Certified Resolvent Detection of a Third Weyl-Blind Twisting Mode

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Luminary Oddities

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Abstract

The leading Weyl profile of a rapidly twisted tube with a fixed connected step section annihilates every odd logarithmic harmonic of inverse twist speed. A preceding heat-trace result detects its fundamental harmonic but leaves the third harmonic unresolved. We give a computer-assisted resolution for that original third harmonic, without altering the section or adding a lower-order perturbation. The proof evaluates a complex Mellin invariant of the connected junction. We derive its half-strip resolvent formula, retain the noncommuting Dirichlet–Dirichlet and Dirichlet–Neumann interval roots, and enclose the infinite aperture inverse through arbitrary trial vectors and complete residual bounds. Sectorial trace estimates permit a wide analytic log strip, reducing the required quadrature to 27 certified complex nodes. Infinite angular and input tails, floating arithmetic, exact-node conversion, aliasing and the two sample tails are included. The resulting Mellin modulus is below $5.653 \cdot 10^{-6}$, whereas cancellation requires modulus above $1.13 \cdot 10^{-5}$. The actual third heat coefficient therefore has modulus greater than $5.08 \cdot 10^{-5}$. The supplied source certificate states its arithmetic assumptions and regenerates its numerical inputs.

Keywords: twisted tube; relative heat trace; Mellin transform; Dirichlet-to-Neumann operator; validated numerics; inverse spectrum.

1 The fixed geometry and the precise new conclusion

Let

$$\omega = \operatorname{int}\big(\overline{(1,3)\times(0,1)} \cup \overline{(3,\sqrt{11})\times(0,2)}\big) \tag{1.1}$$

be a polar section. Its aperture $r = 3$, $0 < \vartheta < 1$, is interior. A Euclidean tube end is parametrized by

$$(a, r, \vartheta) \longmapsto (a, r\cos(\vartheta + \Theta(a)), r\sin(\vartheta + \Theta(a))), \qquad (r, \vartheta) \in \omega.$$

The speed $b(a) = \Theta'(a)$ is smooth, positive and strictly increasing outside a bounded core. Its inverse in the absolute axial coordinate is, for sufficiently large y,

$$R_\epsilon(y) = y\{1 + \epsilon\cos(3\tau\log y)\}, \qquad \tau = \frac{\pi}{\log 2}, \qquad |\epsilon| < \frac{1}{\sqrt{1+9\tau^2}}. \tag{1.2}$$

The strict bound implies $g > 0$ and $g + g' > 0$ for $g(u) = 1 + \epsilon\cos(3\tau u)$. The same fixed polar section is used through the bounded core, with a Dirichlet cap as in the preceding heat theorem [1]. Changing a bounded core-volume normalization changes only a constant in the lower heat profile below.

Theorem 1.1 (Third-mode detection)**.** *Under the arithmetic model specified in Section 6, the source certificate proves the following statement for the Dirichlet tube* (1.1)–(1.2)*. Put* $z_3 = 1 + 3i\tau$*. Its actual one-end heat trace has the expansion*

$$\operatorname{Tr} e^{-H_\epsilon/T^2} = \frac{T^4}{6\pi} + T^3\left\{C_{\mathrm{core}} + \epsilon\,\mathrm{Re}\big(C_3 e^{3i\tau\log T}\big)\right\} + o(T^3), \qquad |C_3| > 5.08\cdot 10^{-5}. \tag{1.3}$$

The leading profile is independent of ϵ*, while the full eigenvalue multiset determines* ϵ*. In particular two distinct admissible amplitudes in this specified one-end family cannot be isospectral. The same conclusion holds when two identical specified ends carry the same amplitude.*

The global two-profile expansion and its coefficient formula are inherited from [1, Theorem 1.1 and the connected-step formula]. Our new assertion is a certified separation of that formula from zero at z_3. We prove the connected resolvent and all numerical error bounds needed for that separation.

For a precise statement of the inherited reduction, let $D(t)$ be the relative heat trace of the connected step

$$\mathcal{S} = \mathrm{int}\big(\overline{(-\infty,0)\times(0,1)} \cup \overline{(0,\infty)\times(0,2)}\big)$$

against the operator with an additional Dirichlet cut at its aperture. Let

$$Z_a(t) = \sum_{n\ge 1} e^{-t(\pi n/a)^2}, \qquad E(t) = D(t) - \tfrac{1}{2}Z_1(t), \qquad \mathcal{M}_E(z) = \tfrac{1}{2}\int_0^\infty t^{(z-1)/2}E(t)\,\mathrm{d}t. \tag{1.4}$$

For $\mathrm{Re}\, z > 0$ these integrals converge absolutely. The global theorem in [1] gives, at every odd lattice frequency $z = 1 + ik\tau$,

$$C(z) = \frac{z}{2\sqrt{\pi}}\{AM_1(z) + 3\mathcal{M}_E(z)\}, \qquad A = \tfrac{7}{2} + \sqrt{11}, \qquad M_1(z) = \frac{\Gamma((z+1)/2)\zeta(z+1)}{2\pi^{z+1}}. \tag{1.5}$$

Thus we need only exclude the particular cancellation value

$$\mathcal{M}_E(z_3) = \mathcal{T} := -\frac{A}{3}M_1(z_3). \tag{1.6}$$

The factor $2^{z+1} = -4$ at odd frequencies explains both the inherited leading blindness and the simplification (1.5).

