# Chollet's permanent inequality for diagonally dominant factors and every graph Laplacian

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**Abstract**

Chollet's conjecture (1982) asserts $\operatorname{per}(H \circ T) \le \operatorname{per} H \operatorname{per} T$ for positive semidefinite matrices. We prove the stronger Bapat–Sunder bound $\operatorname{per}(H \circ T) \le \operatorname{per} T \prod_i H_{ii}$ whenever $H$ is Hermitian with positive diagonal and scaled diagonally dominant, for *every* positive semidefinite $T$. In particular Chollet's inequality holds for every pair of weighted graph Laplacians, with arbitrary magnetic phases, and for all their principal submatrices. This answers the question of Pant and Singh whether $\operatorname{per}(L_G \circ L_G) \le \operatorname{per}(L_G) \prod_v \deg_G(v)$ for every graph $G$; previously it was known for graphs whose blocks are bipartite or cycles. The proof writes the permanent as an expectation over the cycles of a random map and bounds each cycle by one of its edges, which gives a sharper matching bound with an exact equality case. Spectral radius one is the sharp threshold for this bound. A second, entrywise criterion covers correlation matrices whose squared off-diagonal row sums are at most $2/5$. As an application we determine the exact optimum of generalized boson bunching over all internal-state Gram matrices in the same class, together with its minimal internal dimension.

## 1 Introduction

For a square matrix $A$ write $\operatorname{per} A = \sum_\sigma \prod_i A_{i,\sigma(i)}$ (with $\operatorname{per} \varnothing = 1$) and $A \circ B$ for the entrywise (Schur) product. A *correlation matrix* is a Hermitian positive semidefinite matrix with unit diagonal.

**Chollet's conjecture and the Bapat–Sunder bound.** Chollet [3] conjectured in 1982 that

$$\operatorname{per}(A \circ B) \le \operatorname{per} A \operatorname{per} B \qquad (A, B \succeq 0), \tag{1}$$

the permanental analogue of Oppenheim's inequality. It remains open in general; see the survey [16]. Bapat and Sunder [1] proposed the stronger bound $\operatorname{per}(A \circ B) \le \operatorname{per} A \prod_i b_{ii}$, which implies (1) by Lieb's inequality $\operatorname{per} B \ge \prod_i b_{ii}$ [9]. Drury [4] disproved the Bapat–Sunder bound, and further counterexamples are known even with nonnegative entries [11]. It is therefore natural to ask for structural classes on which the strong bound survives.

**Graph Laplacians.** Pant and Singh [10] proved the Bapat–Sunder bound when the retained factor is a real symmetric Z-matrix with bipartite support, and developed a vertex-coalescence calculus for graph Laplacians. They established

$$\operatorname{per}(L_G \circ L_G) \le \operatorname{per}(L_G) \prod_v \deg_G(v) \tag{2}$$

for graphs whose blocks are edges, cycles or bipartite, and Chollet's inequality for blocks that are also allowed to be complete. In their concluding section they ask whether Chollet's inequality holds for the Laplacian of every graph and, more strongly, whether every graph Laplacian satisfies (2) [10, Section 6]. An earlier record of the author [14] extended the admissible block classes.

**Results.** A Hermitian matrix $H$ with positive diagonal is *scaled diagonally dominant* if some positive diagonal congruence $DHD$ has each diagonal entry at least the sum of the moduli of the off-diagonal entries in its row. Equivalently (Section 3), $\rho\big(|D_H^{-1/2} H D_H^{-1/2} - I|\big) \le 1$ with $D_H = \operatorname{diag}(H)$, where $|\cdot|$ is taken entrywise and $\rho$ is the spectral radius. Such matrices are positive semidefinite.

**Theorem 1.** *Let $H$ be Hermitian, positive diagonal and scaled diagonally dominant, and let $T$ be any Hermitian positive semidefinite matrix of the same order. Then*

$$\operatorname{per}(H \circ T) \le \Big(\prod_i H_{ii}\Big) \operatorname{per} T \le \operatorname{per} H \operatorname{per} T.$$

*If $T$ has positive diagonal, the first inequality is an equality exactly when the off-diagonal support of $T$ is a matching on whose edges $|H_{ij}|^2 = H_{ii}H_{jj}$.*

