} B & x \\ x^* & 1 \end{pmatrix}, \qquad H = B - xx^*, \qquad R_t = H + t\,xx^*, \qquad t \geq 1.$$

Define

$$b_m(t) = \max_{0 \leq k \leq m} \frac{t^k}{k+1}, \qquad d_m(t) = \max_{0 \leq s \leq m} \frac{s+1}{t^{2s}}.$$

Then $R_t \geq 0$, $\operatorname{rank} R_t \leq \operatorname{rank} A$, its diagonal is positive, and

$$P(R_t) \leq b_m(t) P(A), \qquad Q(A) \leq d_m(t) Q(R_t). \tag{2}$$

Consequently, an order-m rank-at-most-four bound $Q \leq c_m P^2$ implies

$$Q(A) \leq c_m b_m(t)^2 d_m(t) P(A)^2 \qquad (\operatorname{rank} A \leq 4). \tag{3}$$

Strictness for positive diagonal propagates from the order-m bound.

Proof. Schur congruence gives $H \geq 0$ and $\operatorname{rank} H = \operatorname{rank} A - 1$. Thus $\operatorname{rank} R_t \leq \operatorname{rank} A$ and $(R_t)_{ii} = 1 + (t-1)|x_i|^2 \geq 1$.

Set $z_i = |x_i|^2$ and

$$U = H \circ \overline{H}, \quad V = H \circ \overline{xx^*} + \overline{H} \circ xx^*, \quad W = zz^T.$$

These are PSD: U follows from Schur positivity, each summand of V is a diagonal congruence of H or $\overline{H}$, and W has rank at most one. Expand

$$F(t) = P(H + t\,xx^*) = \sum_{k=0}^{m} \alpha_k t^k, \qquad \Gamma(a,b) = P(U + aV + bW) = \sum_{\substack{r,s \geq 0 \\ r+s \leq m}} \gamma_{rs} a^r b^s.$$

Occupation positivity gives $\alpha_k, \gamma_{rs} \geq 0$.

For $B(i \mid j)$ obtained by deleting row i and column j, border expansion yields

$$P(A) = P(B) + \sum_{i,j} x_i \overline{x_j} \operatorname{per} B(i \mid j) = F(1) + F'(1) = \sum_k (k+1)\alpha_k.$$

In particular the cofactor orientation is unchanged for complex x. The matrix $A \circ \overline{A}$ has upper block $U + V + W$ and border z. The same expansion in direction W gives

$$Q(A) = \Gamma(1,1) + \partial_b \Gamma(1,1) = \sum_{r+s \leq m} (s+1)\gamma_{rs}, \tag{4}$$

$$Q(R_t) = \Gamma(t, t^2) = \sum_{r+s \leq m} t^{r+2s} \gamma_{rs}. \tag{5}$$

The first comparison in (2) follows coefficientwise from $t^k \leq b_m(t)(k+1)$. For the second, $t \geq 1$ ensures

$$s + 1 \leq d_m(t) t^{2s} \leq d_m(t) t^{r+2s}.$$

Apply this to (4)–(5). Normalize R_t using (1), apply its order-m bound, and use nonnegativity of PSD permanents to square the first comparison. This proves (3). Since R_t has positive diagonal, a strict predecessor bound gives strictness. □

4 Small-order closure and the all-orders consequence

We first evaluate the scalar losses exactly. If $3/2 \leq t^2 \leq 2$, then

$$d_m(t) = 2/t^2 \qquad (m \geq 1). \tag{6}$$

The $s = 0$ term is at most $2/t^2$, the $s = 1$ term equals it, and the successive ratios for $s \geq 1$ are at most $(3/2)/t^2 \leq 1$. The sequence $t^k/(k+1)$ has increasing successive ratios $t(k+1)/(k+2)$, so its maximum is at an endpoint. In particular,

$$b_m(t) = \max\{1, t^m/(m+1)\}.$$

Lemma 4.1 (Optimal scalar loss balance)**.** *For each $m \geq 1$, the product $b_m(t)^2 d_m(t)$, $t \geq 1$, attains its minimum at $t_m = (m+1)^{1/m}$. The minimum is*

$$\max_{0 \leq s \leq m} \frac{s+1}{(m+1)^{2s/m}}.$$

For $6 \leq m \leq 9$ it equals $2/(m+1)^{2/m}$.