The predecessor settles $k = 1$ by a Stieltjes estimate for the nonincreasing positive function E. That bound decays algebraically in the frequency. We use its actual connected resolvent rather than replacing an exposed face by a separate angular channel. Dirichlet-to-Neumann trace representations are classical [3]; a smooth-boundary trace theorem alone does not provide the corner-domain and infinite-aperture error estimates used here.

2 Heat bounds and an explicit flatness remainder

The killed Brownian-bridge representation, or finite-box monotone limits, defines D without subtracting two infinite heat traces. The relevant identities established in [1] are

$$E \ge 0, \qquad E \text{ nonincreasing}, \qquad E(0+) = \frac{1}{36}, \qquad E(t) \le \frac{Z_2(t) - Z_1(t)}{2}. \tag{2.1}$$

For clarity, after translating the junction vertex to $(0,0)$, its unbounded local wedge is $\{Y < 0\} \cup \{X > 0\}$. The function E counts surviving loops hitting both internal rays, confined to $-1 < Y < 1$. Scaling a loop changes only the width of this strip, proving monotonicity. The unconfined loop mass is $1/36$, equivalently the corner combination $c(3\pi/2) + c(\pi/2)$ for $c(\alpha) = (\pi^2 - \alpha^2)/(24\pi\alpha)$; see the polygon heat calculation in [2, 1].

Lemma 2.1 (Explicit loop flatness)**.** *For $t > 0$,*

$$0 \le \tfrac{1}{36} - E(t) \le J(t) := \frac{1}{\sqrt{\pi t}} \sum_{n \ge 1} e^{-n^2/t} \le \frac{e^{-1/t}}{\sqrt{\pi t}(1 - e^{-3/t})}. \tag{2.2}$$

Proof. A loop removed by the two horizontal walls hits $X = 0$, visits $Y = 0$, and hits at least one of $Y = 1, -1$. Discard its wedge restriction and separate the horizontal and vertical bridges. The integrated free horizontal diagonal weight of hitting zero is $1/2$. The two vertical events give, by a union bound, the integrated one-dimensional bridge weight of hitting both 0 and 1. Reflection and inclusion–exclusion give that weight as $Z_1(t) + 1/2 - (4\pi t)^{-1/2}$. The theta identity turns it into $J(t)$. Finally $n^2 \ge 1 + 3(n-1)$ bounds its sum geometrically. □

Positive integration of (2.1) gives

$$\int_0^\infty tE(t)\,\mathrm{d}t \le \frac{1}{12}, \qquad \int_0^\infty t^2 E(t)\,\mathrm{d}t \le \frac{1}{15}, \qquad \mathcal{M}_E(1) \le U < 0.022, \tag{2.3}$$

where a convenient explicit choice is

$$U = \frac{\log 18 + (1 - 18^{-3})^{-1}}{18\pi^2}. \tag{2.4}$$

For the last bound use $E \le 1/36$ up to $t_0 = 4\log 18/\pi^2$ and $E \le Z_2/2$ afterwards, then $n^2 \ge 1 + 3(n-1)$. The first two constants follow from the exact odd-integer series $Z_2 - Z_1$. These bounds will control analytic continuation, sample tails and exact-node conversion.

3 The actual connected aperture and its trace

On $L^2(0,1)$ let $A_D = -\partial_\vartheta^2$ have Dirichlet conditions at both ends, and let A_N have a Dirichlet condition at 0 and Neumann condition at 1. Their normalized real bases are

$$e_n = \sqrt{2}\sin(n\pi\vartheta), \qquad f_m = \sqrt{2}\sin((m - \tfrac{1}{2})\pi\vartheta).$$

Write $Q_D(\lambda) = (A_D + \lambda)^{1/2}$ and $Q_N(\lambda) = (A_N + \lambda)^{1/2}$, with principal roots. Zero extension $J_0 : L^2(0,1) \to L^2(0,2)$ and reflection about 1 give

$$J_0^* F(A_{(0,2),D}) J_0 = \tfrac{1}{2}\{F(A_D) + F(A_N)\}. \tag{3.1}$$

It follows by splitting the extension into its even and odd parts; each part has squared norm one half. It holds for the root forms and bounded inverse powers needed below. The aperture operator is therefore

$$\mathfrak{M}(\lambda) = \frac{3Q_D(\lambda) + Q_N(\lambda)}{2}. \tag{3.2}$$

Its closed form has trace domain $H^{1/2}_{00}(0,1)$; this is the zero-extension trace space, including the aperture endpoint conditions. No commutation between the two interval roots is assumed.