**Corollary 2** (Every graph Laplacian)**.** *Let $G$ be a finite graph with symmetric edge weights $w_{ij} \ge 0$, and let $L$ be Hermitian with $|L_{ij}| \le w_{ij}$ for $i \ne j$ and $L_{ii} \ge \sum_{j \ne i} w_{ij}$. This includes the ordinary Laplacian, weighted Laplacians, magnetic Laplacians with arbitrary edge phases, attenuated edges and diagonal slack. Then every principal submatrix $H$ of $L$ and every Hermitian positive semidefinite $T$ satisfy*

$$\operatorname{per}(H \circ T) \le \Big(\prod_i H_{ii}\Big) \operatorname{per} T \le \operatorname{per} H \operatorname{per} T.$$

*In particular* (2) *holds for every finite simple graph, hereditarily, and Chollet's inequality* $\operatorname{per}(L_G \circ L_{G'}) \le \operatorname{per} L_G \operatorname{per} L_{G'}$ *holds for every two graphs on a common vertex set. For a connected graph on at least three vertices,* (2) *is strict.*

Corollary 2 answers the question of [10, Section 6] in the affirmative, in its strong form, and extends it to two different graphs, to weights and to magnetic phases. Theorem 1 follows from a sharper bound, Theorem 3, which replaces $\prod_i H_{ii}$ by a matching product. Section 4 gives a different sufficient condition that is not diagonal dominance, and Section 5 applies the matching bound to boson bunching.

**Method.** The functional-digraph expansion of permanents goes back to Kim and Roush [7]; Lavi [8] uses a signed cycle formula for $\operatorname{per}(I - P)$ with $P$ stochastic, and Chen and Cao [2] study cycle factorizations in a related extremal problem. Here a Perron rescaling turns the permanent of $A \circ B$ into the expectation of a product over the cycles of a *random map*, which can be bounded cycle by cycle.

## 2 The matching bound

For a unit-diagonal $A$ and a correlation matrix $B$, let $G(A, B)$ be the graph with an edge $\{i, j\}$ whenever $B_{ij} \ne 0$ and $A_{ij}$ or $A_{ji}$ is nonzero. For a graph $G$ and $0 \le r \le 1$ put

$$K_G(B; r) = \max_M \prod_{\{i,j\} \in M} \big(1 + r^2 |B_{ij}|^2\big),$$

the maximum over matchings $M$ of $G$, including the empty matching.

**Theorem 3.** *Let $A$ be complex with unit diagonal and $\rho(|A-I|) \le r \le 1$. Every correlation matrix $B$ satisfies*

$$|\operatorname{per}(A \circ B)| \le K_{G(A,B)}(B; r) \le \operatorname{per} B.$$

*Equality $|\operatorname{per}(A \circ B)| = \operatorname{per} B$ holds exactly when the off-diagonal support of $B$ is a matching on whose edges $A_{ij}A_{ji} = 1$. For fixed $B$ and $r$, the best bound valid for all such $A$ is $K_G(B; r)$ with $G$ the full off-diagonal support of $B$. The value $r = 1$ is sharp: for every $t > 1$ there are $A$ with $\rho(|A - I|) = t$ violating the theorem.*

**Lemma 4** (Random-map identity)**.** *Let $R = |A - I|$ with $\rho(R) < 1$, put $v = (I - R)^{-1}\mathbf{1} > 0$ and $p_{ij} = R_{ij}v_j/v_i$. Then $\sum_j p_{ij} \le 1$. Let $X = (X_i)$ be independent, with $X_i = j$ with probability $p_{ij}$ and $X_i$ in a cemetery state with the remaining probability. Let $\mathcal{C}(X)$ be the set of directed cycles of length at least two of the map $i \mapsto X_i$, with rotations identified, and for a directed simple cycle $\gamma$ in the support of $R$ set $z_\gamma = \prod_{(i,j)\in\gamma} A_{ij}B_{ij}/R_{ij}$. Then*

$$\operatorname{per}(A \circ B) = \mathbb{E} \prod_{\gamma \in \mathcal{C}(X)} (1 + z_\gamma).$$