Proof. Put $t_0 = (m+1)^{1/m}$. For $1 \leq t \leq t_0$ we have $b_m(t) = 1$, and $d_m(t)$ is nonincreasing. For $t \geq t_0$ the product equals

$$\max_{0 \leq s \leq m} \frac{(s+1)t^{2(m-s)}}{(m+1)^2},$$

a maximum of nondecreasing functions. Thus t_0 is a minimizer. For $6 \leq m \leq 9$, the integer inequalities

$$3^m \leq 2^m (m+1)^2 \leq 4^m$$

give $3/2 \leq t_m^2 \leq 2$, so (6) supplies the last formula. This optimizes the specified coefficientwise losses, not the best possible matrix constants. □

Proof of Theorem 1.1. Normalize a positive-diagonal input to a correlation matrix; the zero-diagonal case was settled above. Rank at most four makes every order-five input singular, so Proposition 2.1 gives $c_5 = 3/5$ strictly. At $m = 5, t = \sqrt{2}$, both losses are one: $2^k \le (k+1)^2$ and $s+1 \le 2^s$ for $0 \le k, s \le 5$. Hence $c_6 = 3/5$. For $m = 6, 7, 8, 9$, use $t_m = (m+1)^{1/m}$ in Lemma 3.1. Lemma 4.1 gives $c_{m+1} = 2c_m/(m+1)^{2/m}$, proving the product formula. Its factors are greater than one, so c_{10} is the largest of these constants. The displayed rational upper bound is equivalent, after raising positive quantities to the power 126, to

$$48000^{126} < 4795^{126}7^{42}8^{36}3^{63}10^{28}.$$

This exact integer comparison is checked by the supplied scalar verifier. Thus all the constants are below one, and strictness propagates at every step. □

Proof of Corollary 1.2. For $a_\sigma = \prod_i a_{i,\sigma(i)}$ and $b_\sigma = \prod_i b_{i,\sigma(i)}$, Cauchy–Schwarz gives

$$|\operatorname{per}(A \circ B)| \le \Big(\sum_\sigma |a_\sigma|^2\Big)^{1/2}\Big(\sum_\sigma |b_\sigma|^2\Big)^{1/2} = \sqrt{Q(A)Q(B)}.$$

Theorem 1.1 supplies the paired bounds for orders five through ten; [4, Theorem 1.2] supplies them for all $n \ge 11$. The Schur product is PSD, so its permanent is real and nonnegative. For $n = 4$ use Rodtes' theorem [3]. For $n < 4$, apply that theorem to $A \oplus I_{4-n}$ and $B \oplus I_{4-n}$ and use multiplicativity of permanents under direct sums. Rodtes' theorem is unrestricted in rank, so the padding introduces no hypothesis problem.

If both diagonals are positive, $P(A), P(B) > 0$: for instance the Gram symmetric tensor of nonzero vectors is nonzero, since under its identification with a product of nonzero linear forms the polynomial ring has no zero divisors. The strict local bounds and Ma's $\rho_4 < 1$ therefore imply the asserted strict coefficient-one result. A zero diagonal in either factor gives the stated vanishing. □

Remark 4.2. The optimized factors satisfy $c_7 < 63/100$, $c_8 < 7/10$, $c_9 < 4/5$ and $c_{10} < 959/1000$. They improve the earlier sufficient factors without altering the singular-five certificate. The proof's quantitative gain comes from balancing the two border losses, rather than from a new positivity certificate or an assertion of sharp matrix constants.

Dependencies and reproducibility

The singular-five certificate interpretation, its positivity and strictness proofs, the new transfer, and the finite scalar comparisons are given in full here. The all-orders conclusion explicitly reuses Rodtes' order-four theorem, the singular-five theorem of [2], and Ma's Theorem 1.2 for $n \ge 11$; it does not use Ma's tight-nine certificate or Li–Zhao–Luo's unrestricted through-six theorem. The reader package includes the unchanged singular-five certificate and its exact-integer verifier, the complete LaTeX source and a short exact check of the scalar comparisons. The numerical searches used during development are not proof inputs. The large-order conclusion uses Ma's cited theorem with its stated rank and order hypotheses.

AI assistance

ChatGPT assisted research, proof development, drafting and verification. Scott Schubr directed the work, reviewed the mathematical arguments and supporting materials, and takes responsibility for the content, accuracy, completeness, and attribution.

Funding and competing interests

This research was self-funded. The author declares no competing interests.

Data and code availability

The accompanying source package contains the exact certificate data and standalone verification programs needed for the computer-assisted part of the proof. Its reader instructions give the verification commands.

A The exact singular-five certificate

The definitions, exact identity and proof in this appendix reproduce the singular-five input from [2]. Its certificate and checker are unchanged; the controlled-loss continuation is proved in Section 3. The certificate is finite exact mathematical data. We specify its polynomial interpretation before proving Proposition 2.1.