Proposition 3.1 (Resolvent trace formula)**.** *For $\operatorname{Re}\lambda > 0$ put*

$$q(\lambda) = \int_0^\infty te^{-\lambda t}E(t)\,\mathrm{d}t, \qquad C_\lambda = \mathfrak{M}(\lambda)^{-1}.$$

Then

$$\begin{aligned} q(\lambda) &= \operatorname{Tr}(C_\lambda B_\lambda) + \operatorname{Tr}(C_\lambda A_\lambda C_\lambda A_\lambda) - \tfrac{1}{2}\operatorname{Tr}(A_D + \lambda)^{-2}, \\ A_\lambda &= \tfrac{1}{4}(3Q_D^{-1} + Q_N^{-1}), \qquad B_\lambda = \tfrac{1}{8}(3Q_D^{-3} + Q_N^{-3}). \end{aligned} \tag{3.3}$$

All products defining these traces are trace class.

Proof. For real $\lambda \geq 0$ the Poisson extension on the left half-strip is $K_D u(X) = e^{XQ_D} u$, $X < 0$, and on the right it is $K_2 u(X) = e^{-XQ_2} J_0 u$, $X > 0$. Let K combine them. Spectral integration and (3.1) give

$$K^*K = \tfrac{1}{2}(Q_D^{-1} + J_0^* Q_2^{-1} J_0) = \mathfrak{M}' = A_\lambda.$$

Integration by parts first for traces supported away from the aperture endpoints, followed by closure in $H_{00}^{1/2}$, gives

$$(L + \lambda)^{-1} - (L_{\mathrm{cut}} + \lambda)^{-1} = K \mathfrak{M}^{-1} K^*.$$

The added Poisson solution has common aperture trace and matches the two normal fluxes. The coercive closed aperture form gives uniqueness, including at the reflex endpoint. Moreover

$$\operatorname{Tr}(\mathfrak{M}^{-1/2} \mathfrak{M}' \mathfrak{M}^{-1/2}) \leq \frac{2}{3} \left\{ \frac{3}{4} \operatorname{Tr} Q_D^{-2} + \frac{1}{4} \operatorname{Tr} Q_N^{-2} \right\} < \infty,$$

because $Q_N \leq Q_D$ in form order and $\mathfrak{M} \geq 3Q_D/2$. Thus $K\mathfrak{M}^{-1/2}$ is Hilbert–Schmidt and the resolvent correction is trace class. The nonnegative heat-kernel difference and Tonelli identify its trace with $\int e^{-\lambda t} D(t)\, \mathrm{d}t = \operatorname{Tr}(\mathfrak{M}^{-1}\mathfrak{M}')$.

Here $\mathfrak{M}'$ is Hilbert–Schmidt and $\mathfrak{M}'' = -B_\lambda$ is trace class, locally uniformly in λ. Difference quotients in these ideals justify differentiation of the trace: the inverse derivative contains two Hilbert–Schmidt factors, and the other term contains $\mathfrak{M}''$. Subtracting the derivative of the narrow-strip correction $Z_1/2$ proves (3.3) on the positive axis. Holomorphic form inversion and the same locally uniform trace ideals extend it to the right half-plane. □

4 Sectorial continuation and Mellin conditioning

The root construction continues to the slit plane $\lambda \in \mathbb{C} \setminus (-\infty, 0]$, as does (3.3). The following bound makes that continuation quantitative.

Lemma 4.1 (Sector bound)**.** *For $\lambda = \rho e^{i\psi}$, $|\psi| \leq \phi < \pi$, put $c_0 = \cos(\phi/2)$. Then*

$$|q(\rho e^{i\psi})| \leq c_0^{-3}\{q(\rho) + D_2(\rho)\}, \qquad D_2(\rho) = \sum_{n \geq 1} (\pi^2 n^2 + \rho)^{-2}, \tag{4.1}$$

$$\int_0^\infty |q(\rho e^{i\psi})|\, \mathrm{d}\rho \leq c_0^{-3}(2U + 1/6). \tag{4.2}$$

Proof. For $a > 0$, elementary scalar algebra gives

$$\operatorname{Re} \sqrt{a + \lambda} \geq c_0 \sqrt{a + \rho}, \qquad |\sqrt{a + \lambda}| \geq \sqrt{c_0}\sqrt{a + \rho}.$$

For the modulus inequality square $|a + \lambda| \geq \cos(\psi/2)(a + \rho)$; the difference of its squares is $(a - \rho)^2 \sin^2(\psi/2)$. The real-part inequality follows from $|a + \lambda| \geq a \cos\psi + \rho$ and the formula for the real part of a principal square root. Apply functional calculus separately in each root. With $M_\rho = \mathfrak{M}(\rho)$,

$$\operatorname{Re} \mathfrak{M}(\lambda) \geq c_0 M_\rho, \qquad \|M_\rho^{1/2} C_\lambda M_\rho^{1/2}\| \leq c_0^{-1}.$$

This is coercivity of a normalized closed form; its inverse is holomorphic locally on the slit plane.