*Proof.* $Rv = v - \mathbf{1} \le v$ gives $\sum_j p_{ij} \le 1$. For a directed simple cycle $\gamma$ the vector factors telescope, so it occurs with probability $\prod_{(i,j)\in\gamma} p_{ij} = \prod_{(i,j)\in\gamma} R_{ij} =: r_\gamma$. A family of vertex-disjoint cycles involves distinct rows, so it occurs with probability $\prod r_\gamma$. Expanding the product,

$$\mathbb{E} \prod_{\gamma \in \mathcal{C}(X)} (1 + z_\gamma) = \sum_{\mathcal{F}} \prod_{\gamma \in \mathcal{F}} r_\gamma z_\gamma = \sum_{\mathcal{F}} \prod_{\gamma \in \mathcal{F}} \prod_{(i,j)\in\gamma} A_{ij}B_{ij},$$

the sum over families $\mathcal{F}$ of vertex-disjoint directed cycles of length at least two. Cycles through a zero entry of $A$ have probability zero and contribute zero on both sides. These families are exactly the nontrivial cycle structures of permutations, whose fixed points contribute $A_{ii}B_{ii} = 1$. The right side is therefore $\operatorname{per}(A \circ B)$. □

*Proof of Theorem 3. First inequality, $r = 1$.* Suppose first that $\rho(R) < 1$. In Lemma 4, $|A_{ij}|/R_{ij} = 1$ on the support, so $|z_\gamma| = \prod_{(i,j)\in\gamma} |B_{ij}|$. Choose on $\gamma$ an edge $\{i, j\}$ maximizing $|B_{ij}|$. Since $\gamma$ has at least two steps and every $|B_{ij}| \le 1$, we get $|1 + z_\gamma| \le 1 + |B_{ij}|^2$, and the chosen edge lies in $G(A, B)$. Disjoint cycles give disjoint edges, so pointwise $\left|\prod_\gamma (1 + z_\gamma)\right| \le K_{G(A,B)}(B; 1)$; take expectations. If $\rho(R) = 1$, apply this to $I + s(A - I)$, $0 < s < 1$, which has the same support, and let $s \uparrow 1$.

*First inequality, $r < 1$.* For $0 < r < 1$ apply the case $r = 1$ to $I + (A - I)/r$ and the correlation matrix $I + r(B - I)$; their Schur product is $A \circ B$ and the supports are unchanged. If $r = 0$, then $R$ has no directed cycle (a cyclic permutation matrix supported in $R$ would give $\rho(R) > 0$), so $\operatorname{per}(A \circ B) = 1$.

*Second inequality.* For a matching $M$, Lieb's block inequality [9] with the pairs of $M$ and the remaining singletons as blocks gives $\operatorname{per} B \ge \prod_{\{i,j\}\in M}(1 + |B_{ij}|^2) \ge \prod_{\{i,j\}\in M}(1 + r^2|B_{ij}|^2)$.

*Equality.* Equality in Lieb's block inequality for a positive semidefinite matrix with no zero row forces the cross blocks to vanish [6, Eq. (3)]; iterating over the matching partition shows that equality forces the support of $B$ to be a matching. Then $\operatorname{per}(A \circ B) = \prod(1 + A_{ij}A_{ji}|B_{ij}|^2)$ and $\operatorname{per} B = \prod(1 + |B_{ij}|^2)$. Since $\sqrt{|A_{ij}A_{ji}|}$ is the spectral radius of a $2 \times 2$ compression of $R$, it is at most one, and equality of moduli forces $A_{ij}A_{ji} = 1$ on every edge with $B_{ij} \ne 0$.

*Optimality and sharpness.* For a maximizing matching of the support of $B$, let $A$ be block diagonal with $2\times 2$ blocks of off-diagonal entry $r$ on its pairs and unit singletons. Then $\rho(|A-I|) = r$ and $\operatorname{per}(A \circ B) = K_G(B; r)$. For $t > 1$ take $A = \left(\begin{smallmatrix} 1 & t \\ t & 1 \end{smallmatrix}\right)$ and $B = J$: then $\rho(|A - I|) = t$ and $\operatorname{per}(A \circ B) = 1 + t^2 > 2 = \operatorname{per} B$. □

*Remark* 5. The intermediate matching bound fails immediately above radius one even for correlation multipliers. For the $3 \times 3$ matrix $A$ with all off-diagonal entries $t \in (1/2, 1)$ and $B = J$, $A$ is positive definite, $\rho(|A - I|) = 2t$, and $\operatorname{per}(A \circ B) = 1 + 3t^2 + 2t^3 > 2 = K$.