Let $X = (x_{ij})_{1\le i,j\le 5}$ consist of 25 commuting formal variables. The involution $f \mapsto f^*$ sends x_{ij} to x_{ji} and fixes real coefficients. On Hermitian matrices it is complex conjugation. Define the formal polynomials

$$p(X) = \sum_{\sigma\in\mathfrak{S}_5} \prod_i x_{i,\sigma(i)}, \qquad q(X) = \sum_{\sigma\in\mathfrak{S}_5} \prod_i x_{i,\sigma(i)} x_{\sigma(i),i}.$$

They specialize to $P(A)$ and $Q(A)$. Let $G = \mathfrak{S}_5 \times \{1, {}^T\}$ act by simultaneous relabeling of rows and columns and by transpose. Its uniform Reynolds operator is

$$\mathcal{R}f = \frac{1}{240} \sum_{g\in G} gf.$$

For a partition λ of five, let

$$c_\lambda(X) = \sum_{\substack{\sigma\in\mathfrak{S}_5 \\ \mathrm{cyc}(\sigma)=\lambda}} \prod_i x_{i,\sigma(i)}.$$

There are seven such cycle types. Each c_λ is invariant under G.

Proposition A.1 (Exact certificate)**.** *The accompanying file*

`certificate/singular5_rho3_5_certificate.json`

specifies 38 monomial columns b_k, symmetric integer matrices N_k, products w_k of principal determinants, square integer matrices T_k, and integers h_λ, satisfying

$$D(3p^2 - 5q) = \mathcal{R}\left(\sum_{k=1}^{38} w_k\, b_k^* N_k b_k\right) + (\det X) \sum_{\lambda\vdash 5} h_\lambda c_\lambda(X), \qquad D = 260963607394941665280. \quad (7)$$

For every k, the symmetric integer matrix $Z_k = T_k N_k T_k^T$ satisfies

$$(Z_k)_{ii} > \sum_{j\ne i} |(Z_k)_{ij}| \quad \textit{for every row } i. \quad (8)$$

The third block is factor-free: $w_3 = 1$, and its monomial column contains exactly the 119 nonidentity permutation monomials.

Exact verification. The accompanying standalone program `code/verify_certificate.py` expands (7) using integer arithmetic. A feature is encoded by its 25 nonnegative exponents in row-major order. The fields `factors`, `features` and `numerator` specify w_k, b_k and N_k; `congruence` specifies T_k. The seven entries of `multiplier_numerators`, indexed by `multiplier_classes`, specify h_λ.

The program constructs p^2 from all 120^2 ordered permutation pairs, constructs q from all 120 permutations, and expands each principal determinant. It reconstructs the Gram forms, applies the 240 group actions, and checks every coefficient in the resulting degree-ten identity after clearing the factor 240. The complete balanced support has 6,210 monomials, with row and column degree two; no additional monomial is permitted. It separately checks every integer inequality in (8) and the exact 119-feature set

equality. Thus the checks establish a formal polynomial identity and exact positivity witnesses, without sampling matrices or accepting a floating-point tolerance.

The certificate was obtained using the principal-minor feature cone of [7], with the determinant multiplier in (7), semidefinite optimization and rational coefficient repair. Its validity depends on the displayed exact identity and integer witnesses, which the verifier reconstructs independently of the numerical discovery data. □

Remark A.2 (Reproducibility). The certificate is an essential part of the proof. Its SHA-256 is

`f50eae5dfb642b9015d41a7a1d366aab`
`cce6e90e6bfe237ac93694284ad733f0`.

The hash identifies the supplied object; the coefficient and positivity checks prove the assertion. The verifier uses the Python standard library only. From the package root, run

```
python code/verify_certificate.py
```

The same checks remain active with `python -O`. No optimizer, numerical solution, external package, or predecessor certificate is needed for verification.

Proof. Write $A = M$ for the matrix in Proposition 2.1. A real symmetric matrix Z satisfying (8) is positive definite, also on complex vectors, because

$$u^* Z u \geq \sum_i \left(Z_{ii} - \sum_{j \neq i} |Z_{ij}| \right) |u_i|^2 > 0 \quad (u \neq 0).$$

Since $Z_k = T_k N_k T_k^T$ is positive definite and T_k is square, T_k is invertible; otherwise T_k^T would have a nonzero kernel. It follows that every N_k is positive definite.

All principal determinants of A are nonnegative, so $w_k(A) \geq 0$ and $w_k(A) b_k(A)^* N_k b_k(A) \geq 0$. The group transformations preserve Hermitian positive semidefiniteness, so their average is nonnegative. Singularity makes the determinant term in (7) vanish. This proves the weak inequality.

Suppose the diagonal is positive. A singular diagonal matrix with positive diagonal is impossible, so some $a_{ij} \neq 0$ with $i \neq j$. The transposition (ij) contributes the strictly positive feature

$$|a_{ij}|^2 \prod_{\ell \notin \{i,j\}} a_{\ell\ell}$$

to $b_3(A)$. Thus $b_3(A) \neq 0$ and its factor-free positive-definite Gram form is strictly positive. The Reynolds average includes this term with positive weight and all other terms are nonnegative. Hence $3P(A)^2 - 5Q(A) > 0$. The zero-diagonal case was established in Section 2. □

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