Write $F(\rho) = \operatorname{Tr}(M_\rho^{-1} B_\rho) + \operatorname{Tr}[(M_\rho^{-1/2} A_\rho M_\rho^{-1/2})^2]$. Absolute values of the beta rank-one factors of B_λ are bounded by $c_0^{-3/2} B_\rho$, so the first complex trace is at most $c_0^{-5/2}$ times its real counterpart. If H is the positive operator formed from the absolute alpha factors, then $H \leq c_0^{-1/2} A_\rho$. The second complex trace is bounded by $\|H^{1/2} C_\lambda H^{1/2}\|_{\mathfrak{S}_2}^2$. Schatten Hölder with exponents $4, \infty, 4$ bounds this by

$$c_0^{-2} \operatorname{Tr}[(M_\rho^{-1/2} H M_\rho^{-1/2})^2] \leq c_0^{-3} \operatorname{Tr}[(M_\rho^{-1/2} A_\rho M_\rho^{-1/2})^2].$$

The last step uses positive eigenvalue monotonicity, not commuting roots. Finally the DD subtraction has modulus at most $c_0^{-2}D_2(\rho)/2$, and $F(\rho) = q(\rho) + D_2(\rho)/2$. This proves (4.1). Positive Tonelli gives $\int q = \int E = 2\mathcal{M}_E(1)$ and $\int D_2 = 1/6$, proving (4.2). The estimates also prove local uniform trace convergence for the continuation just used. □

Put $b_* = 3\pi/(2\log 2)$ and $s = 1 + ib_*$, so $z_3 = 2s - 1$. Absolute Fubini on the positive axis yields

$$\mathcal{M}_E(z_3) = \frac{1}{2\Gamma(2-s)} \int_0^\infty \lambda^{1-s} q(\lambda)\, \mathrm{d}\lambda.$$

This real-ray formula has amplification exceeding 3300. Rotate strictly inside the right half-plane to $\lambda = c\rho$, $c = 1 - 8i$:

$$\mathcal{M}_E(z_3) = PI, \qquad P = \frac{c^{2-s}}{2\Gamma(2-s)}, \qquad I = \int_0^\infty \rho^{1-s} q(c\rho)\, \mathrm{d}\rho, \qquad |P| < 2. \tag{4.3}$$

Indeed $|q(\lambda)| \le \min(1/12, [36(\operatorname{Re}\lambda)^2]^{-1})$ in that half-plane, so both circular arcs vanish. All powers use the principal logarithm. The Gamma modulus identity gives

$$|P|^2 < \frac{65 e^{2b_* \arctan(1/8)}}{8\pi b_*} < \frac{65 e^{17/10}}{144} < 4,$$

using $6 < b_* < 34/5$, $\pi b_* > 18$ and $e^{17/10} < 6$. No boundary integral on the imaginary axis is needed.

5 A finite certificate for the infinite aperture

We now give the bounds evaluated at each actual complex node $\lambda = (1 - 8i)\rho$. Set

$$d_n = \sqrt{\pi^2 n^2 + \rho}, \quad n_m = \sqrt{\pi^2 (m - \tfrac{1}{2})^2 + \rho}, \quad D = (A_D + \rho)^{1/2}.$$

Then $\operatorname{Re}\mathfrak{M}(\lambda) \ge 3D/2$ and $\|D^{1/2} C_\lambda D^{1/2}\| \le 2/3$.

5.1 Root entries with a leading infinite-tail correction

Direct integration gives the exact overlap

$$O_{nm} = \langle e_n, f_m \rangle = \frac{(-1)^{n+m} 2n}{\pi(n^2 - u_m^2)}, \qquad u_m = m - \tfrac{1}{2}. \tag{5.1}$$

For $n, l \le N$, the DN root entry is $\sum_{m \ge 1} O_{nm} O_{lm} \sqrt{\pi^2 u_m^2 + \lambda}$. Choose $M + 1/2 \ge 2N$ and put $\eta = N^2/(M + 1/2)^2$. The omitted entry after M terms has the explicit center

$$L_{nl} = (-1)^{n+l} \frac{2nl}{\pi M^2} \tag{5.2}$$

and an absolute remainder no greater than

$$\frac{4nl}{\pi} \left\{ \frac{1}{8M^4} + \frac{1}{2M^5} + \frac{n^2 + l^2 + |\lambda|/\pi^2}{4M^4(1-\eta)^2} \right\}. \tag{5.3}$$

To see this, the leading summand is $(-1)^{n+l}(4nl/\pi)u_m^{-3}$. Convex midpoint quadrature gives

$$0 \le \frac{1}{2M^2} - \sum_{m>M} u_m^{-3} \le \frac{1}{8M^4} + \frac{1}{2M^5}, \qquad \sum_{m>M} u_m^{-5} \le \frac{1}{4M^4}.$$

For example bound each midpoint error of x^{-3} on a unit cell by the maximum of its second derivative divided by 24, and bound the resulting decreasing sum by its first term plus its integral. The normalized root differs from 1 by at most $|\lambda|/(\pi^2 u_m^2)$, while the two inverse overlap denominators differ from 1 by at most $(n^2+l^2)/(u_m^2(1-\eta)^2)$. Combining them proves (5.3). This is a bound on the actual omitted entry, not an extrapolation from successive finite sections.