## 3 Diagonal dominance and graph Laplacians

Let $H$ be Hermitian with positive diagonal $d_i$, put $D = \operatorname{diag}(d_i)$ and $A = D^{-1/2} H D^{-1/2}$. A positive diagonal congruence making $H$ diagonally dominant is the same as a vector $v > 0$ with $|A-I|v \le v$. This gives $\rho(|A-I|) \le 1$ by the Collatz–Wielandt bound. Conversely, if $\rho(|A-I|) \le 1$, a positive Perron vector on each connected component of the symmetric matrix $|A - I|$ (value one on singletons) supplies $v$. Then $x^* A x \ge \|x\|^2 - |x|^T |A - I|\,|x| \ge 0$, so $H \succeq 0$.

*Proof of Theorem 1.* If $T$ has a zero diagonal entry it has a zero row and every term vanishes. Otherwise write $T = E^{1/2} B E^{1/2}$ with $B$ a correlation matrix and $E = \operatorname{diag}(T)$. Then $\operatorname{per}(H \circ T) = \prod_i H_{ii} \prod_i T_{ii} \operatorname{per}(A \circ B)$ and $\operatorname{per} T = \prod_i T_{ii} \operatorname{per} B$. Since $H \circ T \succeq 0$ by Schur's theorem, its permanent is real and nonnegative. Theorem 3 with $r = 1$ gives the first inequality and its equality case, translated through the congruences. The second is Lieb's inequality $\operatorname{per} H \ge \prod_i H_{ii}$. □

*Proof of Corollary 2.* $L$ is diagonally dominant, hence so is every principal submatrix (deleting rows and columns only removes off-diagonal terms). Theorem 1 applies to each. For the ordinary Laplacian, $\prod_i H_{ii} = \prod_v \deg_G(v)$ on the full vertex set, giving (2); taking $T = L_{G'}$ gives Chollet's inequality for two graphs. For strictness, a connected graph on at least three vertices has off-diagonal support that is not a matching. □

## 4 A squared-entry criterion

The following condition is not diagonal dominance: it allows many small off-diagonal entries whose absolute row sums exceed one.

**Theorem 6.** *Let $B$ be a correlation matrix with $\sum_{j \ne i} |B_{ij}|^2 \le 2/5$ for every $i$. Then $\operatorname{per}(B \circ \overline{B}) \le \operatorname{per} B$, strictly unless $B = I$. More precisely, with $T = \sum_{i<j} |B_{ij}|^4$,*

$$\log \operatorname{per} B - \log \operatorname{per}(B \circ \overline{B}) \ge T/150.$$

*The same holds for every principal submatrix.*

*Proof.* Put $C_{ii} = 0$ and $C_{ij} = |B_{ij}|^2$, so that $\operatorname{per}(B \circ \overline{B}) = \operatorname{per}(I + C) =: Q$, and $P = \operatorname{per} B$. Thus $T = \sum_{i<j} C_{ij}^2$.

*Lower bound for $P$.* For a matching $M$, Lieb's block inequality gives $P \ge \prod_{e \in M} (1 + C_e)$. If $\lambda C$ (as an edge vector) lies in the matching polytope, write it as a convex combination of matchings and average the logarithms: $\log P \ge \lambda \sum_{i<j} C_{ij} \log(1 + C_{ij})$. By Edmonds' theorem [5], the polytope consists of nonnegative edge vectors with vertex sums at most one and internal sum at most $(|U| - 1)/2$ on every odd vertex set $U$. Take $\lambda = 5/3$. Vertex sums of $\lambda C$ are at most $2/3$.

For odd $|U| = m \geq 3$ the internal sum is at most half the sum of the vertex sums, $m/3 \leq (m-1)/2$. Since $\log(1+x) \geq \frac{5}{2} \log \frac{7}{5}\, x$ on $[0, 2/5]$,

$$\log P \geq \tfrac{25}{6} \log \tfrac{7}{5}\, T \geq \tfrac{314}{225} T,$$

using the first four terms of the alternating series of $\log(7/5)$.