5.2 Arbitrary trial columns and their full residuals

Use the input list $g_i = (e_1, \dots, e_K, f_1, \dots, f_K)$. For arbitrary finite DD trial columns v_i, define the bilinear matrices

$$X_{ij} = g_i^T C_\lambda g_j, \qquad B_{ij} = g_i^T v_j + v_i^T g_j - v_i^T \mathfrak{M} v_j, \tag{5.4}$$
$$r_i = g_i - \mathfrak{M} v_i, \qquad |X_{ij} - B_{ij}| \le \frac{2}{3}\sqrt{w_i w_j}, \qquad w_i = \langle r_i, D^{-1} r_i \rangle.$$

Both roots, hence the inverse, are complex symmetric under transpose in these real bases. Expansion gives $X_{ij} - B_{ij} = r_i^T C_\lambda r_j$; coercivity and Cauchy–Schwarz prove (5.4). The disks are two-sided even on the diagonal. A numerical linear solve merely proposes v_i.

All residual quantities can be enclosed using finite root entries. For $v = \sum_{n \le N} v_n e_n$ and $\rho_i = \|r_i\|^2$,

$$w_i \le \sum_{n \le N} \frac{|\langle e_n, r_i \rangle|^2}{d_n} + \frac{\rho_i - \sum_{n \le N} |\langle e_n, r_i \rangle|^2}{d_{N+1}}. \tag{5.5}$$

When evaluated by intervals, the subtracted sum is bounded below and ρ_i above. To obtain ρ_i, use

$$\|\mathfrak{M} v\|^2 = \tfrac{1}{4}\{9\|Q_D v\|^2 + \|Q_N v\|^2 + 6\,\mathrm{Re}\langle Q_D v, Q_N v \rangle\}. \tag{5.6}$$

Here $\|Q_D v\|^2 = \sum |q_n|^2 |v_n|^2$, with $q_n = \sqrt{\pi^2 n^2 + \lambda}$. The distinct DN norm is

$$\|Q_N v\|^2 = \int_0^1 |v'|^2 \, \mathrm{d}\vartheta + \rho \|v\|^2 + \sum_{m \ge 1} \left\{ \sqrt{(\pi^2 u_m^2 + \rho)^2 + 64\rho^2} - (\pi^2 u_m^2 + \rho) \right\} |f_m^T v|^2. \tag{5.7}$$

Writing $S_v = \sum_{n \le N} n|v_n|$ and $a = M - 1/2$, the omitted correction in (5.7) is at most

$$\frac{32(8\rho)^2 S_v^2}{45\pi^4 a^5}. \tag{5.8}$$

Indeed $|O_{nm}| \le 8n/(3\pi u_m^2)$ for the omitted modes, and $\sqrt{A^2 + 64\rho^2} - A \le 64\rho^2/(2A)$. The decreasing sixth-power sum proves (5.8). The cross term in (5.6) uses only the finite DN root block. For a DN input, $f_m^T \mathfrak{M} v = (3/2) f_m^T Q_D v + (\sqrt{\pi^2 u_m^2 + \lambda}/2) f_m^T v$. These identities enclose the full residual, including its part outside the DD trial span.

5.3 Absolute input-factor tails of cubic order

Assign the combined input coefficients

$$\alpha(e_n) = \frac{3}{4q_n}, \quad \alpha(f_m) = \frac{1}{4r_m}, \qquad \beta(e_n) = \frac{3}{8q_n^3}, \quad \beta(f_m) = \frac{1}{8r_m^3}, \quad r_m = \sqrt{\pi^2 u_m^2 + \lambda}.$$

The truncated trace is

$$F_K = \sum_{i \le 2K} \beta_i X_{ii} + \sum_{i,j \le 2K} \alpha_i \alpha_j X_{ij} X_{ji}. \tag{5.9}$$

These are complex products, not absolute squares. Absolute values enter only the following error bounds. Let $H = \sum_i |\alpha_i| g_i g_i^*$ and $H_>$ its omitted input factors. Set

$$t_K = \frac{3}{4d_{K+1}^2} + \frac{1}{4n_{K+1}^2}, \qquad H_> \le t_K D. \tag{5.10}$$

The omitted beta trace is at most

$$B_> = \frac{1}{12\pi^4}\left\{\frac{1}{K^3} + \frac{1}{3(K-1/2)^3}\right\}. \tag{5.11}$$

This follows from C_λ's normalized norm $2/3$ and the separate DD/DN fourth-power spectral sums.

For the high–high squared trace, Schatten Hölder and (5.10) give

$$HH \le \frac{4}{9} t_K \operatorname{Tr}(D^{-1} H_>) \le \frac{4t_K}{9\pi^2}\left\{\frac{3}{4K} + \frac{1}{4(K-1/2)}\right\}. \tag{5.12}$$

Indeed the positive normalized $H_>$ has norm at most t_K; its squared trace is bounded by norm times trace. No interval-root commutation is used. For each low input, let $T_i = \langle v_i, H_> v_i\rangle$. Then

$$LH \le \sum_{i \le 2K} |\alpha_i| \left\{\sqrt{T_i} + \tfrac{2}{3}\sqrt{t_K w_i}\right\}^2. \tag{5.13}$$

This follows by applying $H_>^{1/2}$ to $C_\lambda g_i = v_i + C_\lambda r_i$ and using normalized coercivity. The complete squared-trace tail is at most $2LH + HH$. The finite sums defining T_i use DD coefficients above K and DN coefficients from $K+1$ through M. Their remaining DN sum is at most

$$\frac{4S_{v_i}^2}{9\pi^3(M-1/2)^4}. \tag{5.14}$$

For (5.14), multiply the squared overlap bound by $1/(4\pi u_m)$ and sum u_m^{-5}. Thus $B_> + 2LH + HH$ is a fully finite-data bound on the omitted trace. The final DD subtraction in (3.3) is summed through L, with remaining modulus at most $1/(6\pi^4 L^3)$ after its factor $1/2$.

6 Arithmetic enclosure and a runnable certificate

The computation uses binary64 round-to-nearest basic arithmetic and correctly rounded real square roots. It assumes those properties for the pinned NumPy implementation; it is not a formal verification of hardware or the Python runtime. All reported runs must satisfy the finite/range checks in the supplied source. Scalar integer powers of floating values are explicit repeated multiplications. Complex reciprocal centers use real divisions by $x^2 + y^2$.

Put $u = 2^{-53}$ and $\gamma_k = ku/(1-ku)$. A complex entrywise ball $(\widehat{A}, R_A)$ encloses A when $|A - \widehat{A}| \le R_A$. The source uses the following rules, with every positive upper expression additionally inflated by $1 + \gamma_{64}$ and rounded with `nextafter` toward $+\infty$. An absolute 10^{-250} floor covers negligible underflow terms in the checked computation.

Operation	Propagated bound before final outward inflation
Complex sum	$R_A + R_B + \gamma_8(\lvert\widehat{A}\rvert + \lvert\widehat{B}\rvert)$
Entry product	$\lvert\widehat{A}\rvert R_B + R_A\lvert\widehat{B}\rvert + R_A R_B + \gamma_{16}\lvert\widehat{A}\rvert\lvert\widehat{B}\rvert$
Length-k matrix product	$\lvert\widehat{A}\rvert R_B + R_A\lvert\widehat{B}\rvert + R_A R_B + \gamma_{8k+32}\lvert\widehat{A}\rvert\lvert\widehat{B}\rvert$
Positive length-k dot	Its computed value divided by $1 - \gamma_{2k+8}$
Reciprocal of z	$R/(\lvert\widehat{z}\rvert(\lvert\widehat{z}\rvert - R))$ plus a γ_{64} center allowance, using lower moduli
Squared modulus	$2\lvert\widehat{z}\rvert R + R^2$ plus a γ_{16} center allowance

All absolute center factors in this table are themselves bounded upwards. The matrix-product bound is componentwise. The familiar product-of-$(1+\delta)$ estimate proves the γ rules; regrouping a conventional sum of k real products has at most $2k$ elementary factors on any path. A complex product and its two real sums fit the conservative $8k + 32$ allowance. Fused operations do not enlarge these bounds.

Proof-relevant products call NumPy 2.3.5 `einsum('ik,kj->ij',...,optimize=False)`. Its pinned source dispatches to conventional C sum-products [5]. A generic BLAS interface would not justify that componentwise model [6]; BLAS is used only to propose trial columns. Their accuracy is immaterial by (5.4).

The overlap denominators are exactly representable quarter-integers in the checked index range. The stored binary64 π value is bounded using rational Machin series. Principal roots are evaluated without subtraction:

$$h_0 = \sqrt{a^2 + b_0^2}, \quad u_0 = \sqrt{(h_0 + a)/2}, \quad \sqrt{a + ib_0} = u_0 + i\frac{b_0}{2u_0}, \qquad a > 0.$$

There are fewer than 32 conditioned elementary factors after the input error is included; the implementation allows γ_{128}. The stable DN correction uses $b_0^2/(\sqrt{a^2 + b_0^2} + a)$ and allows γ_{512}. The finite DN matrix includes a $\gamma_{8M+2048}$ absolute-product allowance, covering root/overlap inputs and the conventional dot. Insertion of (5.2) and evaluation of (5.3) have separate outward allowances. The number reserves exceed the finite-operation counts in these positive formulas.

For the final scalar assembly the source uses outward decimal intervals, rational bounds for π and logarithms, and sine/cosine Taylor polynomials with explicit remainder. Floating node centers and radii are converted by exact hexadecimal or exact decimal expansion; a nearest-decimal display is never treated as an outward radius. The public driver regenerates its inputs, checks every acceptance inequality, and returns the actual bound; it does not accept a saved success label in place of recomputation.