*Upper bound for $Q$.* Expanding $\operatorname{per}(I + C)$ over cycle families and allowing overlaps, $Q \leq \prod_\gamma (1 + w_\gamma)$ over directed simple cycles $\gamma$ of length at least two up to rotation, with $w_\gamma$ the product of $C$-entries along $\gamma$. A simple $k$-cycle appears $k$ times in $\operatorname{tr} C^k$, and the other closed walks contribute nonnegatively, so $\log Q \leq \sum_{k \geq 2} \operatorname{tr}(C^k)/k$. The row sums give $\rho(C) \leq b = 2/5$, so $\operatorname{tr} C^k \leq b^{k-2} \operatorname{tr} C^2 = 2b^{k-2} T$ and

$$\log Q \leq \frac{2T}{b^2} \Big( \log \frac{1}{1-b} - b \Big) = \tfrac{25}{2} \big( \log \tfrac{5}{3} - \tfrac{2}{5} \big) T \leq \tfrac{25}{18} T,$$

since $\log \frac{5}{3} = 2 \sum_{j \geq 0} 4^{-2j-1}/(2j+1) < \frac{1}{2} + \frac{1}{90}$. Finally $\frac{314}{225} - \frac{25}{18} = \frac{1}{150}$. The row condition is inherited by principal submatrices. □

For a weighted graph with positive weighted degrees $d_i$, the normalized Laplacian $D^{-1/2} L D^{-1/2}$ has squared off-diagonal row sums $\sum_j w_{ij}^2/(d_i d_j)$, so Theorem 6 applies whenever $w_{ij}/d_j \leq 2/5$ on all ordered edges.

## 5 Generalized boson bunching

Send one boson into each of $n$ occupied input modes of a passive linear interferometer with transfer matrix $U$, and let $V$ be the submatrix of $U$ on a set $S$ of output modes and the occupied inputs. With $H = V^* V$ and internal-state overlaps $A_{ij} = \langle \phi_i, \phi_j \rangle$, the probability that all particles exit in $S$ is $P_S(A) = \operatorname{per}(A \circ H)$ [15, 12]; for identical particles it is $\operatorname{per} H$. If some $H_{ii} = 0$ the probability is zero; otherwise put $B_{ij} = H_{ij}/\sqrt{H_{ii} H_{jj}}$.

**Theorem 7.** *If $\rho(|A - I|) \leq r \leq 1$, then*

$$P_S(A) \leq \Big( \prod_i H_{ii} \Big) K_{G(A,B)}(B; r) \leq \operatorname{per} H,$$

*and the middle quantity with $G$ the full support of $B$ is attained.*

1. *For $r < 1$ the internal space has dimension at least $n$, and the bound is attained in dimension $n$.*
2. *For $r = 1$ the bound is attained in internal dimension $\lceil n/2 \rceil$, and no smaller dimension is possible in the class $\rho(|A - I|) \leq 1$.*
3. *If $n = 2d$ and the internal dimension is $d$, every admissible internal Gram matrix is, up to phases and permutation, a direct sum of $d$ identical pairs.*

*Proof.* The bound is Theorem 3 after normalization. For $r < 1$, $A$ is positive definite. For $r = 1$, extend a maximizing matching to $\lfloor n/2 \rfloor$ pairs by pairing unmatched vertices. Any added pair has $B$-entry zero, since otherwise the matching product could be increased. Prepare identical states within each pair and orthogonal states across pairs. The Gram matrix has rank $\lceil n/2 \rceil$ and attains the bound. Minimality holds because $\operatorname{tr} A = n$ and $\|A - I\| \leq \rho(|A - I|) \leq 1$ force every eigenvalue

into $[0,2]$, so at least $\lceil n/2 \rceil$ are nonzero. If $n = 2d$ and the rank is $d$, all eigenvalues are 0 or 2, so $A - I$ is unitary and $\sum_j |A_{ij} - \delta_{ij}|^2 = 1$. With $R = |A - I|$ this gives $\operatorname{tr} R^2 = n$, and since $\rho(R) \le 1$, $R^2 = I$. The rows of the nonnegative symmetric $R$ are then orthonormal with disjoint supports, so each has a single entry one, which gives the pairs. □

Pioge, Novo and Cerf [13] give no-enhancement results for several overlap models, including orthogonal groups. Theorem 7 determines the exact optimum over all internal Gram matrices in the spectral-radius class, which depends only on the overlaps; for example $\sum_{j \ne i} |A_{ij}| \le 1$ for all $i$ suffices.

**Verification.** Theorem 3 and Corollary 2 were also tested numerically (Ryser permanents) on 4 000 random complex instances of order at most seven with multipliers of spectral radius in $[0.2, 1]$, and on 3 000 random graphs of order at most eight with ordinary or random magnetic phases. No violation occurred; equality was observed only in the cases characterized above.

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