7 A global analytic quadrature with 27 actual nodes

Set

$$f(x) = e^{(2-s)x} q(ce^x), \qquad h = \tfrac{2}{5}, \qquad d = \tfrac{3}{2}, \qquad \theta = \arctan 8.$$

Lemma 4.1 makes f analytic in the closed strip $|\operatorname{Im} x| \le d$, since $\theta + d < 3 < \pi$. Its shifted boundary norms satisfy

$$\int_{\mathbb{R}} |f(x + iy)|\,\mathrm{d}x \le \frac{e^{b_* y}}{\sqrt{65}} \cos\!\left(\frac{y - \theta}{2}\right)^{-3} (2U + 1/6), \qquad y = \pm d. \tag{7.1}$$

Their sum is less than 1000. A simple rational check uses $1.4 < \theta < 1.5$, $b_* < 34/5$, $e^{10.2} < 27000$, $\cos((\theta - d)/2) > .99$, $\cos((\theta + d)/2) > .07$, $\sqrt{65} > 8$, and $2U + 1/6 < .211$.

The strip-trapezoid theorem [4, Theorem 5.1 and the asymmetric boundary extension after (5.15)], or direct Fourier contour shift, gives

$$\left| I - h \sum_{j \in \mathbb{Z}} f(jh) \right| < \frac{1000}{e^{2\pi d/h} - 1} < \frac{1}{16000000}. \tag{7.2}$$

For completeness, positive and negative nonzero Fourier aliases are bounded using the two boundary integrals times $e^{-2\pi d|n|/h}$; summing them gives (7.2). The sector bound yields uniform decay $O(e^x)$ at $-\infty$ and $O(e^{-x/2})$ at $+\infty$, justifying both shifts and Poisson summation. After multiplication by $|P| < 2$, the alias error is below $1.25 \cdot 10^{-7}$, independently of numerical quadrature agreement.

7.1 The low and high infinite sample sums

Let $r_j = e^{jh}$. Since (2.3) implies $|q(c\rho) - q(0)| \le |c|\rho/15 < 3\rho/5$, the low grid is

$$h \sum_{j \le -18} f(jh) = L_0 + \varepsilon_-, \qquad L_0 = q(0)\frac{he^{(2-s)(-18)h}}{1 - e^{-(2-s)h}}, \tag{7.3}$$
$$|\varepsilon_-| \le \frac{3h}{5}\frac{r_{-18}^2}{1 - e^{-2h}}.$$

The certificate recomputes $q(0)$ from a certified real point $q(2^{-39})$, enlarged by $2^{-39}/15$. It verifies the fixed rational outer interval

$$q(0) \in [0.00550048488672214,\ 0.00585907094669114], \tag{7.4}$$

which is retained throughout the complex geometric sum.

By Lemma 2.1, for $\operatorname{Re}\lambda = a > 0$,

$$\left| q(\lambda) - \frac{1}{36\lambda^2} \right| \le S(a), \qquad S(a) = \sum_{n \ge 1} \left(\frac{n}{a} + \frac{1}{2a^{3/2}} \right) e^{-2n\sqrt{a}}. \tag{7.5}$$

Indeed integrate $te^{-at}J(t)$ termwise, differentiating the Gaussian-Laplace identity

$$\int_0^\infty t^{-1/2} e^{-at - n^2/t}\, \mathrm{d}t = \sqrt{\pi/a}\, e^{-2n\sqrt{a}}.$$

If $v = e^{-2\sqrt{a}}$, then

$$S(a) = \frac{v}{a(1-v)^2} + \frac{v}{2a^{3/2}(1-v)},$$
$$\int_R^\infty S(a)\, \mathrm{d}a = \frac{e^{-2\sqrt{R}}}{\sqrt{R}(1 - e^{-2\sqrt{R}})}.$$

The last equality integrates $-\frac{\mathrm{d}}{\mathrm{d}a}(e^{-2n\sqrt{a}}/\sqrt{a})$ termwise. Therefore the high grid is

$$h \sum_{j \ge 10} f(jh) = H_0 + \varepsilon_+, \qquad H_0 = \frac{he^{-10sh}}{36c^2(1 - e^{-sh})}, \tag{7.6}$$
$$|\varepsilon_+| \le hRS(R) + \int_R^\infty S(a)\, \mathrm{d}a, \qquad R = e^{10h}.$$

Here $aS(a)$ decreases, so the first term plus a decreasing-function integral bounds the sampled residual. These two tail replacements leave precisely $j = -17, \dots, 9$ as actual aperture evaluations.

7.2 Exact grid inputs and final error combination

The intended $r_j = e^{2j/5}$ is enclosed by scalar intervals; the solver receives its recorded exact binary64 approximation $\widehat{r}_j$. The correction to $q(cr_j)$ is at most

$$9 \min\left\{ \tfrac{1}{15}, \frac{1}{18 \min(r_j, \widehat{r}_j)^3} \right\} |r_j - \widehat{r}_j|, \tag{7.7}$$

using the derivative Laplace integral, (2.3) and $|c| < 9$. Each positive term is evaluated outwards. If the actual node disk is $|q(c\widehat{r}_j) - z_j| \le \epsilon_j$, its contribution is $hr_j\epsilon_j$ before P. The complete complex sum is

$$S_0 = L_0 + H_0 + \sum_{j=-17}^{9} hr_j e^{-ib_* jh} z_j. \tag{7.8}$$

The outer box for $q(0)$, all scalar intervals, the alias, both sample-tail errors, the node disks and (7.7) give an enclosure of I around (7.8). In particular a norm upper bound gives $|\mathcal{M}_E(z_3)| \le 2 \sup |I|$.

8 Numerical separation and its spectral consequence

The source-only computation uses $N = 512$, $K = 128$, $M = 4096$ at each of the 27 complex nodes. The scalar acceptance inequalities and every numerical input are regenerated. Table 1 gives conservative outward error bounds for the designated computation; the complete machine-readable output retains its actual intervals.

Source of error in Mellin units	Strict upper bound
Global analytic alias	$1.2500 \cdot 10^{-7}$
Low infinite sample replacement	$4.8590 \cdot 10^{-7}$
High infinite sample replacement	$4.2960 \cdot 10^{-7}$
Sum of actual node disk radii with weights	$2.2970 \cdot 10^{-6}$
Exact-node input conversion	10^{-12}

Table 1: Complete categories of analytic and node errors. The interval width from (7.4) remains in the complex sum, rather than being discarded or counted as a successful point approximation.

The resulting conservative comparison is

$$|\mathcal{M}_E(z_3)| < 5.653 \cdot 10^{-6}, \qquad |\mathcal{T}| > 1.13 \cdot 10^{-5}. \tag{8.1}$$

For the target, the absolutely convergent Euler product gives $|\zeta(2+2ib_*)| \ge \zeta(4)/\zeta(2) = \pi^2/15$, and

$$|\Gamma(1 + ib_*)|^2 = \frac{\pi b_*}{\sinh(\pi b_*)}.$$

Rational Machin and positive atanh series enclose π and $\log 2$; positive exponential series with geometric tails verify the strict target inequality in (8.1). All decisive scalar tests use exact fractions or directed intervals, not printed decimal rounding. From (1.5), $|z_3| > 12$ and $2\sqrt{\pi} < 4$ imply

$$|C_3| > 9\{1.13 \cdot 10^{-5} - 5.653 \cdot 10^{-6}\} > 5.08 \cdot 10^{-5}.$$

This proves the numerical assertion in Theorem 1.1 under the stated arithmetic model.

The literal interior integral can also be certified, not just the whole-ray bound. Subtract $100^{-s}/(36c^2 s)$ from the full I enclosure and add radius

$$\frac{1}{12 \cdot 10^{10}} + \frac{e^{-20}}{10(1 - e^{-20})}$$

to cover the omitted continuous low interval and the high residual. The driver reports a complex disk for $\int_{10^{-10}}^{100} \rho^{1-s} q(c\rho)\, \mathrm{d}\rho$ separately. That disk need not exclude zero: the desired test concerns (1.6), not nonvanishing of this finite integral by itself.

To finish the spectral conclusion, insert the nonzero multiplier into the inherited expansion (1.3). The full spectrum determines the heat trace. Put $u = \log T$ and average its normalized lower profile against $e^{-3i\tau u}$ over an interval of length $2\pi/(3\tau)$. The constant core term integrates to zero and the limit extracts $\epsilon C_3/2$. The $o(1)$ remainder is uniformly small on each such interval once its left endpoint tends to infinity. Nonzero C_3 therefore determines ϵ. Identical two ends double both profiles, with the same conclusion.

9 Reproduction, relation to prior work and limitations

The accompanying source is a mathematical certificate, not a numerical plot or a saved internal approval. A clean run regenerates the tiny-real input for (7.4), all 27 actual connected-aperture disks, the exact scalar comparisons, and the global and finite-interior enclosures. The only third-party numerical dependency is NumPy 2.3.5; the scalar interval layer uses the Python standard library. The tested reference environment is CPython 3.12 with IEEE binary64 arithmetic. The README specifies commands and the implementation contract. A new platform must satisfy that contract and pass the same inequalities; bit-identical proposed trial vectors are not required for a new valid run. The stored reference output, if supplied, identifies one run and cannot replace source recomputation.

This paper is a computational supplement to [1], in the same twisted-tube result family. Its inherited global theorem and leading blindness are explicitly credited. Contour rotation, Dirichlet-to-Neumann trace methods, Schatten inequalities and strip-trapezoid quadrature are established mechanisms [3, 4]; our contribution is their complete, model-specific infinite-aperture certificate resolving the previously open third frequency. Related waveguide spectral asymptotics or inverse boundary-data problems do not by themselves supply this fixed connected-junction Mellin separation.

The result is ideal asymptotic audibility of the original third mode under the stated arithmetic model.

AI assistance. AI tools assisted the derivations, implementation, literature checking and independent internal verification. The mathematical statements, proofs and runnable certificate are provided for scrutiny.

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